SYMMETRIC MULTIVARIATE WAVELETS

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For arbitrary matrix dilation \( M \) whose determinant is odd or equal to \( \pm 2 \), we describe all symmetric interpolatory masks generating dual compactly supported wavelet systems with vanishing moments up to arbitrary order \( n \). For each such mask, we give explicit formulas for a dual refinable mask and for wavelet masks such that the corresponding wavelet functions are real and symmetric/antisymmetric. We proved that an interpolatory mask whose center of symmetry is different from the origin cannot generate wavelets with vanishing moments of order \( n > 0 \). For matrix dilations \( M \) with \( |\det M| = 2 \), we also give an explicit method for construction of masks (non-interpolatory) \( m_0 \) symmetric with respect to a semi-integer point and providing vanishing moments up to arbitrary order \( n \). It is proved that for some matrix dilations (in particular, for the quincunx matrix) such a mask does not have a dual mask. Some of the constructed masks were successfully applied for signal processes.

**Keywords:** Wavelet system; matrix dilation; Unitary Extension Principle; interpolatory mask; symmetric/antisymmetric wavelet function.

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1. Introduction

It is well known that the order of vanishing moments is one of the most important factors for success of wavelets in applications. In particular, vanishing moment condition is necessary for smoothness of wavelets and guarantees the approximation order. There is a well-known method on how to provide this property in the one-dimensional case with dyadic dilation: a generating mask \( m_0 \) should be represented in the form

\[
m_0(x) = (1 + e^{2\pi i x})^k T(x)
\]

(see, e.g., Ref. 9). Situation is essentially different in the multi-dimensional case. Zero properties of masks cannot
be described by means of factorization because no Euclid algorithm for multivariate polynomials exists. Different characterizations of masks providing vanishing moments are known, in particular, so-called sum rule. The sum rule is appropriate to check vanishing moment property for a given mask. However, finding masks by means of the sum rule (as well as some other characterizations) is possible only numerically, especially, when it is difficult to construct masks providing vanishing moments with some special properties. In this aspect, a polyphase criterium for vanishing moments given in Ref. 26 is more convenient. This criterium gives an explicit general form for all masks providing vanishing moment of arbitrary order \( n \), and we have a chance to extract desirable ones from this formula.

The symmetry/antisymmetry of refinable and wavelet functions (which is equivalent to the symmetry/antisymmetry of the Fourier coefficients of their masks) plays a very important role in applications (see Sec. 4.2 for details). A lot of papers were devoted to methods for construction of one-dimensional symmetric/antisymmetric wavelets. A detailed history of question has given in Ref. 24. In Ref. 10, the symmetric group of a mask such that associated refinable function with a general matrix dilation has a certain kind of symmetry was investigated.

Our goal is to describe all interpolatory symmetric masks generating wavelets with an arbitrary number of vanishing moments and to give explicit formulas for the corresponding symmetric/antisymmetric wavelet functions. We succeeded for matrix dilations whose determinant is odd or equals ±2.

Throughout the paper we will use the following notations.

\( N \) is the set of positive integers, \( \mathbb{R}^d \) denotes the \( d \)-dimensional Euclidean space, \( x = (x_1, \ldots, x_d), y = (y_1, \ldots, y_d) \) are its elements (vectors), \( \langle x, y \rangle = \sum_{k=1}^{d} x_k y_k \), \( |x| = \sqrt{(x,x)} \), \( 0 = (0, \ldots, 0) \in \mathbb{R}^d \); \( \mathbb{Z}^d \) is the integer lattice in \( \mathbb{R}^d \); \( \delta_{ab} \) denotes Kronecker delta; \( \mathbb{T}^d \) is the unit \( d \)-dimensional torus. For \( x, y \in \mathbb{R}^d \), we write \( x > y \) if \( x_j > y_j, j = 1, \ldots, d; \mathbb{Z}^d_+ = \{ x \in \mathbb{Z}^d : x \geq 0 \} \). If \( a, b \in \mathbb{R}^d \), we set \( a^b = \prod_{j=1}^{d} a_j^{b_j} \).

For \( \alpha \in \mathbb{Z}^d_+ \), denote by \( o(\alpha) \) the set of odd coordinates of \( \alpha \); set \( |\alpha| = \sum_{j=1}^{d} \alpha_j \), \( D^{\alpha} f = \prod_{j=1}^{d} \frac{\delta^{[\alpha]}_j}{\delta^{[1]}_1 \cdots \delta^{[\alpha]}_d}, \alpha! = \prod_{j=1}^{d} \alpha_j!, \Pi_{\alpha}(x) = \prod_{j=1}^{d} (1 - e^{2\pi i x_j})^{\alpha_j} \). If \( \alpha, \beta \in \mathbb{Z}^d_+ \), we set \( \binom{\alpha}{\beta} = \frac{\alpha!}{\beta! \cdot (\alpha - \beta)!} \).

Let \( M \) be a non-degenerate \( d \times d \) integer matrix. We say that numbers \( k, n \in \mathbb{Z}^d \) are congruent modulo \( M \) (write \( k \equiv n \pmod{M} \)) if \( k - n = M\ell, \ell \in \mathbb{Z}^d \). The integer lattice \( \mathbb{Z}^d \) is split into cosets with respect to the introduced relation of congruence. The number of cosets is equal to \(|\det M| := m \) (see, e.g., Ref. 21 or Ref. 34). Let us take an arbitrary representative from each coset, call them digits and denote the set of digits by \( D(M) = \{ s_0, \ldots, s_{m-1} \} \). We will consider that \( s_0 \equiv 0 \pmod{M} \). The conjugate matrix to \( M \) will be denoted by \( M^* \), \( I_d \) is the unit \( d \times d \) matrix.

For a \( d \times d \) integer matrix \( M \) whose eigenvalues are bigger than 1 in module (matrix dilation), we will consider wavelets constructed in the framework of multiresolution analysis. Let a MRA in \( L^2(\mathbb{R}^d) \) be generated by a scaling
function $\varphi$ which satisfies the refinement equation
\[ \hat{\varphi}(x) = m_0(M^{*-1}x)\hat{\varphi}(M^{*-1}x), \]
where $m_0 \in L_2(\mathbb{T}^d)$ is its mask (refinable mask). For any $m_{\nu} \in L_2(\mathbb{T}^d)$, there exists a unique set of functions $\mu_{\nu k} \in L_2(\mathbb{T}^d)$, $k = 0, \ldots, m - 1$, (polyphase representatives of $m_{\nu}$) so that
\[ m_{\nu}(x) = \frac{1}{\sqrt{m}} \sum_{k=0}^{m-1} e^{2\pi i(\nu k, x)} \mu_{\nu k}(M^*x). \] (1.1)

The functions $\mu_{\nu k}$ can be expressed by
\[ \mu_{\nu k}(x) = \frac{1}{\sqrt{m}} \sum_{s \in D(M^*)} e^{-2\pi i (M^{-1} \nu k, x + s)} m_{\nu}(M^{-1}(x + s)). \] (1.2)

(see, e.g., Ref. 21). It is clear that a function $m_{\nu}$ is a trigonometric polynomial if and only if its polyphase representatives are also trigonometric polynomials.

A refinable mask $m_0$ is said to be interpolatory if $\mu_{00} \equiv \text{const.}$

Now, let another MRA be generated by a scaling function $\hat{\varphi}$ with a mask $\tilde{m}_0$ such that the integer shifts of $\varphi, \tilde{\varphi}$ are biorthonormal. According to the Unitary Extension Principle, to construct biorthonormal wavelets we should find wavelet masks $m_{\nu}, \tilde{m}_{\nu} \in L_2(\mathbb{T}^d)$, $\nu = 1, \ldots, m - 1$, so that the polyphase matrices
\[ \mathcal{M} := \{ \mu_{\nu k} \}_{\nu, k=0}^{m-1}, \quad \tilde{\mathcal{M}} := \{ \tilde{\mu}_{\nu k} \}_{\nu, k=0}^{m-1}, \]
satisfy
\[ \mathcal{M}\tilde{\mathcal{M}}^* = I_m, \] (1.3)

and define wavelet functions by
\[ \tilde{\psi}^{(\nu)}(x) = m_{\nu}(M^{*-1}x)\tilde{\varphi}(M^{*-1}x), \quad \tilde{\varphi}^{(\nu)}(x) = \tilde{m}_{\nu}(M^{*-1}x)\tilde{\varphi}(M^{*-1}x), \] (1.4)

The corresponding dual systems consisting of the functions
\[ \psi^{(\nu)}_{jk} = m_{\nu}^{j/2}(M^j \cdot + k), \quad \tilde{\psi}^{(\nu)}_{jk} = m_{\nu}^{j/2}(\tilde{M}^j \cdot + k), \]
\[ \nu = 1, \ldots, m - 1, \quad j \in \mathbb{Z}, \quad k \in \mathbb{Z}^d, \]

are biorthonormal.

To construct biorthogonal compactly supported wavelet bases we start with finding trigonometric polynomials $m_0, \tilde{m}_0$ such that $m_0(0) = \tilde{m}_0(0) = 1$ and $\sum_{k=0}^{m-1} \mu_{0k}\tilde{\mu}_{0k} \equiv 1$. The functions
\[ \hat{\varphi}(x) = \prod_{j=1}^{\infty} m_0(M^{*-j}x), \quad \tilde{\hat{\varphi}}(x) = \prod_{j=1}^{\infty} \tilde{m}_0(M^{*-j}x) \]
are Fourier transforms of compactly supported distributions $\varphi, \tilde{\varphi} \in S'$, respectively. Next we find polynomial wavelet masks. The wavelet functions $\psi^{(\nu)}_j$,
Theorem 2. Let \( \tilde{\psi}^{(v)} \), \( v = 1, \ldots, m - 1 \), defined by (1.4) are also compactly supported distributions. If \( \varphi, \tilde{\varphi} \) are in \( L_2(\mathbb{R}^d) \) and the integer shifts of \( \varphi, \tilde{\varphi} \) are biorthogonal, then the systems \( \{ \psi_{jk}^{(v)} \}, \{ \tilde{\psi}_{jk}^{(v)} \} \) form biorthogonal bases. These properties should be checked individually for each concrete pair of dual wavelet systems. Some other properties may be described in terms of masks. We are going to discuss how to arrange construction of refinable and wavelet masks to provide some desirable properties of the corresponding dual wavelet systems. Note that even if a pair of dual wavelet systems does not form biorthogonal bases, it may be useful for applications because the reconstruction of signals holds by construction, because it may happen that these systems form dual frames, and so on.

