Approximation by frame-like wavelet systems

A. Krivoshein, M. Skopina *

St. Petersburg State University, Russian Federation

**Abstract**

A wide class of MRA-based wavelet systems which are not frames in \( L_2(\mathbb{R}^d) \), generally speaking, is studied. Frame-type expansions over a pair of dual wavelet systems (with the series converging in different senses) and their approximation order are investigated.

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1. Introduction

During last ten years wavelet frames are actively studied in the literature (see [7–9,12,14,18–20,22,26,34,36,37,39,40] and the references therein). They are also of great interest for different engineering applications, especially for signal and image processing (see, e.g., [1,5,6,15]).

A family of functions \( \{f_n\}_{n \in \mathbb{N}} \) (\( \mathbb{N} \) is a countable index set) in a Hilbert space \( H \) is called a frame in \( H \) if there exist \( A, B > 0 \) such that \( A \|f\|^2 \leq \sum_{n \in \mathbb{N}} |\langle f, f_n \rangle|^2 \leq B \|f\|^2 \) for all \( f \in H \). An important property of a frame \( \{f_n\}_{n \in \mathbb{N}} \) in \( H \) is the following: every \( f \in H \) can be decomposed as \( f = \sum_{n \in \mathbb{N}} \langle f, f_n \rangle f_n \), where \( \{f_n\}_{n \in \mathbb{N}} \) is a dual frame in \( H \). Good approximation properties of the frame decomposition is usually desirable. Finding wavelet functions generating a pair of dual wavelet frames \( \{\psi_{jk}\}, \{\tilde{\psi}_{jk}\} \) with required approximation order is a complicated problem, especially in the multidimensional case. Though a general scheme for the construction of MRA-based wavelet frames is known, its realization leads to dual wavelet systems but not necessary to dual frames. In particular, it is not easy to provide vanishing moments for all wavelet functions \( \psi^{(v)}, \tilde{\psi}^{(v)} \), which is a necessary condition for the systems \( \{\psi_{jk}\}, \{\tilde{\psi}_{jk}\} \) to be frames in \( L_2(\mathbb{R}^d) \). However, engineers often do not take care of this and successfully apply such “frames” (which are really not frames) for signal processing (see, e.g., [1]). B. Han and Z. Shen [18] considered a class of MRA-based dual wavelet systems \( \{\psi_{jk}\}, \{\tilde{\psi}_{jk}\} \) for dyadic dilation which are respectively frames in the Sobolev spaces \( W^s_2(\mathbb{R}^d), W^{-s}_2(\mathbb{R}^d) \) but not a pair of dual frames in \( L_2(\mathbb{R}^d) \). M. Ehler [11] extended this result to a wider class of dilation matrices.

The goal of this paper is to study frame-type decomposition for a wide class of compactly supported MRA-based dual wavelet systems \( \{\psi_{jk}\}, \{\tilde{\psi}_{jk}\} \), where \( \psi^{(v)}, \tilde{\psi}^{(v)} \) are functions or distributions. Approximation order for such systems is also studied.

© This research was supported by Grant 09-01-00162 of RFBR. The second author is also supported by DFG Project 436 RUS 113/951.

* Corresponding author.

E-mail addresses: san_san@inbox.ru (A. Krivoshein), skopina@MS1167.spb.edu (M. Skopina).

1063-5203/$ – see front matter © 2011 Published by Elsevier Inc.
doi:10.1016/j.acha.2011.02.003

1.1. Notations

\( \mathbb{N} \) is the set of positive integers, \( \mathbb{R}^d \) denotes the \( d \)-dimensional Euclidean space, \( x = (x_1, \ldots, x_d) \), \( y = (y_1, \ldots, y_d) \) are its elements (vectors), \( (x, y) = x_1 y_1 + \cdots + x_d y_d \), \( |x| = \sqrt{(x, x)} \), \( \mathbf{0} = (0, \ldots, 0) \in \mathbb{R}^d \). \( \mathbb{B}_R := \{x \in \mathbb{R}^d : |x| < R \} \), \( R > 0 \), \( \mathbb{Z}^d \) is the integer lattice in \( \mathbb{R}^d \). For \( x, y \in \mathbb{R}^d \), we write \( x > y \) if \( x_j > y_j \), \( j = 1, \ldots, d \), \( x \geq y \) if \( x_j \geq y_j \), \( j = 1, \ldots, d \); \( \mathbb{Z}^d_+ := \{x \in \mathbb{Z}^d : x \geq \mathbf{0}\} \). If \( \alpha, \beta \in \mathbb{Z}^d_+ \), \( a, b \in \mathbb{R}^d \), we set \( [\alpha] = \sum_{j=1}^d \alpha_j, [\alpha]! = \prod_{j=1}^d \alpha_j! \). \( \tilde{\sigma}(\alpha, \beta) \). If \( f \in \mathbb{R}^d \), \( \alpha \in \mathbb{Z}^d \), \( \alpha \geq 0 \) then \( \langle \tilde{\sigma}(\alpha, \beta); f \rangle = \sum_{j=0}^\infty \frac{(-1)^j}{j!} \int |\alpha_j|! [\alpha_j]! \sum_{k=0}^j \frac{1}{k!} \frac{\partial^k f}{\partial x^k}(\alpha) \right) \), \( \tilde{\sigma}(\alpha, \beta) \) denotes its Fourier transform defined by \( \langle \tilde{\sigma}(\alpha, \beta); f \rangle = \sum_{j=0}^\infty \frac{(-1)^j}{j!} \int |\alpha_j|! [\alpha_j]! \sum_{k=0}^j \frac{1}{k!} \frac{\partial^k f}{\partial x^k}(\alpha) \). Note that if \( f \in \mathbb{S}' \), \( g \in \mathbb{S} \) then \( (f, g) := \int_{\mathbb{R}^d} f g \). If \( f \) is \( 1 \)-periodic (with respect to each variable) function and \( F \in L_1([0, 1]^d) \), then \( \hat{F}(k) = \int_{[0, 1]^d} F(x) e^{-2\pi i (k, x)} \). The subset of \( L_\infty(\mathbb{R}^d) \) consisting of all \( f \) such that the restrictions of \( f \) onto \( \mathbb{R}^d \setminus \mathbb{B}_R \) tend to zero in \( L_\infty \)-norm as \( R \to \infty \) is denoted by \( L_\infty(\mathbb{R}^d) \).