Throughout the paper we will consider only polynomial refinable and wavelet masks \( m_0, \tilde{m}_0, \nu = 0, \ldots, m - 1 \).

Definition 1. We say that a wavelet system \( \{ \psi_{jk}^{(v)} \} \) has vanishing moments up to order \( n, n \in \mathbb{Z}_+ \), (has \( VM^n \) property in the sequel) if \( D^\beta \tilde{\psi}^{(v)}(0) = 0 \) for all \( \beta \in \mathbb{Z}_+^d \), \( |\beta| \leq n \) and all \( v = 1, \ldots, m - 1 \).

It easily follows from (1.4) and Leibniz formula that \( VM^n \) property holds if and only if

\[
D^\beta(m_\nu(M^* x)|_{x=0} = 0, \nu = 1, \ldots, m - 1, \quad \forall \beta \in \mathbb{Z}_+^d, \quad |\beta| \leq n. \quad (1.5)
\]

We will say that a mask \( m_0 \) generates dual wavelet systems \( \{ \psi_{jk}^{(v)} \}, \{ \tilde{\psi}_{jk}^{(v)} \} \) if there exist masks \( \tilde{m}_0, \tilde{m}_1, \ldots, \tilde{m}_{m_1-1}, m_1, \ldots, m_{m-1} \) satisfying (1.3).

The following statement summarizes the results given in Ref. 26.

Theorem 2. Let \( D(M) = \{ s_0, \ldots, s_{m-1} \}, s_0 = 0. \) Then

(i) an interpolatory mask \( m_0 \) generates dual wavelet systems \( \{ \psi_{jk}^{(v)} \}, \{ \tilde{\psi}_{jk}^{(v)} \} \) with \( VM^n \) property for \( \{ \tilde{\psi}_{jk}^{(v)} \} \) if and only if

\[
D^\beta m_0 k(0) = \frac{1}{\sqrt{m}} (-2\pi i M^{-1} s_k)^{\beta}, \quad k = 0, \ldots, m - 1, \quad \beta \in \mathbb{Z}_+^d, \quad |\beta| \leq n; \quad (1.6)
\]

(ii) under condition (1.6) for \( m_0 \), construction of dual wavelet systems may be realized by means of the following polyphase matrices

\[
M = \begin{pmatrix}
\frac{1}{\sqrt{m}} & \mu_{01} & \mu_{02} & \cdots & \mu_{0,m-1} \\
-\mu_{01} & \sqrt{m}(1 - |\mu_{01}|^2) & -\sqrt{m}\mu_{01}\mu_{02} & \cdots & -\sqrt{m}\mu_{01}\mu_{0,m-1} \\
-\mu_{02} & -\sqrt{m}\mu_{02}\mu_{01} & \sqrt{m}(1 - |\mu_{02}|^2) & \cdots & -\sqrt{m}\mu_{02}\mu_{0,m-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-\mu_{0,m-1} & -\sqrt{m}\mu_{0,m-1}\mu_{01} & -\sqrt{m}\mu_{0,m-1}\mu_{02} & \cdots & \sqrt{m}(1 - |\mu_{0,m-1}|^2)
\end{pmatrix}. \quad (1.7)
\]
exist such masks with $c$ for applications it suffices to consider only even real masks.

It follows that $M \equiv 0 \pmod{2}$, which contradicts to $s_i \equiv -s_i$ for $M \equiv 0 \pmod{2}$.

Proof. Assume that desirable digits $s_0, \ldots, s_{m-1}$ are chosen. If $2L < m - 1$, then there exists $s \in \mathbb{Z}^d$ such that $s \equiv s_0, \ldots, s_{2L} \pmod{M}$. Since $s_{2k-1} = -s_{2k}$, $k = 1, \ldots, L$, we have $s \not\equiv -s_0, \ldots, -s_{2L} \pmod{M}$. Set $s_{2L+1} = s$, $s_{2L+2} = -s$ and check that $s_{2L+2} \not\equiv s_0, \ldots, s_{2L+1} \pmod{M}$. Assume that $s_{2L+2} \equiv s_{2L+1} \pmod{M}$.

It follows that $M^{-1}(2s) \in \mathbb{Z}^d$. Since $m$ is odd, this is equivalent to $M^{-1}s \in \mathbb{Z}^d$, which contradicts to $s \not\equiv 0 \pmod{M}$. Now we assume that $s_{2L+2} \equiv s_i \pmod{M}$, $0 \leq i \leq 2L$. In this case, $-s \equiv s_i \pmod{M}$, which contradicts to $s \not\equiv -s_i \pmod{M}$.

2. Matrix Dilations with Odd Determinant

We will discuss how to construct real refinable functions $\varphi$ which are symmetric with respect to the origin and provide $VM^n$ property. It is clear that the corresponding refinable mask $m_0$ should be real and even. First of all, let us construct at least one real even refinable mask satisfying (1.6).

**Proposition 3.** If the determinant $m$ of a matrix dilation $M$ is odd, then there exists a set of digits $s_0, \ldots, s_{m-1}$ so that $s_0 = 0$, $s_{2k-1} = -s_{2k}$, $k = 1, \ldots, \frac{m-1}{2}$.

Proof. Assume that desirable digits $s_0, \ldots, s_{2L}$ are chosen. If $2L < m - 1$, then there exists $s \in \mathbb{Z}^d$ such that $s \equiv s_0, \ldots, s_{2L} \pmod{M}$. Since $s_{2k-1} = -s_{2k}$, $k = 1, \ldots, L$, we have $s \not\equiv -s_0, \ldots, -s_{2L} \pmod{M}$. Set $s_{2L+1} = s$, $s_{2L+2} = -s$ and check that $s_{2L+2} \not\equiv s_0, \ldots, s_{2L+1} \pmod{M}$. Assume that $s_{2L+2} \equiv s_{2L+1} \pmod{M}$.

It follows that $M^{-1}(2s) \in \mathbb{Z}^d$. Since $m$ is odd, this is equivalent to $M^{-1}s \in \mathbb{Z}^d$, which contradicts to $s \not\equiv 0 \pmod{M}$. Now we assume that $s_{2L+2} \equiv s_i \pmod{M}$, $0 \leq i \leq 2L$. In this case, $-s \equiv s_i \pmod{M}$, which contradicts to $s \not\equiv -s_i \pmod{M}$.

$$\tilde{M} = \begin{pmatrix} \sqrt{m} \left(1 - \sum_{k=1}^{m-1} |\mu_{0k}|^2\right) & \mu_{01} & \mu_{02} & \ldots & \mu_{0,m-1} \\ -\mu_{01} & 1/\sqrt{m} & 0 & \ldots & 0 \\ -\mu_{02} & 0 & 1/\sqrt{m} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\mu_{0,m-1} & 0 & 0 & \ldots & 1/\sqrt{m} \end{pmatrix},$$

and both the systems $\{\psi_{jk}^{(m)}\}$, $\{\tilde{\psi}_{jk}^{(m)}\}$ have $VM^n$ property in this case.
Let $s_0, \ldots, s_{m-1}$ be as in Proposition 3. Given $n \in \mathbb{Z}_+^d$, consider real trigonometric polynomials $g_\beta = g_{\beta,n}$, $\beta \in \mathbb{Z}_+^d$, $[\beta] \leq n$, such that $D^\gamma g_\beta(0) = \delta_{\beta\gamma}$ for all $\gamma \in \mathbb{Z}_+^d$, $[\gamma] \leq n$. Recursive formulas for such functions $g_\beta$ are given in Ref. 26. Set

$$G_0 = 1, \quad G_\beta(x) = \frac{1}{2}(g_{\beta}(x) + (-1)^{[\beta]} g_{\beta}(-x)), \quad \beta > 0.$$  

It is clear that $D^\gamma G_\beta(0) = \delta_{\beta\gamma}$ for all $\gamma \in \mathbb{Z}_+^d$, $[\gamma] \leq n$ and that $G_\beta$ is an even (odd) function whenever $[\beta]$ is even (odd). Set

$$\mu_{00}^*(x) \equiv \frac{1}{\sqrt{m}}, \quad (2.1)$$

$$\mu_{0,2k-1}^*(x) = \mu_{0,2k}^*(x) = \frac{1}{\sqrt{m}} \sum_{[\beta] \leq n} G_\beta(x)(-2\pi i M^{-1}s_k)^\beta, \quad k = 1, \ldots, \frac{m-1}{2}. \quad (2.2)$$

It is not difficult to see that (1.6) is valid. Since $\text{Re} \mu_{0,2k-1}^*$ is even and $\text{Im} \mu_{0,2k-1}^*$ is odd, the function $\text{Re}(e^{2\pi i(s_{2k-1},x)}\mu_{0,2k-1}^*(M^*x))$ is even. It follows that the corresponding mask

$$m_0^*(x) = \frac{1}{\sqrt{m}} \sum_{k=0}^{m-1} e^{2\pi i(s_k,x)} \mu_{0k}^*(M^*x)$$

$$= \frac{1}{\sqrt{m}} + \frac{1}{\sqrt{m}} \sum_{k=1}^{m-1} \left( e^{2\pi i(s_{2k-1},x)} \mu_{0,2k-1}^*(M^*x) + e^{-2\pi i(s_{2k-1},x)} \mu_{0,2k-1}^*(M^*x) \right) \quad (2.3)$$

is a real even trigonometric polynomial.

To describe the whole class of real even interpolatory masks satisfying (1.6) we need some auxiliary statements.