1.2. Preliminaries

A function/distribution \( \varphi \) is called refinable if there exists a \( \mathcal{F} \)-periodic function \( m_0 \in L_2([0, 1]^d) \) (mask, also refinable mask) such that

\[
\hat{\varphi}(\xi) = m_0(M^{-1} \xi) \hat{\varphi}(M^{-1} \xi).
\]

This condition is called refinement equation. It is well known (see, e.g. [33, Theorem 2.4.4]) that for any trigonometric polynomial \( m_0 \) satisfying \( m_0(\mathbf{0}) = 1 \) there exists a unique solution (up to a factor) of the refinement equation (3) in \( \mathcal{F}' \). The solution is compactly supported and given by

For any 1-periodic function \( m_v \in L_2([0, 1]^d) \), there exists a unique set of 1-periodic function \( \mu_{v,k} \in L_2([0, 1]^d) \), \( k = 0, \ldots, m - 1 \) (polynomials of order \( k \)) such that

\[
m_v(\xi) = \frac{1}{\sqrt{m}} \sum_{k=0}^{m-1} e^{2\pi i (sk, \xi)} \mu_{v,k}(M^s \xi).
\]

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\[
m_v(\xi) = \frac{1}{\sqrt{m}} \sum_{k=0}^{m-1} e^{2\pi i (sk, \xi)} \mu_{v,k}(M^s \xi).
\]

It is not difficult to see that \( m_v \) is a trigonometric polynomial if and only if its polyphase components are trigonometric polynomials.

A method for the construction of MRA-based wavelet frames (Unitary/Mixed Extension Principle) was developed in [36, 37]. To construct a pair of dual wavelet frames one starts with two refinable functions \( \varphi, \tilde{\varphi} \in L_2(\mathbb{R}^d) \) with masks \( m_0, \tilde{m}_0 \) respectively, finds wavelet masks \( m_v, \tilde{m}_v \in L_2([0, 1]^d) \), \( v = 1, \ldots, r, r \geq m - 1 \), that satisfy the following conditions.

\[
\mathcal{M} := \begin{pmatrix}
\mu_{0,0} & \cdots & \mu_{0,m-1} \\
\vdots & \ddots & \vdots \\
\mu_{r,0} & \cdots & \mu_{r,m-1}
\end{pmatrix}, \quad \tilde{\mathcal{M}} := \begin{pmatrix}
\tilde{\mu}_{0,0} & \cdots & \tilde{\mu}_{0,m-1} \\
\vdots & \ddots & \vdots \\
\tilde{\mu}_{r,0} & \cdots & \tilde{\mu}_{r,m-1}
\end{pmatrix}
\]

satisfy

\[
\mathcal{M}^T \tilde{\mathcal{M}} = I_m,
\]

and define wavelet functions by

\[
\hat{\psi}^{(v)}(\xi) = m_v(M^{s-1} \xi) \hat{\varphi}(M^{s-1} \xi), \quad \tilde{\hat{\psi}}^{(v)}(\xi) = \tilde{m}_v(M^{s-1} \xi) \hat{\varphi}(M^{s-1} \xi).
\]

If wavelet functions \( \psi^{(v)}, \tilde{\psi}^{(v)}, v = 1, \ldots, r \), are constructed as above, then \( \{\psi_{jk}^{(v)}, \tilde{\psi}_{jk}^{(v)}\} \) is said to be (MRA-based) dual wavelet systems. If we want to have compactly supported wavelet functions, all the masks should be trigonometric polynomials. In what follows we will discuss only such constructions. On the other hand, each wavelet function \( \psi^{(v)}, \tilde{\psi}^{(v)} \) should have vanishing moment of order 0. This condition is necessary (see [39, Theorem 1]) and sufficient (see [19, Theorems 2.2, 2.3]) for a compactly supported systems \( \{\psi_{jk}^{(v)}, \tilde{\psi}_{jk}^{(v)}\} \) with \( \psi^{(v)}, \tilde{\psi}^{(v)} \in L_2(\mathbb{R}^d) \) to be a pair of dual frames in \( L_2(\mathbb{R}^d) \).