**Lemma 4.** A trigonometric polynomial $T(x) = \sum_l h_l e^{2\pi i(l, x)}$ has real Fourier coefficients $h_l$ if and only if $\text{Re} T$ is an even function and $\text{Im} T$ is an odd function. The polynomial $T$ is a real even function if and only if $h_l \in \mathbb{R}$ and $h_{-l} = h_l$ for all $l \in \mathbb{Z}$.

The proof is evident.

**Lemma 5.** Let $n \in \mathbb{Z}_+^d$. A general form for all real even (odd) trigonometric polynomials $T$ such that $D^\beta T(0) = 0$, $\beta \in \mathbb{Z}_+^d$, $[\beta] \leq n$, is given by

$$T(x) = \sum_{\alpha \in \mathbb{Z}_+^d} \left( A_\alpha(x) \text{Re} \Pi_\alpha(x) + B_\alpha(x) \text{Im} \Pi_\alpha(x) \right), \quad (2.4)$$

where $A_\alpha$ and $B_\alpha$ are arbitrary respectively real even (odd) and real odd (even) trigonometrical polynomials.
Proof. Let $T(x)$ be defined by (2.4), $\alpha, \beta \in \mathbb{Z}^d_+, [\alpha] = n + 1, [\beta] \leq n$. It follows from

$$D^3 \Pi_\alpha(0) = D^3 \left( \prod_{j=1}^{d} (1 - e^{2\pi i x_j})^{\alpha_j} \right) \bigg|_{x=0} = 0,$$

that

$$D^3 (\text{Re} \Pi_\alpha(x)) \bigg|_{x=0} = 0, \quad D^3 (\text{Im} \Pi_\alpha(x)) \bigg|_{x=0} = 0. \quad (2.5)$$

Since this is true for all $\beta \in \mathbb{Z}^d_+, [\beta] \leq n$, due to Leibniz formula, we obtain $D^3(T(x))|_{x=0} = 0$ for all $\beta \in \mathbb{Z}^d_+, [\beta] \leq n$. Due to Lemma 4, it is clear that $T$ is a real even (odd) function.

Now let $T$ be a real even trigonometric polynomial, such that $D^3 T(0) = 0$ for all $\beta \in \mathbb{Z}^d_+, [\beta] \leq n$. By Taylor formula, there exist trigonometric polynomials $T_\alpha, \alpha \in \mathbb{Z}^d_+, [\alpha] = n + 1$, such that

$$T(x) = \sum_{\alpha \in \mathbb{Z}^d_+, [\alpha] = n+1} \Pi_\alpha(x) T_\alpha(x)$$

$$= \sum_{\alpha \in \mathbb{Z}^d_+, [\alpha] = n+1} (\text{Re} T_\alpha(x) \text{Re} \Pi_\alpha(x) - \text{Im} T_\alpha(x) \text{Im} \Pi_\alpha(x))$$

$$+ i(\text{Im} T_\alpha(x) \text{Re} \Pi_\alpha(x) + \text{Re} T_\alpha(x) \text{Im} \Pi_\alpha(x)). \quad (2.6)$$

Since $T$ is real, the imaginary part of the right hand side is identical zero. So, (2.6) may be rewritten as

$$T(x) = \sum_{\alpha \in \mathbb{Z}^d_+, [\alpha] = n+1} (A_\alpha(x) \text{Re} \Pi_\alpha(x) + B_\alpha(x) \text{Im} \Pi_\alpha(x))$$

$$= \sum_{\alpha \in \mathbb{Z}^d_+, [\alpha] = n+1} (\tilde{A}_\alpha(x) \text{Re} \Pi_\alpha(x) + \tilde{B}_\alpha(x) \text{Im} \Pi_\alpha(x)),$$

where

$$A_\alpha(x) = \frac{1}{2}(\text{Re} T_\alpha(x) + \text{Re} T_\alpha(-x)), \quad \tilde{A}_\alpha(x) = \frac{1}{2}(\text{Re} T_\alpha(x) - \text{Re} T_\alpha(-x)),$$

$$B_\alpha(x) = \frac{1}{2}(\text{Im} T_\alpha(-x) - \text{Im} T_\alpha(x)), \quad \tilde{B}_\alpha(x) = -\frac{1}{2}(\text{Im} T_\alpha(x) + \text{Im} T_\alpha(-x)).$$

It is clear that $A_\alpha, \tilde{B}_\alpha$ are even and $\tilde{A}_\alpha, B_\alpha$ are odd functions. So, the left hand side is an even polynomial and the right hand side is an odd polynomial. Hence both the sides are identically equal to zero, i.e. (2.4) holds.

For odd $T$, the proof is similar.
Corollary 6. Let \( n \in \mathbb{Z}_+ \). A general form for all real trigonometric polynomials \( T \) with real Fourier coefficients such that \( D^\beta T(0) = 0 \), for all \( \beta \in \mathbb{Z}^d \), \( |\beta| \leq n \), is given by

\[
T(x) = \sum_{\alpha \in \mathbb{Z}^d \atop |\alpha| = n+1} ((A_\alpha(x) + iC_\alpha(x)) \text{Re} \Pi_\alpha(x) + (B_\alpha(x) + iD_\alpha(x)) \text{Im} \Pi_\alpha(x)),
\]

where \( A_\alpha, \ D_\alpha \) are arbitrary real even trigonometric polynomials, \( B_\alpha, C_\alpha \) are arbitrary real odd trigonometric polynomials.

Proof follows immediately from Lemmas 4 and 5.

Lemma 7. Let the determinant \( m \) of a matrix dilation \( M \) be odd, digits \( s_0, \ldots, s_{m-1} \) be as in Proposition 3. If a mask \( m_0 \) is a real even trigonometric polynomial, \( \mu_{0,k} \), \( k = 0, \ldots, m-1 \), are its polyphase representatives, then \( \mu_{0,0} \) is a real even trigonometric polynomial, \( \mu_{0,k} \), \( k = 1, \ldots, m-1 \), are trigonometric polynomials with real Fourier coefficients, and \( \mu_{0,2k} = \mu_{0,2k-1}, k = 1, \ldots, \frac{m-1}{2} \).

Proof. Let \( m_0(x) = \sum_l h_l e^{2\pi i (x,l)} \). Due to Lemma 4, the coefficients \( h_l \) are real and \( h_0 = h_{-1} \). Since \( \mu_{0,0}(x) = \frac{1}{\sqrt{m}} \sum_{l \in \mathbb{Z}^d} h_l e^{2\pi i (l,x)} \), by Lemma 4, \( \mu_{0,0} \) is a real even function. For \( k = 1, \ldots, \frac{m-1}{2} \), taking into account that \( s_{2k-1} = -s_{2k} \), we have

\[
\mu_{0,2k-1}(x) = \frac{1}{\sqrt{m}} \sum_{l \in \mathbb{Z}^d} h_{Ml+s_{2k-1}} e^{2\pi i (l,x)}
\]

\[
= \frac{1}{\sqrt{m}} \sum_{l \in \mathbb{Z}^d} h_{-Ml+s_{2k-1}} e^{-2\pi i (l,x)}
\]

\[
= \frac{1}{\sqrt{m}} \sum_{l \in \mathbb{Z}^d} h_{Ml+s_{2k}} e^{-2\pi i (l,x)}
\]

\[
= \mu_{0,2k}(x).
\]

Theorem 8. Let the determinant \( m \) of a matrix dilation \( M \) be odd. A general form for all real even interpolatory masks \( m_0 \) generating dual compactly supported wavelet systems \( \{\psi_{jk}^{(v)}\}, \{\tilde{\psi}_{jk}^{(v)}\} \), with \( V M^u \) property is given by

\[
m_0(x) = m_0^*(x) + \sum_{k=1}^{m-1} \sum_{\alpha \in \mathbb{Z}^d \atop |\alpha| = n+1} (\text{Re} \Pi_\alpha(M^* x) A_{ak}(M^* x) \cos 2\pi(x, s_{2k})

+ C_{ak}(M^* x) \sin 2\pi(x, s_{2k})) + \text{Im} \Pi_\alpha(M^* x) B_{ak}(M^* x) \cos 2\pi(x, s_{2k})

+ D_{ak}(M^* x) \sin 2\pi(x, s_{2k})),
\]

where \( s_0, \ldots, s_{m-1} \) is a set of digits as in Proposition 3, \( m_0^*(x) \) is defined by (2.1)–(2.3), \( A_{ak}, D_{ak} \) are real even trigonometrical polynomials, \( B_{ak}, C_{ak} \) are real odd trigonometrical polynomials.
Proof. Let an interpolatory mask $m_0$ be a real even trigonometric polynomial. Assume that $m_0$ generates dual wavelet systems $\{\psi_{jk}^\nu\}, \{\tilde{\psi}_{jk}^\nu\}$ with $VM^n$ property. This means that there exists a set of digits $\{s_0, \ldots, s_{m-1}\}$ for which the polyphase matrices

$$
M := \{\mu_{\nu k}\}_{\nu, k=0}^{m-1}, \quad \tilde{M} := \{\tilde{\mu}_{\nu k}\}_{\nu, k=0}^{m-1},
$$
satisfy (1.3) and the corresponding wavelet masks $\tilde{m}_\nu$ are such that

$$
D^\beta(\tilde{m}_\nu (M^{*-1} x))|_{x=0} = 0, \quad \nu = 1, \ldots, m - 1,
$$
for all $\beta \in \mathbb{Z}_+^d$, $[\beta] \leq n$. Note that this generating property of $m_0$ does not depend on a set of digits. Indeed, if $\{s'_0, \ldots, s'_{m-1}\}$ is another set of digits such that $s_k \equiv s'_k \pmod{M}$, $k = 0, \ldots, m - 1$, then the corresponding polyphase matrices are

$$
M' := \{\mu'_{\nu k}\}_{\nu, k=0}^{m-1}, \quad \tilde{M}' := \{\tilde{\mu}'_{\nu k}\}_{\nu, k=0}^{m-1},
$$

where $\mu'_{\nu k}(x) = e^{2\pi i x (M^{-1} s_k - s'_k)} \mu_{\nu k}(x)$, $\tilde{\mu}'_{\nu k}(x) = e^{2\pi i x (M^{-1} s_k - s'_k)} \tilde{\mu}_{\nu k}(x)$. It is clear that $M'M' = I_m$. So, taking into account Proposition 3, we can assume that $s_0 = 0$, $s_{2k-1} = -s_{2k}$, $k = 1, \ldots, m/2$. The polyphase representatives of $m_0 - m_0^*$ are $\mu_{00} - \mu_{00}^* = 0$, $\mu_{0k} - \mu_{0k}^*$, $k = 1, \ldots, m - 1$. Due to Theorem 2, we have $D^\beta(\mu_{0k}(x) - \mu_{0k}^*(x))|_{x=0} = 0$, $k = 1, \ldots, m - 1$, for all $\beta \in \mathbb{Z}_+^d$, $[\beta] \leq n$. It follows from Corollary 6 and Lemma 4 that

$$
\mu_{0,2k}(x) - \mu_{0,2k}^*(x) = \sum_{\alpha \in Z^d} \alpha \in n+1 ((A'_{\alpha k}(x) + iC'_{\alpha k}(x)) \Re \Pi_\alpha(x)
$$

$$
+ (B'_{\alpha k}(x) + iD'_{\alpha k}(x)) \Im \Pi_\alpha(x), \quad k = 1, \ldots, m - 1,
$$

(2.8)

where $A'_{\alpha k}$, $D'_{\alpha k}$ are real even trigonometrical polynomials, $C'_{\alpha k}$, $B'_{\alpha k}$ are real odd trigonometrical polynomials, and, due to Lemma 7,

$$
\mu_{0,2k-1}(x) - \mu_{0,2k-1}^*(x) = \mu_{0,2k}(x) - \mu_{0,2k}^*(x), \quad k = 1, \ldots, m - 1.
$$

(2.9)

To prove (2.7) it remains to combine (2.8) and (2.9) with

$$
m_0(x) - m_0^*(x) = \frac{1}{\sqrt{m}} \sum_{k=0}^{m-1} e^{2\pi i s_k x} (\mu_{0k}(M^* x) - \mu_{0k}^*(M^* x))
$$

and to set $A_{\alpha k} = 2A'_{\alpha k}$, $B_{\alpha k} = 2B'_{\alpha k}$, $C_{\alpha k} = -2C'_{\alpha k}$, $D_{\alpha k} = -2D'_{\alpha k}$. Now let a mask $m_0$ is defined by (2.7). It is not difficult to see that the polyphase representatives of $m_0 - m_0^*$ are given by (2.8) and (2.9) and $\mu_{00} - \mu_{00}^* = 0$. Due to Corollary 6, $D^\beta(\mu_{0k}(x) - \mu_{0k}^*(x))|_{x=0} = 0$ for all $\beta \in \mathbb{Z}_+^d$, $[\beta] \leq n$ and all $k = 1, \ldots, m - 1$. Since

$$
D^\beta(\mu_{0k}^*(x))|_{x=0} = \frac{1}{\sqrt{m}} (-2\pi i M^{-1} s_k)^\beta, \quad \beta \in \mathbb{Z}_+^d, \quad [\beta] \leq n, \quad k = 1, \ldots, m - 1,
$$
that such wavelet functions will not be real. We are going to construct real symmetric-wavelet systems with \( VM \) property. Construction of wavelets may be realized by means of the polyphase matrixes \( M, MV \) defined by (1.7) and (1.8). But such wavelet functions will not be real. We are going to construct real symmetric/antisymmetric wavelet functions. It follows from (1.1) that the corresponding wavelet masks should be real even/pure imaginary odd trigonometric polynomials.

Let a set of digits \( \{ s_0, \ldots, s_{m-1} \} \) is chosen as in Proposition 3. Set

\[
\mathcal{M}' := \begin{pmatrix}
Q_0 \\
(Q_1 + Q_2) \\
(Q_1 - Q_2) \\
(Q_3 + Q_4) \\
(Q_3 - Q_4) \\
\vdots \\
(Q_{m-2} + Q_{m-1}) \\
(Q_{m-2} - Q_{m-1})
\end{pmatrix}, \quad \tilde{\mathcal{M}'} := \begin{pmatrix}
\tilde{Q}_0 \\
\frac{1}{2}(\tilde{Q}_1 + \tilde{Q}_2) \\
\frac{1}{2}(\tilde{Q}_1 - \tilde{Q}_2) \\
\frac{1}{2}(\tilde{Q}_3 + \tilde{Q}_4) \\
\frac{1}{2}(\tilde{Q}_3 - \tilde{Q}_4) \\
\vdots \\
\frac{1}{2}(\tilde{Q}_{m-2} + \tilde{Q}_{m-1}) \\
\frac{1}{2}(\tilde{Q}_{m-2} - \tilde{Q}_{m-1})
\end{pmatrix},
\]

where \( Q_k, \tilde{Q}_k, k = 0, \ldots, m - 1 \), is the \( k \)th row of \( \mathcal{M}, \tilde{\mathcal{M}} \), respectively. It is clear that \( \mathcal{M}'\tilde{\mathcal{M}}' = I_m \). Taking into account that \( \mu_{0,2k-1} = \overline{\mu_{0,2k}}, k = 1, \ldots, \frac{m-1}{2} \), we have

\[
\begin{align*}
\mu_{2\nu-1,0} &= -2 \text{Re} \mu_{0,2\nu-1}, \\
\mu_{2\nu-1,2\nu-1} &= \sqrt{m}(1 - |\mu_{0,2\nu-1}|^2 - \mu_{0,2\nu-1}^2), \\
\mu_{2\nu-1,2\nu} &= \sqrt{m}(1 - |\mu_{0,2\nu-1}|^2 - \mu_{0,2\nu-1}^2), \\
\mu_{2\nu-1,2k-1} &= -2\sqrt{m} \text{Re} (\mu_{0,2\nu-1})\mu_{0,2k-1}, \quad k = 1, \ldots, \frac{m-1}{2}, \quad k \neq \nu, \\
\mu_{2\nu-1,2k} &= -2\sqrt{m} \text{Re} (\mu_{0,2\nu-1})\overline{\mu_{0,2k-1}}, \quad k = 1, \ldots, \frac{m-1}{2}, \quad k \neq \nu, \\
\mu_{2\nu,0} &= 2\text{Im} \mu_{0,2\nu-1}, \\
\mu_{2\nu,2\nu-1} &= \sqrt{m}(1 - |\mu_{0,2\nu-1}|^2 + \mu_{0,2\nu-1}^2), \\
\mu_{2\nu,2\nu} &= -\sqrt{m}(1 - |\mu_{0,2\nu-1}|^2 + \mu_{0,2\nu-1}^2),
\end{align*}
\]
For the case $m = \pm 2$. It is not difficult to see that $\mu_{2\nu-1,0}$ is a real even trigonometric polynomial, $\mu_{2\nu-1,2k-1} = \mu_{2\nu-1,2k}$, $k = 1, \ldots, \frac{m-1}{2}$; $\mu_{2\nu,0}$ is a pure imaginary odd trigonometric polynomial, $\mu_{2\nu,2k-1} = -\mu_{2\nu,2k}$, $k = 1, \ldots, \frac{m-1}{2}$, real and imaginary parts of $\mu_{\nu,k}$, $\nu, k = 1, \ldots, m-1$ are, respectively, even and odd trigonometric polynomials. Since $s_0 = 0$, $s_{2k-1} = -s_{2k}$, $k = 1, \ldots, \frac{m-1}{2}$, it follows from (1.3) that the masks $m_{2\nu-1}$ and $m_{2\nu}$, $\nu = 1, \ldots, \frac{m-1}{2}$, are, respectively, real even and pure imaginary odd.

Similarly, for $\nu = 1, \ldots, \frac{m-1}{2}$,

$$
\tilde{\mu}_{2\nu-1,0} = -\text{Re} \mu_{0,2\nu-1},
$$
$$
\tilde{\mu}_{2\nu-1,2\nu-1} = \mu_{2\nu-1,2\nu} = \frac{1}{2\sqrt{m}},
$$
$$
\tilde{\mu}_{2\nu-1,2k-1} = \tilde{\mu}_{2\nu-1,2k} = 0, \quad k = 1, \ldots, \frac{m-1}{2}, \quad k \neq \nu,
$$
$$
\tilde{\mu}_{2\nu,0} = \text{Im} \mu_{0,2\nu-1},
$$
$$
\tilde{\mu}_{2\nu,2\nu-1} = -\tilde{\mu}_{2\nu,2\nu} = \frac{1}{2\sqrt{m}},
$$
$$
\tilde{\mu}_{2\nu,2k-1} = \tilde{\mu}_{2\nu,2k} = 0, \quad k = 1, \ldots, \frac{m-1}{2}, \quad k \neq \nu.
$$

It follows that $\tilde{m}_{2\nu-1}$ and $\tilde{m}_{2\nu}$, $\nu = 1, \ldots, \frac{m-1}{2}$, are, respectively, real even and pure imaginary odd trigonometric polynomials. It is also clear that $\tilde{m}_0$ is a real even trigonometric polynomial. Hence, the corresponding wavelet functions $\psi^{(\nu)}$, $\tilde{\psi}^{(\nu)}$ are symmetric/antisymmetric with respect to the origin.