If wavelet functions \( \psi^{(v)}, v = 1, \ldots, r \), are defined by (7) and \( \hat{\varphi}(0) \neq 0 \), then the \( VM^0 \) property for the corresponding wavelet system can be characterized in terms of the wavelet masks as follows

\[
D^\beta (m_v(M^{s-1}x)|_{[x]} = 0, \quad v = 1, \ldots, r, \forall \beta \in \mathbb{Z}_+^d, |\beta| \leq n.
\]

It is well known that in the case \( r = m - 1 \), (8) depends only on the refinable mask \( \tilde{m}_0 \), and does not depend on matrix extension (which is not unique). However, it is not true if \( r > m - 1 \), in this case (8) depends on \( m_0, \tilde{m}_0 \) and matrix extension.

A criterion is given in

**Theorem A.** (See [39, Theorem 7.]) Let \( n \in \mathbb{Z}_+, r \geq m - 1, m_0, \tilde{m}_0, v = 0, \ldots, r, \) be trigonometric polynomials, \( \mu_{v,k}, \tilde{\mu}_{v,k} \), \( k = 0, \ldots, m - 1 \), \( v, k = 0, \ldots, r \), be their polyphase components respectively. If \( \mathcal{N}, \tilde{\mathcal{N}} \) are \( r \times r \) matrices whose entries are trigonometric polynomials \( \mu_{v,k}, \tilde{\mu}_{v,k}, v = 0, \ldots, r, \) respectively (i.e., first \( (m - 1) \) columns consist of the above polyphase components), and whose columns form a biorthonormal system, then (8) holds if and only if

\begin{align*}
(A1) & \text{ there exist complex numbers } \tilde{\lambda}_\gamma, \gamma \in \mathbb{Z}_+^d, |\gamma| \leq n, \tilde{\lambda}_0 = 1, \text{ such that } \\
& D^\beta \tilde{\mu}_{0k}(0) = \frac{1}{\sqrt{m}} \sum_{0 \leq \beta \leq \gamma} \tilde{\lambda}_\gamma \left( \frac{\beta}{\gamma} \right) (-2\pi i M^{s-1} \tilde{s}_k)^{\gamma - \beta} \forall \beta \in \mathbb{Z}_+^d, |\beta| \leq n, k = 0, \ldots, m - 1; \\
(A2) & D^\beta \mu_{0k}(0) = 0, \forall \beta \in \mathbb{Z}_+^d, |\beta| \leq n, k = m, \ldots, r.
\end{align*}

So, to provide a desirable number of vanishing moments for wavelet functions (in particular \( VM^0 \) for both wavelet systems) one should have refinable masks \( m_0, \tilde{m}_0 \) such that \( A1 \) and a similar condition for \( \mu_{0k} \) is satisfied. However, this is not enough to satisfy \( A2 \). Note that in the one-dimensional case it is much easier to satisfy \( A2 \) than in the multidimensional case. Construction of univariate dual wavelet frames with vanishing moments was developed, e.g., in [8]. A method for construction of multivariate refinable masks satisfying \( A1, A2 \) and an algorithm for matrix extension is given in [39]. Using this method, one can start with any smooth refinable function \( \varphi \) with a mask \( m_0 \). Condition \( A1 \) (with \( n \) depending on the order of smoothness) is satisfied because of
Theorem B. (See [10, Theorem 10].) Let \( t \) be a trigonometric polynomial, \( \tau_k, k = 0, \ldots, m - 1 \), be its polyphase components. The following conditions are equivalent

\[(B1) \text{ there exist complex numbers } \lambda_\gamma, \gamma \in \mathbb{Z}_+^d, |\gamma| \leq n, \lambda_0 = 1, \text{ such that} \]
\[
D^\beta \tau_k(0) = \frac{1}{\sqrt{m}} \sum_{0 \leq \gamma \leq \beta} \lambda_\gamma \left( \frac{\beta}{\gamma} \right) (-2\pi i M^{-1} s_k)^{\beta - \gamma} \quad \forall \beta \in \mathbb{Z}_+^d, |\beta| \leq n, k = 0, \ldots, m - 1;
\]

\[(B2) D^\beta (M_s^{-1} x)|_{x=0} = 0 \text{ for all } s \in D(M_s^*), s \neq 0, \text{ and for all } \beta \in \mathbb{Z}_+^d, |\beta| \leq n.\]

Next, one finds a proper dual refinable mask \( \tilde{m}_0 \) and wavelet masks \( m_\nu, \tilde{m}_\nu, \nu = 1, \ldots, r \), define \( \tilde{\varphi} \) by
\[
\tilde{\varphi}(\xi) = \prod_{j=1}^{\infty} \tilde{m}_0(M^{s-j} \xi)
\]
and wavelet functions \( \psi^{(\nu)}, \tilde{\psi}^{(\nu)} \) by (7). This leads to dual frames if only \( \tilde{\varphi} \in L_2(\mathbb{R}^d) \), which is not guaranteed, unfortunately.

Thus the construction of dual compactly supported wavelet frames is much more complicated than finding dual wavelet masks according to Mixed Extension Principle. If wavelet masks \( m_\nu, \tilde{m}_\nu, \nu = 1, \ldots, r \) are found in this way and \( \psi^{(\nu)}, \tilde{\psi}^{(\nu)} \) are defined by (7), we say that \( \psi^{(\nu)}, \tilde{\psi}^{(\nu)} \) are wavelet functions associated with \( \varphi, \tilde{\varphi} \) (or with its masks \( m_0, \tilde{m}_0 \)). It is not difficult to construct compactly supported wavelet functions associated with arbitrary trigonometric polynomials \( m_0, \tilde{m}_0 \) (see the proof of Lemma 14). Wavelet functions associated with a pair of refinable functions generate the corresponding pair of MRA-based dual wavelet systems \( \{\psi^{(\nu)}_j\}, \{\tilde{\psi}^{(\nu)}_j\} \) which are not necessary frames in \( L_2(\mathbb{R}^d) \) and, moreover, consist of tempered distributions, generally speaking. Nevertheless, we can consider frame-type decomposition with respect to these systems of appropriate functions \( f \) (e.g., \( f \in S \) if \( \tilde{\psi}^{(\nu)} \in S' \); \( f \in L_\rho \) if \( \tilde{\psi}^{(\nu)} \in L_\rho, \frac{1}{p} + \frac{1}{q} = 1 \)).

Let \( \{\psi^{(\nu)}_j\}, \{\tilde{\psi}^{(\nu)}_j\} \) be dual wavelet systems and \( A \) be a class of functions \( f \) for which \( \langle f, \psi^{(\nu)}_{0k} \rangle, \langle f, \tilde{\psi}^{(\nu)}_{jk} \rangle \) have meaning. We say that \( \{\psi^{(\nu)}_j\} \) is frame-like if
\[
f = \sum_{i=-\infty}^{\infty} \sum_{\nu=1}^{r} \sum_{k \in \mathbb{Z}^d} \langle f, \tilde{\psi}^{(\nu)}_{jk} \rangle \psi^{(\nu)}_j \quad \forall f \in A, \quad (9)
\]
and \( \{\psi^{(\nu)}_j\} \) is almost frame-like if
\[
f = \sum_{k \in \mathbb{Z}^d} \langle f, \psi^{(\nu)}_{0k} \rangle \varphi_{0k} + \sum_{i=0}^{\infty} \sum_{\nu=1}^{r} \sum_{k \in \mathbb{Z}^d} \langle f, \tilde{\psi}^{(\nu)}_{jk} \rangle \psi^{(\nu)}_j \quad \forall f \in A, \quad (10)
\]
where the series in (9) and (10) converge in some natural sense.

1.3. Paper outline

The paper is organized as follows. In Section 2 the scaling operator \( Q_j \) in different spaces is studied. In particular, the limit of \( Q_j(f, \varphi, \tilde{\varphi}) \) as \( j \to \pm \infty \) in different senses is investigated for arbitrary appropriate functions/distributions \( \varphi, \tilde{\varphi} \). In Section 3 we assume that \( \varphi, \tilde{\varphi} \) are refinable and consider associated dual wavelet systems \( \{\psi^{(\nu)}_j\}, \{\tilde{\psi}^{(\nu)}_j\} \). We investigate if these systems are frame-like or almost frame-like in different situations, and study approximation order of the frame-type decompositions (9), (10). In Section 4 several examples are presented.

2. Scaling approximation

First, we study the scaling operators \( Q_j \) in the space \( S' \).

Lemma 1. Let \( \varphi \in S' \) be compactly supported, \( f \in S, j \in \mathbb{Z} \),
\[
G_j(\xi) = G_j(\varphi, f, \xi) = \sum_{l \in \mathbb{Z}^d} \hat{f}(M^{sj}(\xi + l)) \hat{\varphi}(\xi + l).
\]

Then \( G_j \) is a bounded 1-periodic function, in particular, \( G_j \in L_2[-\frac{1}{2}; \frac{1}{2}]^d \), and
\[
\langle f, \varphi_{jk} \rangle = m^{1/2} \hat{G}_j(k).
\]
Moreover, if \( j \geq 0 \), then
\[
|G_j(\xi)| \leq C_{\varphi, f, M}, \quad \forall \xi \in \mathbb{R}^d,
\]
(11)
and for any \( N \in \mathbb{N} \) there exists a constant \( C_{N, \varphi, f, M} > 0 \) such that
\[
|G_j(\xi) - \hat{f}(M^{-j}\xi)\hat{\varphi}(\xi)| \leq C_{N, \varphi, f, M}\|M^{-j}\|_N^N
\]
(12)
for all \( \xi \in [-\frac{1}{2}; \frac{1}{2}]^d \).