3. Matrix Dilations $M$ with $|\det M| = 2$

For the case $m = \pm 2$, a method for construction of real even interpolatory masks generating dual wavelet systems with vanishing moments up to arbitrary order $n$ is given in Ref. 14. In particular, if there exists a vector $s \in \mathbb{Z}^d$ so that $M^{*^{-1}}s = (1/2, \ldots, 1/2)$, then such a mask may be defined by

$$
m_0^*(x) = \sum_{k=1}^d \sum_{j_k=0}^{n-1} f \left( \frac{j_1}{n-1}, \ldots, \frac{j_d}{n-1} \right) \prod_{l=1}^d \left( \frac{n-1}{j_l} \right) \sin^{2j_l} \pi x_l \cos^{2n-2j_l-2} \pi x_l,
$$
Let

\[ f(t) := \begin{cases} 
1, & t_1 + \cdots + t_d < \frac{d}{2}, \\
\frac{1}{2}, & t_1 + \cdots + t_d = \frac{d}{2}, \\
0, & t_1 + \cdots + t_d > \frac{d}{2}.
\end{cases} \]

We give another method of finding \( m_0^* \) and describe all such masks. For construction, we need 2-periodic function

\[ \theta_n(u) = \sum_{l=0}^{L} c_k \cos(2l + 1)u, \quad n = 1, 2, \ldots. \tag{3.1} \]

such that

\[ \theta_n(0) = 1, \quad \frac{d^k \theta_n}{d x^k}(0) = 0, \quad k = 1, \ldots, n. \tag{3.2} \]

Note that \( \theta_n(u) \) is a real even function and \( \theta_n(u + 1) = -\theta_n(u) \) for all \( u \in \mathbb{R} \). It is not difficult to find such functions numerically. For example, we can take

\[ \theta_1(u) = \cos \pi u, \quad \theta_2(u) = \theta_3(u) = \frac{1}{8}(9 \cos \pi u - \cos 3\pi u). \]

\[ \theta_4(u) = \theta_5(u) = \frac{25}{192} \left( 9 \cos \pi u - \frac{3}{2} \cos 3\pi u + \frac{9}{50} \cos 5\pi u \right). \]

Let \( D(M) = \{0, s\} \) be an arbitrary set of digits,

\[ \mu^*_0(x) = \frac{1}{\sqrt{2}}, \quad \mu^*_1(x) = \frac{1}{\sqrt{2}} e^{-2\pi i (M^{-1}s, x)} \prod_{j \in \{2M^{-1}s\}} \theta_n(x_j). \]

Note that \( \mu_01 \) is well defined because \( 2M^{-1}s \in \mathbb{Z}^d \). It is clear that \( \mu_{01} \) is a trigonometric polynomial and (1.6) is valid. It remains to note that

\[ m_0^*(x) = \frac{1}{2} + \frac{1}{2} \prod_{j \in \{2M^{-1}s\}} \theta_n((M^*x)_j) \tag{3.3} \]

is a real even function.

**Theorem 9.** Let \( M \) be a matrix dilation with \( |\det M| = 2 \). A general form for all real even interpolatory masks \( m_0 \) generating dual compactly supported wavelet systems \( \{\psi_{jk}^{(r)}\}, \{\tilde{\psi}_{jk}^{(r)}\} \), with VM Property is given by

\[ m_0(x) = m_0^*(x) + \sum_{\alpha \in \mathbb{Z}^d \atop |\alpha| = n + 1} (\text{Re} \Pi_{\alpha}(M^*x))(A_{\alpha}(M^*x) \cos 2\pi(x, s)) \]

\[ + C_{\alpha}(M^*x) \sin 2\pi(x, s) + \text{Im} \Pi_{\alpha}(M^*x)(B_{\alpha}(M^*x) \cos 2\pi(x, s)) \]

\[ + D_{\alpha}(M^*x) \sin 2\pi(x, s)) \]

where \( s \) is a nonzero digit of \( M \), \( m_0^*(x) \) is defined by (3.3), \( A_{\alpha}, D_{\alpha} \) are real even trigonometrical polynomials, \( B_{\alpha}, C_{\alpha} \) are real odd trigonometrical polynomials.
Proof. Let an interpolatory mask $m_0$ be a real even trigonometric polynomial. Assume that $m_0$ generates a dual wavelet systems $\{\psi_{jk}\}, \{\tilde{\psi}_{jk}\}$ with $VM^n$ property. This means that there exists a set of digits $\{s_0, s_1\}$ for which the polyphase matrices

$$M := \begin{pmatrix} \mu_{00} & \mu_{01} \\ \mu_{10} & \mu_{11} \end{pmatrix}, \quad \tilde{M} := \begin{pmatrix} \tilde{\mu}_{00} & \tilde{\mu}_{01} \\ \tilde{\mu}_{10} & \tilde{\mu}_{11} \end{pmatrix},$$

satisfy (1.3) and the corresponding wavelet mask $\tilde{\mu}_1$ is such that

$$D^2(\tilde{m}_1(M^{*-1}x))|_{x=0} = 0,$$

for all $\beta \in \mathbb{Z}_+$, $[\beta] \leq n$. It was shown in the proof of Theorem 8 that this generating property of $m_0$ does not depend on a set of digits. So, we can take $D(M) = \{0, s\}$, where $s$ is the same vector which was chosen for construction of $m^*_0$. The polyphase representatives of $m_0 - m^*_0$ are $\mu_{00} - \mu^*_0 \equiv 0$, $\mu_{01} - \mu^*_0$. Due to Theorem 2, we have $D^2(\mu_0(x) - \mu^*_0(x))|_{x=0} = 0$, for all $\beta \in \mathbb{Z}_+$, $[\beta] \leq n$. It follows from Corollary 6 and Lemma 4 that

$$\mu_0(x) - \mu^*_0(x) = \sum_{\alpha \in \mathbb{Z}^d} ((A'_\alpha(x) + iC'_\alpha(x))\text{Re} \, \Pi_\alpha(x)$$

$$+ (B'_\alpha(x) + iD'_\alpha(x))\text{Im} \, \Pi_\alpha(x),)$$

where $A'_\alpha, D'_\alpha$ are real even trigonometrical polynomials, $B'_\alpha, C'_\alpha$ are real odd trigonometrical polynomials. Combining (3.5) with (1.1), we have

$$m_0(x) - m^*_0(x)$$

$$= \sum_{\alpha \in \mathbb{Z}^d} (\text{Re} \, \Pi_\alpha(x)(A'_\alpha(M^*x) \cos 2\pi(x, s) - C'_\alpha(M^*x) \sin 2\pi(x, s))$$

$$+ \text{Im} \, \Pi_\alpha(x)(B'_\alpha(M^*x) \cos 2\pi(x, s) - D'_\alpha(M^*x) \sin 2\pi(x, s))$$

$$+ i \sum_{\alpha \in \mathbb{Z}^d} (\text{Re} \, \Pi_\alpha(x)(C'_\alpha(M^*x) \cos 2\pi(x, s) + A'_\alpha(M^*x) \sin 2\pi(x, s))$$

$$+ \text{Im} \, \Pi_\alpha(x)(D'_\alpha(M^*x) \cos 2\pi(x, s) + B'_\alpha(M^*x) \sin 2\pi(x, s)).$$

Since the left-hand side is real, the image part of the right-hand side equals 0. To prove (3.4) it remains to set $A_\alpha = A'_\alpha, B_\alpha = B'_\alpha, C_\alpha = -C'_\alpha, D_\alpha = -D'_\alpha$.

Now let a mask $m_0$ is defined by (3.4). It is not difficult to see that

$$\mu_0(x) = \mu^*_0(x) + \sum_{\alpha \in \mathbb{Z}^d} \text{Re} \, \Pi_\alpha(x)((A_\alpha(x) - iC_\alpha(x))$$

$$+ (A_\alpha(x) + iC_\alpha(x))e^{-2\pi i(M^{-1}(2s), x)}$$
where \( P \) wavelet functions become of the following statement. It does not exist interpolatory masks due to Leibniz formula, we obtain \( D^n \mu_0 = D^n \mu_0^* \) for all \( \alpha \in \mathbb{Z}^d \), \( |\alpha| \leq n \). By Theorem 2, the mask \( m_0 \) generates dual compactly supported wavelet systems \( \{\psi_{jk}\}, \{\tilde{\psi}_{jk}\} \), with \( VM^n \) property.

Construction of dual wavelet systems may be realized by means of the following polyphase matrices

\[
\mathcal{M} = \begin{pmatrix}
\frac{1}{\sqrt{2}} & \mu_01(x) \\
-\mu_01(x)e^{-2\pi i(2M^{-1}s,x)} & \sqrt{2}(1 - |\mu_01(x)|^2)e^{-2\pi i(2M^{-1}s,x)}
\end{pmatrix},
\]

\[
\tilde{\mathcal{M}} = \begin{pmatrix}
\sqrt{2}(1 - |\mu_01(x)|^2) & \mu_01(x) \\
-\mu_01(x)e^{-2\pi i(2M^{-1}s,x)} & \frac{1}{\sqrt{2}}e^{-2\pi i(2M^{-1}s,x)}
\end{pmatrix}.
\]

It is clear that \( \tilde{m}_0 \) is a real even trigonometric polynomial and

\[
m_1(x) = e^{-2\pi i(s,x)} \left( \frac{1}{\sqrt{2}}e^{2\pi i(s,x)\mu_01(M^*x)} + 1 - |\mu_01(M^*x)|^2 \right) = e^{-2\pi i(s,x)} P(x),
\]

\[
\tilde{m}_1(x) = e^{-2\pi i(s,x)} \left( \frac{1}{\sqrt{2}}e^{2\pi i(s,x)\mu_01(M^*x)} + \frac{1}{2} \right) = e^{-2\pi i(s,x)} Q(x),
\]

where \( P \) and \( Q \) are real even trigonometric polynomials. Hence, the corresponding wavelet functions \( \psi, \tilde{\psi} \) are symmetric with respect to the the point \(-s\).

The synthesis of refinable masks whose Fourier coefficients are symmetric with respect to a semi-integer point (even supported filters in the engineering terminology) is very important. As it was shown in Ref. 31, even supported filters usually have better characteristics in image processing applications. The most important reason is that even supported filters can implement half-sample delay. This is necessary for synthesis of Hilbert transforms and complex wavelets and in other situations. So, we are interested in finding refinable masks \( m_0 \) providing vanishing moments for \( \tilde{\psi}_{jk} \) such that \( e^{\pi i(c,x)}m_0(x) \) is a real even function and each coordinate of a vector \( c \in \mathbb{Z}^d \) is odd. Unfortunately, such a mask can not be interpolatory because of the following statement.