**Proof.** By the Paley–Wiener theorem for tempered distributions, there exist \( N_0 \in \mathbb{N} \) and \( C_{\varphi} > 0 \) so that \(|\hat{\varphi}(\xi)| \leq C_{\varphi}|\xi|^{N_0} \) for all \( \xi \notin [-\frac{1}{2}; \frac{1}{2}]^d \).

Clearly, it suffices to check (12) for big enough \( N \). Let \( N > N_0 + d, N \in \mathbb{N} \), \( C_{N, f} = \sup_{\xi \in \mathbb{R}^d} |\xi|^N|\hat{f}(\xi)| \). If \( \xi \in [-\frac{1}{2}; \frac{1}{2}]^d \), then
\[
\left| \sum_{l \in \mathbb{Z}^d, l \neq 0} \hat{f}(M^{-j}(\xi + l))\hat{\varphi}(\xi + l) \right| \leq C_{\varphi} \sum_{l \in \mathbb{Z}^d, l \neq 0} \frac{|\xi + l|^N|\hat{f}(M^{-j}(\xi + l))|}{|\xi + l|^{N_0}}
\]
\[
\leq C_{\varphi} C_{N, f}\|M^{-j}\|_N^N \sum_{l \in \mathbb{Z}^d, l \neq 0} \frac{1}{|\xi + l|^{N - N_0}} = C_{N, \varphi, f}\|M^{-j}\|_N^N,
\]
which yields the boundedness of \( G_j \) for all \( j \in \mathbb{N} \), and (12) for all \( N \in \mathbb{N} \) and \( j \geq 0 \). Since
\[
|\hat{f}(M^{-j}\xi)\hat{\varphi}(\xi)| \leq C_{\varphi}\|M^{-j}\|_N^N \left|\hat{f}(M^{-j}\xi)\right|\|M^{-j}\|_N^N \leq C_{N_0, f}C_{\varphi}\|M^{-j}\|_N^N,
\]
taking into account (2) and (12), we obtain (11). Besides, we proved that the series
\[
\sum_{l \in \mathbb{Z}^d} |\hat{f}(M^{-j}(\xi + l))\hat{\varphi}(\xi + l)|
\]
uniformly converges on \([-\frac{1}{2}; \frac{1}{2}]^d \) to a bounded function, which implies
\[
\langle f, \varphi_{jk} \rangle = \langle \hat{f}, \hat{\varphi}_{jk} \rangle = m^{-j/2} \int_{\mathbb{R}^d} \hat{f}(\xi)\hat{\varphi}(M^{-j}\xi)e^{-2\pi i(k,M^{-j}\xi)} d\xi
\]
\[
= m^{j/2} \int_{[-\frac{1}{2}; \frac{1}{2}]^d} G_j(\xi)e^{-2\pi i(k,\xi)} d\xi = m^{j/2}\hat{G}_j(k),
\]
where \( k \in \mathbb{Z}^d \). \( \square \)

**Theorem 2.** Let \( \varphi, \hat{\varphi} \in S' \) be compactly supported, \( f \in S \). Then
(a) for every \( j \in \mathbb{Z} \), the series \( \sum_{k \in \mathbb{Z}^d} \langle f, \varphi_{jk} \rangle \varphi_{jk} \) converges unconditionally in \( S' \), in particular, \( Q_j(\varphi, \hat{\varphi}, f) \in S' \);
(b) \( Q_j(\varphi, \hat{\varphi}, f) \) tends to \( \hat{\varphi}(0)\hat{\varphi}(0)f \) in \( S' \) as \( j \to +\infty \).