**Theorem 10.** It does not exist interpolatory masks \( m_0 \) providing \( VM^n \) property, \( n \geq 1 \), for \( \{\tilde{\psi}_{jk}\} \) such that \( e^{\pi i(c,x)}m_0(x) \) is an even function, \( c \in \mathbb{R}^d, c \neq 0 \).
Proof. Let $D(M) = \{0, s_1, \ldots, s_{m-1}\}$. Assume that an interpolatory mask $m_0$ generates dual wavelet systems $\{\psi_{jk}\}, \{\tilde{\psi}_{jk}\}$ with $VM^n$ property for $\{\tilde{\psi}_{jk}\}, n \geq 1,$ and there exists $e \in \mathbb{R}^d$ such that $e^{\pi i (c,x)}m_0(x)$ is even. Set

$$F(x) = e^{\pi i (c,M^{-1}x)}m_0(M^{-1}x).$$

Since $F$ is even, $\frac{\partial F}{\partial x_l}(0) = 0$ for all $l = 1, \ldots, d$. On the other hand, using (1.1) and (1.6), we have

$$\frac{\partial F}{\partial x_l}(0) = \frac{1}{m} \frac{\partial}{\partial x_l} \left( \frac{1}{m} e^{\pi i (M^{-1}c,x)} + \sum_{k=1}^{m-1} e^{\pi i (M^{-1}(c+2sk),x)} \mu_{0k}(x) \right) \bigg|_{x=0}$$

$$= \frac{1}{m} \left( \pi i M^{-1}c \right)_l + \sum_{k=1}^{m-1} \left( \pi i M^{-1}(c+2sk) \right)_l + \left( -2\pi i M^{-1}sk \right)_l$$

$$= \pi i (M^{-1}c)_l$$

for all $l = 1, \ldots, d$. This yields $c = 0$.

Now we will give a method for construction of masks $m_0$ providing $VM^n$ property for $\{\tilde{\psi}_{jk}\}$ such that $e^{\pi i (c,x)}m_0(x)$ is a real even function, for a matrix dilation $M$ with $|\det M| = 2$ and for $c \in Z^d$.

We need the following statement which is proved in Ref. 26.

Theorem 11. Let a polynomial mask $m_0$ generate dual compactly supported wavelet systems $\{\psi_{jk}\}, \{\tilde{\psi}_{jk}\}$ (by means of Unitary Extension Principle). The following conditions are equivalent:

(i) $VM^n$ property is valid for $\{\tilde{\psi}_{jk}\}$;

(ii) there exist complex numbers $\lambda_\beta, \gamma \in Z^d_+, |\beta| \leq n$, such that $\lambda_0 = 1,$

$$D^\alpha \mu_{0k}(0) = \frac{1}{\sqrt{m}} \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} \lambda_\beta (-2\pi i M^{-1}sk)^{\alpha - \beta} \quad (3.6)$$

for all $\alpha \in Z^d_+, [\alpha] \leq n$.

First we assume that $c \not\equiv 0 \pmod{M}$. Chose $D(M) = \{0, -\}c$ as a set of digits. Let

$$\lambda_\beta = (-\pi i M^{-1}c)^\beta, \quad \beta \in Z^d_+, \quad |\beta| \leq n,$$

$$\mu_{00}(x) = \mu_{0n}(x) = \frac{1}{\sqrt{2}} \sum_{0 \leq [\beta] \leq n} \lambda_\beta G_\beta(x)$$

$$+ \sum_{\alpha \in Z^d, |\alpha| = n+1} ((A_\alpha(x) + iC_\alpha(x)) \Re \Pi_\alpha(x) + \Im \Pi_\alpha(x)),$$

$$+ iD_\alpha(x))) \Im \Pi_\alpha(x),$$
where the functions $G_\beta$ are defined as above, $A_\alpha$, $D_\alpha$ are arbitrary real even trigonometric polynomials, $B_\alpha$, $C_\alpha$ are arbitrary real odd trigonometric polynomials. By construction, $\text{Re} \mu_{00}$ is a real even trigonometric polynomial and $\text{Im} \mu_{00}$ is a real odd trigonometric polynomial. It is not difficult to see that

$$D^\alpha \mu_{00}(0) = \frac{1}{\sqrt{2}} \lambda_\alpha,$$

$$D^\alpha \mu_{01}(0) = \frac{1}{\sqrt{2}} \sum_{\alpha}^{n} \sum_{0 \leq \gamma \leq \beta} \frac{\alpha}{\beta} \lambda_\beta (2\pi i M^{-1} c)^{\alpha - \beta}$$

for all $\alpha \in \mathbb{Z}^d$, $[\alpha] \leq n$, i.e. (3.6) is valid. Evidently, $\text{Re}(e^{\pi i(c,x)} \mu_{00}(M^* x))$ is a real even function. It remains to note that

$$e^{\pi i(c,x)} m_0(x) = \frac{1}{\sqrt{2}} e^{\pi i(c,x)} (\mu_{00}(M^* x) + e^{-2\pi i(c,x)} \mu_{01}(M^* x))$$

$$= \frac{1}{\sqrt{2}} (e^{\pi i(c,x)} \mu_{00}(M^* x) + e^{\pi i(c,x)} \mu_{00}(M^* x))$$

$$= \sqrt{2} \text{Re}(e^{\pi i(c,x)} \mu_{00}(M^* x)).$$

Now consider the case $c \equiv 0 \pmod{M}$. Let $D(M) = \{0, s\}$,

$$\lambda_\beta = (-\pi i M^{-1} c)^\beta, \quad [\beta] \leq n,$$

and let 2-periodic function $\theta_n$, $n = 1, 2, \ldots$ satisfy (3.1) and (3.2). Set

$$\mu_{00}(x) = \frac{1}{\sqrt{2}} e^{-\pi(M^{-1} c, x)} \prod_{j \in \{0(M^{-1} c)\}} \theta_n(x_j),$$

$$\mu_{01}(x) = \frac{1}{\sqrt{2}} e^{-\pi(M^{-1}(c + 2s), x)} \prod_{j \in \{0(M^{-1}(c + 2s))\}} \theta_n(x_j).$$

Note that these functions are well defined because $M^{-1} c, 2M^{-1} s \in \mathbb{Z}^d$. It is clear that $\mu_{00}, \mu_{01}$ are trigonometric polynomials and

$$D^\alpha \mu_{00}(0) = \frac{1}{\sqrt{2}} \lambda_\alpha,$$

$$D^\alpha \mu_{01}(0) = \frac{1}{\sqrt{2}} \sum_{0 \leq \beta \leq \alpha} \frac{\alpha}{\beta} \lambda_\beta (-2\pi i M^{-1} s)^{\alpha - \beta},$$

i.e. (3.6) is valid. It remains to note that

$$e^{\pi i(c,x)} m_0(x) = \frac{1}{2} \left( \prod_{j \in \{0(M^{-1} c)\}} \theta_n((M^* x)_j) + \prod_{j \in \{0(M^{-1}(c + 2s))\}} \theta_n((M^* x)_j) \right)$$

is a real even function.

So, we know how to construct a mask $m_0$ satisfying (3.6) and such that $e^{\pi i(c,x)} m_0(x)$ is a real even function, for arbitrary matrix dilation $M$ with
|det $M| = 2$ and for arbitrary $c \in \mathbb{Z}^d$. Unfortunately, it may happen that such a mask does not generate wavelets. To construct wavelets we need a dual mask $\tilde{m}_0$ so that

$$\mu_{00}\tilde{\mu}_{00} + \mu_{01}\tilde{\mu}_{01} \equiv 1.$$ 

Due to the Hilbert theorem, a dual mask exists if and only if the functions $\mu_{00}, \mu_{01}$ do not have common zeros.

**Proposition 12.** Let $M$ be a matrix dilation with $|\det M| = 2$, $\mu_{00}, \mu_{01}$ be polyphase representatives of a mask $m_0$ with respect to a set of digits $\{0, s\}, c \in \mathbb{Z}^d, c \equiv 0 \pmod{M}$ and let there exist $a \in \mathbb{Z}^d$ such that the numbers $(M^{-1}c, a), (M^{-1}(c + 2s), a)$ are odd. If $e^{\pi i(c,x)}m_0(x)$ is an even function then

$$\mu_{00}\left(\frac{a}{2}\right) = \mu_{01}\left(\frac{a}{2}\right) = 0.$$ 

**Proof.** Let $P(x) := e^{\pi i(M^{-1}c,x)}m_0(Mx^{-1}x), l \in \mathbb{Z}^d$. Since $P$ is even and $P(x-2l) = P(x)$ for all $x \in \mathbb{R}^d$, the function $P(\cdot + l)$ is even. By (1.2),

$$e^{\pi i(M^{-1}c,x)}\mu_{00}(x) = \frac{1}{\sqrt{2}}\left(P(x) + e^{-\pi i(M^{-1}c,s^*)P(x+s^*)}\right),$$

where $s^* \in \mathbb{Z}^d, s^* \neq 0 \pmod{M*}$. So, $\mu_{00}(x) = e^{-\pi i(M^{-1}c,x)}Q(x)$, where $Q$ is even. Using oddness of $(M^{-1}c, a)$, we have

$$Q\left(-\frac{a}{2}\right) = \mu_{00}\left(-\frac{a}{2}\right) e^{-\pi i(M^{-1}c,\frac{a}{2})}$$

$$= -\mu_{00}\left(a - \frac{a}{2}\right) e^{\pi i(M^{-1}c,a-\frac{a}{2})}$$

$$= -\mu_{00}\left(\frac{a}{2}\right) e^{\pi i(M^{-1}c,\frac{a}{2})} = -Q\left(\frac{a}{2}\right).$$

On the other hand, $Q\left(-\frac{a}{2}\right) = Q\left(\frac{a}{2}\right)$ because $Q$ is even. It follows that $Q\left(\frac{a}{2}\right) = 0$, and hence $\mu_{00}(\frac{a}{2}) = 0$. Similarly, using oddness of $(M^{-1}(c + 2s), a)$ and taking into account that, due to (1.1), $e^{\pi i(M^{-1}(c+s)a)}\mu_{01}(x)$ is an even function, we can prove that $\mu_{01}(\frac{a}{2}) = 0$. \hfill \square

For $M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, it is not difficult to check that the hypotheses of Proposition 12 is fulfilled for any $c \in \mathbb{Z}^2$ with odd coordinates and for any digit $s$ (we can take $a = c$), i.e. if $e^{\pi i(c,x)}m_0(x)$ is an even function, then $m_0$ does not generate dual wavelet systems.

4. Applications

Construction of multivariate wavelets is important for various engineering applications. A wide variety of multi-dimensional wavelet masks (filter banks or FBs in the engineering terminology) with finite or infinite support, having the property of
perfect reconstruction or near perfect reconstruction, of orthogonal and biorthogonal types, currently receive much attention for use in different application areas. Compression of images and video, computer tomography, texture characterization, high-definition television are the best known of them.

Other emerging applications include multispectral remote sensing images processing, M-D visualization systems (for future three-dimensional – 3D-TV systems, processing of light and acoustic fields), for M-D interpolation (accurate and effective implementation of rotation, and of other operations).

4.1. Nonseparable versus separable systems

Nonseparable decimation matrices and FBs are preferable because M-D signals by their nature are nonseparable.\textsuperscript{35,36} Nonseparable FBs have better characteristics than their separable counterparts (which consist of products of 1D FBs along each dimension). The number of degrees of freedom is also much bigger for nonseparable FBs.

In tomography the 2D separable wavelets impose a rectangular tiling of the frequency plane, which is not well suited to the radial band-limited assumption of the image.\textsuperscript{4} The application of nonseparable multiresolution tomography to 2D wavelets allows to respect the geometry of the system by tiling the frequency plane in a diamond-shaped fashion that is more respectful to the radial band-limited assumptions. Local tomography using these nonseparable bases shows an improvement in terms of PSNR. Another successful application of nonseparable wavelets in 3D rotational angiography is given in Ref. 5.

In certain cases, it is desirable to use nonseparable subsampling to obtain useful 2D wavelet representations. For example, nonseparable wavelet orthonormal bases can be used for texture discrimination and fractal analysis.\textsuperscript{16} The selection of nonseparable decomposition filters has a significant influence on the result of texture characterization. In Refs. 22 and 33 it is shown that nonseparable wavelets have features invariant to rotation of the texture image. That is why the classification and segmentation results are better if to use nonseparable symmetric wavelets.

It is supposed that one of the main drawbacks of nonseparable filtration is its high computational complexity. Our investigation and the results from Ref. 23 have shown that in general this is not true. The real complexity highly depends on many factors. Among some of them one should mention the size of the signal to be processed, the support of the wavelets (the size of FIR filter), the architecture of computing system and others. For example on multi-processor systems comparing convolution-based algorithms for different distributions, separable wavelet filters are more efficient for the block distribution for small values of kernel (filter) size $L$. But for nonseparable filters, the block distribution is more efficient if the image size $n$ is large for a practical range of $L$. It is important to note that the results reported for Intel Paragon are specific to the parameters of the architecture. One can observe the high impact of architectural parameters (e.g., ratio of computation to communication speed) on the relative performance of the algorithms.
The convolution-based algorithm using block distribution is faster than the algorithms using row distribution as in the case of the separable wavelet filters. However, in contrast with the separable case, the execution time of the convolution-based algorithms increases faster than the execution time for FFT-based (Fast Fourier Transform) algorithms with increasing size of the wavelet filter.

For example, the relatively large startup time for message passing on the Intel Paragon (40–45 ms) adversely affects the performance of the matrix transpose operation required by the separable 2D DWT algorithms using row distribution. The results may be different for a machine with a very low message startup time.

In high-definition television applications, in conversion between progressive and interlaced video, in motion parameter estimation, 3D nonseparable filters are widely used. The 3D matched filtering approach results in nonseparable filters that produce the best SNR improvement among all linear filters, provided the object spatial signature is known.

If to mention IIR filters (filters with infinite support, or with infinite impulse response), in Ref. 1 it is noted that one has to take advantage of well known superior frequency behavior at a low computational cost obtainable from nonseparable IIR filters.

4.2. Linear phase and symmetry

As it is well known any kind of symmetry for impulse response of the filters might be translated in the terms of linearity of its phase response (linear phase or LP). The human visual system is known to be sensitive to phase distortion. Thus linearity of phase response is a more desirable property of multi-dimensional subband filters than their 1D counterparts, due to the fact that typical 2D signals such as images are known to be more sensitive to phase during compression stage.

Since phase distortion can be avoided by applying filters with the LP property, it is desirable that all filters composing filter banks have the LP property when the system is applied to image processing. Hence, the LP property of filter banks is particularly significant for the subband coding of images. Several 1D linear-phase paraunitary filter banks (LPPUFBs) have been developed so far. The lattice and the modulation-based structures in particular have received a lot of attention because they enable us to design LPPUFBs in a systematic way, and some of them enable fast implementation. On the other hand, symmetric filter banks make it easier to work with the edges of the image, to implement different techniques of image extension, which is necessary for any coding system.

4.3. Desirable filters

From the implementation point of view it is necessary to achieve separable decimation after a few steps of iterations. This requirement might be translated into

\[ M^k = qI_d. \]
There are some reasons behind of it. When nonseparable filter banks and decimation matrices are used in multirate systems for compression of M-D signals by using of hierarchical coding algorithms (like set partitioning in hierarchical trees and other coding algorithms\(^8\)), the problem is that without the requirement (4.1) it would be impossible to get a similarity between different levels of wavelet decomposition of the image after every \(k\) steps. In Ref. 32 it is shown that from the point of view of keeping the property of signal isotropy after its decimation the requirement (4.1) should also hold true.

Let us consider \(2 \times 2\) matrix dilations \(M\) satisfying the following condition

\[
M^2 = mI_2. \tag{4.2}
\]

**Proposition 13.** A \(2 \times 2\) matrix \(M\) with integer entries and \(|\det M| = 3\) satisfies (4.2) if and only if

\[
M = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}, \quad A^2 + BC = 3. \tag{4.3}
\]

**Proof.** Assume that (4.2) with \(m = 3\) holds for a matrix \(M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\). It follows that \(B(A + D) = C(A + D) = 0\), i.e. either \(B = C = 0\) or \(A = -D\). If \(B = C = 0\), then, by (4.2), \(A^2 = D^2 = 3\), which is impossible for for \(A, D \in \mathbb{Z}\). Hence, \(D = -A\) and, due to (4.2), \(A^2 + BC = 3\). Inversely, it is not difficult to see that any matrix defined by (4.3) satisfies (4.2).

\(\square\)

Note that the eigenvalues of a matrix defined by (4.3) are \(\pm \sqrt{3}\) (i.e. strictly bigger than 1). So, any such a matrix can be used as a matrix dilation.

Let \(M\) be defined by (4.3) and at least one of the numbers \(A, C\) be not divisible by 3. Check that the vectors \(s_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\), \(s_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}\), \(s_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}\) form a set of digits for \(M\). Indeed, \(M^{-1}s_1 = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}\), \(M^{-1}s_2 = \begin{pmatrix} -1/2 \\ 1/2 \end{pmatrix}\), \(M^{-1}(s_1 - s_2) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\), and these vectors are not in \(\mathbb{Z}^2\). If both \(A\) and \(C\) are divisible by 3, then \(B\) is not divisible by 3 (otherwise the determinant is divisible by 9). In this case, the vectors \(s_0' = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\), \(s_1' = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\), \(s_2' = \begin{pmatrix} 0 \\ -1 \end{pmatrix}\) form a set of digits for \(M\).

Let \(M\) be defined by (4.3), \(D(M) = \{s_0, s_1, s_2\}\). Using the results of Sec. 2, let us find polyphase representatives of refinable and wavelet masks generating dual wavelet systems with \(VM^2\) property.

Choosing \(A_{ok}, B_{ok}, C_{ok}, D_{ok} \equiv 0\) in (2.7) and \(G_{00} \equiv 1\), \(G_{10}(x) = \frac{1}{2\pi} \sin 2\pi x_1\), \(G_{01}(x) = \frac{1}{2\pi} \sin 2\pi x_2\), \(G_{11}(x) = \frac{1}{4\pi^2} \sin 2\pi x_1 \sin 2\pi x_2\), \(G_{20}(x) = \frac{1}{2\pi} \sin^2 2\pi x_1\), \(G_{02}(x) = \frac{1}{4\pi^2} \sin^2 2\pi x_2\) in (2.2), we obtain

\[
\mu_{00} = \frac{1}{\sqrt{3}}, \quad \tilde{\mu}_{00} = \frac{1}{\sqrt{3}} \left(1 - \frac{\sigma^4}{162}\right),
\]

\[
\mu_{01} = \tilde{\mu}_{01} = \mu_{02} = \tilde{\mu}_{02} = \frac{1}{\sqrt{3}} \left(1 - \frac{i}{3} \sigma - \frac{1}{18} \sigma^2\right),
\]
\[ \mu_{10} = 2\tilde{\mu}_{10} = -\frac{2}{\sqrt{3}} \left( 1 - \frac{\sigma^2}{18} \right), \]
\[ \mu_{11} = \tilde{\mu}_{12} = \frac{1}{\sqrt{3}} \left( 1 + \frac{2\sigma^2}{9} - \frac{\sigma^4}{162} + \frac{i}{3} \left( 2\sigma - \frac{\sigma^3}{9} \right) \right), \]
\[ \tilde{\mu}_{11} = \tilde{\mu}_{12} = \tilde{\mu}_{21} = -\tilde{\mu}_{22} = \frac{1}{2\sqrt{3}}, \]
\[ \mu_{20} = 2\tilde{\mu}_{20} = -\frac{2i\sigma}{3\sqrt{3}}, \]
\[ \mu_{21} = \frac{1}{\sqrt{3}} \left( 3 - \frac{2\sigma^2}{9} - \frac{i}{3} \left( 2\sigma - \frac{\sigma^3}{9} \right) \right), \]
\[ \mu_{22} = \frac{1}{\sqrt{3}} \left( -3 + \frac{2\sigma^2}{9} - \frac{i}{3} \left( 2\sigma - \frac{\sigma^3}{9} \right) \right), \]

where \( \sigma(x) = A\sin 2\pi x_1 + C\sin 2\pi x_2. \)