**Proof.** Let \( \tilde{f} \in S \). To prove (a) it suffices to check that the series \( \sum_{k \in \mathbb{Z}^d} \langle f, \varphi_{jk} \rangle \varphi_{jk}, \tilde{f} \rangle \) is absolutely convergent, but Lemma 1 yields that
\[
\sum_{k \in \mathbb{Z}^d} |\langle f, \varphi_{jk} \rangle \varphi_{jk}, \tilde{f} \rangle| \leq m^j \|G_j(\varphi, f, \cdot)\|_2 \|G_j(\varphi, \tilde{f}, \cdot)\|_2 < \infty.
\]
To prove (b) we should check that
\[
\lim_{j \to +\infty} \sum_{k \in \mathbb{Z}^d} \langle f, \varphi_{jk} \rangle \varphi_{jk}, \tilde{f} \rangle = \hat{\varphi}(0)\hat{\varphi}(0)(f, \tilde{f}) = \hat{\varphi}(0)\hat{\varphi}(0) \int_{\mathbb{R}^d} \tilde{f}(\xi)\tilde{f}(\xi) d\xi.
\]
Due to Lemma 1,
\[\sum_{k \in \mathbb{Z}^d} \langle f, \tilde{\phi}_{j,k} \rangle \langle \tilde{\psi}_{j,k} \rangle = m_j \int_{-\frac{1}{2}^{d}}^{\frac{1}{2}^{d}} G_j(\tilde{\phi}, f, \xi) \tilde{G}_j(\phi, \tilde{f}, \xi) \, d\xi \]
\[= m_j \int_{-\frac{1}{2}^{d}}^{\frac{1}{2}^{d}} \hat{f}(M^s \xi) \hat{\phi}(\xi) \hat{G}(M^s \xi) \hat{\phi}(\xi) \, d\xi + m_j \int_{-\frac{1}{2}^{d}}^{\frac{1}{2}^{d}} \hat{f}(M^s \xi) \hat{\psi}(\xi) \hat{H}(M^s \xi) \hat{\psi}(\xi) \, d\xi + m_j \int_{-\frac{1}{2}^{d}}^{\frac{1}{2}^{d}} H_j(\tilde{\phi}, f, \xi) \hat{f}(M^s \xi) \hat{\phi}(\xi) \, d\xi + m_j \int_{-\frac{1}{2}^{d}}^{\frac{1}{2}^{d}} H_j(\tilde{\phi}, f, \xi) H_j(\tilde{f}, \xi) \, d\xi,\] (13)

where
\[H_j(\tilde{\phi}, f, \xi) = G_j(\tilde{\phi}, f, \xi) - \hat{f}(M^s \xi) \hat{\phi}(\xi), \quad H_j(\tilde{\phi}, \tilde{f}, \xi) = G_j(\tilde{\phi}, \tilde{f}, \xi) - \hat{f}(M^s \xi) \hat{\phi}(\xi).\]

By change of variable in the first term of the right-hand side of (13), we have
\[m_j \int_{-\frac{1}{2}^{d}}^{\frac{1}{2}^{d}} \hat{f}(M^s \xi) \hat{\phi}(\xi) \hat{G}(M^s \xi) \hat{\phi}(\xi) \, d\xi = \int_{\mathbb{R}^d} X_{M^s(-\frac{1}{2}^{d})}(\xi) \hat{f}(\xi) \hat{G}(M^s \xi) \hat{\phi}(\xi) \hat{M}^{s-1} \hat{\xi} \hat{M}^{s-1} \hat{\xi} \hat{M} \hat{\phi}(\xi) \, d\xi.\]

Due to the Paley–Wiener theorem for tempered distributions and (2), there exist positive numbers \(N, \bar{N}\) and \(C, \bar{C}\) such that
\[|\hat{\phi}(M^{s-1} \xi)| \leq C (1 + |\xi|)^{N}, \quad |\hat{\psi}(M^{s-1} \xi)| \leq \bar{C} (1 + |\xi|)^{\bar{N}}\] (14)
for all \(\xi \in \mathbb{R}^d\) and all \(j > 0\). Hence
\[|X_{M^s(-\frac{1}{2}^{d})}(\xi) \hat{f}(\xi) \hat{G}(M^s \xi) \hat{\phi}(\xi) \hat{M}^{s-1} \hat{\xi} \hat{M}^{s-1} \hat{\xi} \hat{M} \hat{\phi}(\xi)| \leq C \bar{C} (1 + |\xi|)^{N+\bar{N}} |\hat{f}(\xi)| |\hat{G}(\xi)|.\]

Since, due to (2),
\[\lim_{j \to +\infty} X_{M^s(-\frac{1}{2}^{d})}(\xi) \hat{f}(\xi) \hat{G}(M^s \xi) \hat{\phi}(\xi) \hat{M}^{s-1} \hat{\xi} \hat{M}^{s-1} \hat{\xi} \hat{M} \hat{\phi}(\xi) = \hat{\phi}(0) \hat{f}(0)\]
for any \(\xi \in \mathbb{R}^d\) and \((1 + |\xi|)^{N+\bar{N}} |\hat{f}(\xi)| |\hat{G}(\xi)|\) is summable on \(\mathbb{R}^d\), using Lebesgue's dominated convergence theorem, we obtain
\[\lim_{j \to +\infty} \left( m_j \int_{-\frac{1}{2}^{d}}^{\frac{1}{2}^{d}} \hat{f}(M^s \xi) \hat{\phi}(\xi) \hat{G}(M^s \xi) \hat{\phi}(\xi) \, d\xi - \hat{\phi}(0) \hat{f}(0) \int_{\mathbb{R}^d} |\hat{f}(\xi)| |\hat{G}(\xi)| \, d\xi \right) = 0.\] (15)

It follows from Lemma 1 that
\[\sup_{\xi \in [-\frac{1}{2}^{d}, \frac{1}{2}^{d}]} |H_j(\tilde{\phi}, \tilde{f}, \xi)| \leq C_{1,\psi, \tilde{f}, M} \|M^{s-1}\|.

Hence, by change of variable and using (14), we have
\[m_j \int_{-\frac{1}{2}^{d}}^{\frac{1}{2}^{d}} |\hat{f}(M^s \xi) \hat{\phi}(\xi) H_j(\tilde{\phi}, \tilde{f}, \xi) | \, d\xi \leq C_{1,\psi, \tilde{f}, M} \|M^{s-1}\| m_j \int_{-\frac{1}{2}^{d}}^{\frac{1}{2}^{d}} |\hat{f}(M^s \xi) \hat{\phi}(\xi) | \, d\xi \]
\[= C_{1,\psi, \tilde{f}, M} \|M^{s-1}\| \int_{M^s(-\frac{1}{2}^{d})} |\hat{f}(\xi) \hat{\phi}(M^{s-1} \xi) | \, d\xi \leq C_{1,\psi, \tilde{f}, M} \|M^{s-1}\| \int_{\mathbb{R}^d} |\hat{f}(\xi) | (1 + |\xi|)^{\bar{N}} \, d\xi.

Thus, due to (2), the second term of the right-hand side of (13) tends to 0 as \(j \to +\infty\). Similarly, the third and the forth terms also tend to 0. \(\square\)
Corollary 3. Let \( \varphi, \tilde{\varphi} \in S' \) be compactly supported. \( f \in S, j \in \mathbb{Z}, Q_j := Q_j(\varphi, \tilde{\varphi}, f) \). Then \( \tilde{Q}_j \) is a function on \( \mathbb{R}^d \) and

\[
\tilde{Q}_j(\xi) = \hat{\varphi}(M^{-j}\xi) \sum_{l \in \mathbb{Z}^d} \tilde{f}(\xi + M^j l) \overline{\hat{\varphi}(M^{-j}\xi + l)} \quad \text{a.e.}
\]  

(16)

If, moreover, \( \hat{\varphi} \in L_p(\mathbb{R}^d), 1 \leq p \leq \infty \), then \( \tilde{Q}_j \in L_p(\mathbb{R}^d) \).

Proof. Due to condition (a) of Theorem 2, we have

\[
\tilde{Q}_j(\xi) = \sum_{k \in \mathbb{Z}^d} \langle f, \hat{\varphi}_j \rangle \hat{\varphi}_j = \sum_{k \in \mathbb{Z}^d} \hat{G}_j(k) e^{2\pi i k \cdot (M^{-j}\xi)} \hat{\varphi}(M^{-j}\xi),
\]

where \( \hat{G}_j(\xi) = \hat{G}_j(\hat{\varphi}, f, \xi) \) is a function from Lemma 1. Since \( G_j \in L_2(\mathbb{R}^d) \), by the Carleson theorem, \( G_j(M^{-j}\xi) = \sum_{k \in \mathbb{Z}^d} \hat{G}_j(k) e^{2\pi i k \cdot (M^{-j}\xi)} \) for almost all \( \xi \in \mathbb{R}^d \), which proves (16). The boundedness of \( G_j \) yields the second statement. □

Note that identity (16) can be found in the literature for some special cases, in particular, it appears in [9] for the case \( \varphi, \tilde{\varphi} \in L_2(\mathbb{R}^d) \) and in [27] for the case \( \varphi \in L_2(\mathbb{R}^d), \tilde{\varphi} \) is a tempered distribution. We did not see this formula for arbitrary tempered distributions.

Next we are going to consider the scaling operator \( Q_j \) in the spaces \( L_p \). First, we give several simple lemmas.

Lemma 4. Let \( 1 \leq p < \infty, \frac{1}{p} + \frac{1}{q} = 1, f \in L_p(\mathbb{R}^d), \varphi \in L_q(\mathbb{R}^d), \varphi \) be compactly supported. Then

\[
\left( \sum_{k \in \mathbb{Z}^d} \left| \langle f, \varphi_j \rangle \right|^p \right)^{1/p} \leq C_{p,q} m^{\frac{d}{p} - \frac{1}{2}} \| f \|_p.
\]

(17)

Proof. Assume that \( \text{sup} \varphi \subset [-N, N]^d \). Then

\[
\left( \sum_{k \in \mathbb{Z}^d} \left| \langle f, \varphi_j \rangle \right|^p \right)^{1/p} = \left( \sum_{k \in \mathbb{Z}^d} \left| m^{-j/2} \int_{[-N,N]^d - k} f(M^{-j} x) \varphi(x + k) \, dx \right|^p \right)^{1/p} \\
\leq m^{-j/2} \| \varphi \|_q \left( \sum_{k \in \mathbb{Z}^d} \int_{[-N,N]^d - k} |f(M^{-j} x)|^p \, dx \right)^{1/p} \\
\leq m^{-\frac{j}{2}} \| \varphi \|_q (2N + 1)^\frac{d}{q} \| f(M^{-j} \cdot) \|_p = m^{\frac{j}{2} - \frac{d}{2}} \| \varphi \|_q (2N + 1)^\frac{d}{q} \| f \|_p.
\]

Note that identity (16) can be found in the literature for some special cases, in particular, it appears in [9] for the case \( \varphi, \tilde{\varphi} \in L_2(\mathbb{R}^d) \) and in [27] for the case \( \varphi \in L_2(\mathbb{R}^d), \tilde{\varphi} \) is a tempered distribution. We did not see this formula for arbitrary tempered distributions.