If \( D(M) = \{s_0, s_1, s_2\} \), then the functions \( \mu_{\nu k}, \tilde{\mu}_{\nu k}, \nu, k = 0, 1, 2, \) may be defined by the same formulas with \( \sigma(x) = B\sin 2\pi x_1 - A\sin 2\pi x_2. \)

The corresponding masks are the following:

if \( D(M) = \{s_0, s_1, s_2\} \), then

\[ m_0(x) = \frac{1}{3} \left( 1 + 2\cos 2\pi x_1 \left( 1 - \frac{1}{18} \rho^2(x) \right) + \frac{2\sin 2\pi x_1 \rho(x)}{3} \right), \]
\[ m_1(x) = \frac{2}{3} \left( \frac{\rho^2(x)}{18} - 1 + \cos 2\pi x_1 \left( 1 + \frac{2}{9} \rho^2(x) - \frac{1}{162} \rho^4(x) \right) \right) - \frac{\sin 2\pi x_1}{3} \left( 2\rho(x) - \frac{\rho^3(x)}{9} \right), \]
\[ m_2(x) = \frac{2i}{3} \left( -\frac{\rho(x)}{3} - \frac{\cos 2\pi x_1}{3} \left( 2\rho(x) - \frac{\rho^3(x)}{9} \right) + \sin 2\pi x_1 \left( 3 - \frac{2\rho^2(x)}{9} \right) \right), \]
\[ \tilde{m}_0(x) = \frac{1}{3} \left( 1 - \frac{\rho^4(x)}{162} + 2\cos 2\pi x_1 \left( 1 - \frac{1}{18} \rho^2(x) \right) + \frac{2\sin 2\pi x_1 \rho(x)}{3} \right), \]
\[ \tilde{m}_1(x) = \frac{1}{3} \left( \frac{1}{18} \rho^2(x) - 1 + \cos 2\pi x_1 \right), \]
\[ \tilde{m}_2(x) = \frac{i}{3} \left( -\frac{\rho(x)}{3} + \sin 2\pi x_1 \right), \]

where \( \rho(x) = A\sin 2\pi (Ax_1 + Cx_2) + C\sin 2\pi (Bx_1 - Ax_2); \)
if \( D(M) = \{s_0', s_1', s_2'\} \), then

\[
m_0(x) = \frac{1}{3} \left( 1 + 2 \cos 2\pi x_2 \left( 1 - \frac{1}{18} \rho^2(x) \right) + \frac{2 \sin 2\pi x_2 \rho(x)}{3} \right),
\]

\[
m_1(x) = \frac{2}{3} \left( \frac{\rho^2(x)}{18} - 1 + \cos 2\pi x_2 \left( 1 + \frac{2}{9} \rho^2(x) - \frac{1}{162} \rho^4(x) \right) \right)
- \frac{\sin 2\pi x_2}{3} \left( 2\rho(x) - \frac{\rho^3(x)}{9} \right),
\]

\[
m_2(x) = \frac{2i}{3} \left( -\rho(x) - \cos 2\pi x_2 \left( 2\rho(x) - \frac{\rho^3(x)}{9} \right) - \sin 2\pi x_2 \left( 3 - \frac{2\rho^2(x)}{9} \right) \right),
\]

\[
\tilde{m}_0(x) = \frac{1}{3} \left( 1 - \rho^4(x) + 2 \cos 2\pi x_2 \left( 1 - \frac{1}{18} \rho^2(x) + \frac{2 \sin 2\pi x_2 \rho(x)}{3} \right) \right),
\]

\[
\tilde{m}_1(x) = \frac{1}{3} \left( \frac{1}{18} \rho^2(x) - 1 + \cos 2\pi x_2 \right),
\]

\[
\tilde{m}_2(x) = \frac{i}{3} \left( -\rho(x) + \sin 2\pi x_2 \right),
\]

where \( \rho(x) = B \sin 2\pi (Ax_1 + Cx_2) - A \sin 2\pi (Bx_1 - Ax_2) \).

For \( M = \begin{pmatrix} 3 & -2 \\ 3 & -3 \end{pmatrix} \), the vectors \( \{s_0', s_1', s_2'\} \) form a set of digits. So, we can take the corresponding formulas for masks and their polyphase representatives, respectively, with \( \rho(x) = 3 \sin 2\pi (2x_1 + 3x_2) - 2 \sin 6\pi (x_1 + x_2) \) and \( \sigma(x) = -2 \sin 2\pi x_1 - 3 \sin 2\pi x_2 \).

Now let \( M = \begin{pmatrix} 2 & 2 \\ -1 & -2 \end{pmatrix} \), \( c = (1, 1) \), we will find a refinable mask \( m_0 \) providing \( VM^2 \) property and such that \( e^{\pi i (c, x)} m_0(x) \) is a real even function. Due to the results of Sec. 3, the polyphase representatives of \( m_0 \) may be given by

\[
\mu_{00}(x) = \mu_{01}(x) = \frac{1}{\sqrt{2}} \left( 1 - \frac{1}{32} (4 \sin 2\pi x_1 - 3 \sin 2\pi x_2)^2 \right.
- \frac{i}{4} (4 \sin 2\pi x_1 - 3 \sin 2\pi x_2) \bigg),
\]

A dual mask \( \tilde{m}_0 \) may be defined by its polyphase representatives

\[
\tilde{\mu}_{00}(x) = \mu_{01}(x) = \frac{1}{\sqrt{2}} \left( 1 - \frac{i}{8} (4 \sin 2\pi x_1 - 3 \sin 2\pi x_2) \right),
\]

The polyphase matrices

\[
M = \begin{pmatrix} \mu_{00} & \mu_{01} \\ \overline{\mu}_{00} & \overline{\mu}_{01} \end{pmatrix}, \quad \widetilde{M} = \begin{pmatrix} \tilde{\mu}_{00} & \tilde{\mu}_{01} \\ -\mu_{00} & \mu_{01} \end{pmatrix}
\]
generate dual wavelet systems \( \{ \psi_{jk} \} \), \( \{ \tilde{\psi}_{jk} \} \). The corresponding masks are

\[
m_0(x) = e^{-\pi i (x_1 + x_2)} \left( \cos \pi (x_1 + x_2) \left( 1 - \frac{1}{32} \rho^2(x) \right) + \frac{1}{4} \sin \pi (x_1 + x_2) \rho(x) \right),
\]

\[
\tilde{m}_0(x) = e^{-\pi i (x_1 + x_2)} \left( \cos \pi (x_1 + x_2) + \frac{1}{8} \sin \pi (x_1 + x_2) \rho(x) \right),
\]

\[
m_1(x) = i e^{-\pi i (x_1 + x_2)} \left( \frac{1}{8} \cos \pi (x_1 + x_2) \rho(x) - \sin \pi (x_1 + x_2) \right),
\]

\[
\tilde{m}_1(x) = i e^{-\pi i (x_1 + x_2)} \left( \frac{1}{4} \cos \pi (x_1 + x_2) \rho(x) - \sin \pi (x_1 + x_2) \left( 1 - \frac{1}{32} \rho^2(x) \right) \right),
\]

where \( \rho(x) = 4 \sin 2\pi(2x_1 - x_2) - 3 \sin 4\pi(x_1 - x_2) \).

### 4.4. Hierarchical and embedded block coding

A new procedure for the synthesis of nonseparable filtering system was developed.\(^8,30\) It allows to code images and other M-D objects by using of hierarchical and embedded block coding techniques, where there must be a similarity between the wavelet decompositions at different levels. For arbitrary decimation matrices there is no similarity between them. The result of its application for the case of three-channel filter banks developed in this paper is shown in Fig. 1.

![Fig. 1. Two-level decomposition for a three-channel nonseparable multirate system.](image-url)
As it can be seen from Fig. 2 nonseparable FBs give better results than separable filters in compression of 2D signals.\textsuperscript{17} The comparison is made between the filters with the same number of vanishing moments (three) for each dimension: 4-channel separable biorthogonal filters (bior3.3 — separable FBs with separable decimation matrix), 2-channel nonseparable biorthogonal FBs based on Bernstein polynomials (bern3.3 — nonseparable FBs for nonseparable decimation matrix) and 4-channel nonseparable orthogonal FBs (5.3 — nonseparable FBs for separable decimation matrix).

The results of coding test for JPG2000 coder (j2k) developed coding technique is given in Fig. 3. It should be noticed that this coder does not have any arithmetic coder, that might improve coding results.\textsuperscript{20}

4.5. Software implementation

There were developed three generations of software implementation:

(1) Portable and feature-rich (slow) version written in plain C and ported to DSP.

It featured separable and nonseparable convolutions, several padding types, color space conversions, etc. Written in plain C; it was ported to DSP with some DSP-specific tools and presented in a device at the exhibition DSPA2006.
Fig. 3. Coding test for JPEG2000 coder and hierarchical coder.

Fig. 4. Performance comparison for three software generations.
(2) x86-optimized (P-IV/fast) written in C++ using templates and built using Intel C++ optimizing compiler unique in the way it is coded. Complete rewrite of the DWT code. Uses the power of C++ optimizing compiler and templates to build the DWT subroutine for very specific requirements during compile-time. Built with Intel P-IV NetBurst architecture in mind.

(3) GPU-based version (very fast), uses modern video cards GPU power and parallel architecture, requires SM3.0-capable video card. Uses a modern GPU (Graphics Processing Unit) to perform floating-point calculations. Filtering is a highly parallel process, ideally fits video card parallel architectures. SM3.0-capable hardware is widely available, scalable and flexible (shaders can be generated and loaded at runtime, for each specific task). It frees the CPU for other tasks.

For P-IV with 2.4 GHz, 768 MB, DDR 433 MHz RAM and for an image 512 × 512, 8bpp, grayscale, and separable 4-tap filters the results are given in Fig. 4. GPU-based solution uses GeForce 6600GT, 350/800 MHz, 128 MB RAM. The image is 512 × 512 32bpp RGBA with R16FG16FB16FA16F intermediate RTs used.

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