Next we are going to consider the scaling operator \( Q_j \) in the spaces \( L_p \). First, we give several simple lemmas.

Lemma 5. Let \( f \in L_\infty(\mathbb{R}^d), \varphi \in L_1(\mathbb{R}^d), \varphi \) be compactly supported. Then

\[
\sup_{k \in \mathbb{Z}^d} \left| \langle f, \varphi_j \rangle \right| \leq C_{\infty,q} m^{-\frac{d}{2}} \| f \|_\infty.
\]

The proof of this lemma is evident.

Lemma 6. Let \( 1 < p < \infty, \frac{1}{p} + \frac{1}{q} = 1, \varphi \in L_q(\mathbb{R}^d), f \in L_p(\mathbb{R}^d) \) and \( \varphi \) be compactly supported. Then

\[
\lim_{j \to -\infty} m^{\frac{j}{2} - \frac{d}{2}} \left( \sum_{k \in \mathbb{Z}^d} \left| \langle f, \varphi_j \rangle \right|^p \right)^{1/p} = 0.
\]

Proof. First we assume that \( f \) is continuous on \( \mathbb{R}^d \) and \( \text{sup} f \subset B_R, R > 0 \). Using Hölder’s inequality, we obtain

\[
m^{\frac{j}{2} - \frac{d}{2}} \left| \langle f, \varphi_j \rangle \right| = m^{\frac{j}{2}} \left( \int_{\mathbb{R}^d} |f(x)| \left| \varphi(M^j x + k) \right| \, dx \right)^{1/2} \leq m^{\frac{j}{2}} \| f \|_p \left( \int_{\mathbb{R}^d} \left| \varphi(M^j x + k) \right|^q \, dx \right)^{1/2} \\
\leq m^{\frac{j}{2}} \| f \|_p \left( \int_{M^j B_R + k} \left| \varphi(y) \right|^q \, dy \right)^{1/2}.
\]
Since $\varphi$ is compactly supported, there exists only a finite number of $k \in \mathbb{Z}^d$ such that the latter integral not equals zero for at least one $j < 0$. So, it suffices to check that
\[
\lim_{j \to -\infty} \int_{M^j B_R + k} |\varphi(y)|^q \, dy = 0,
\]
for any $k \in \mathbb{Z}^d$. Let $X_{M^j B_R + k}$ denote the characteristic function of $M^j B_R + k$, then
\[
\int_{M^j B_R + k} |\varphi(y)|^q \, dy = \int_{\mathbb{R}^d} X_{M^j B_R + k}(y) |\varphi(y)|^q \, dy.
\]
If $y \neq k$, then $\lim_{j \to -\infty} X_{M^j B_R + k}(y) = 0$. Hence, using Lebesgue’s dominated convergence theorem, we have
\[
\lim_{j \to -\infty} \int_{\mathbb{R}^d} X_{M^j B_R + k}(y) |\varphi(y)|^q \, dy = 0.
\]
Now let $f \in L_p(\mathbb{R}^d)$. Given $\varepsilon > 0$, we find a compactly supported continuous $\tilde{f}$, such that $\|f - \tilde{f}\|_p < \varepsilon$. Using Minkowski’s inequality and Lemma 4, we obtain
\[
m^{\frac{1}{2} - \frac{1}{p}} \left( \sum_{k \in \mathbb{Z}^d} |(\tilde{f}, \varphi_{jk})|^p \right)^{\frac{1}{p}} \leq m^{\frac{1}{2} - \frac{1}{p}} \left( \sum_{k \in \mathbb{Z}^d} |(\tilde{f}, \varphi_{jk})|^p \right)^{\frac{1}{p}} + C_{p, \varphi, \varepsilon}.
\]

**Theorem 7.** Let $1 \leq p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $\varphi \in L_q(\mathbb{R}^d)$ and $\varphi \in L_p(\mathbb{R}^d)$ be compactly supported functions, $f \in L_p(\mathbb{R}^d)$. Then

(a) for every $j \in \mathbb{Z}$, the series $\sum_{k \in \mathbb{Z}^d} (f, \varphi_{jk}) \varphi_{jk}$ converges unconditionally in $L_p(\mathbb{R}^d)$, in particular, $Q_j(\varphi, \tilde{f}, f) \in L_p(\mathbb{R}^d)$;

(b) $\|Q_j(\varphi, \tilde{f}, f)\|_p \leq C_{p, \varphi, f} \|f\|_p$;

(c) if $p > 1$, then $\lim_{j \to -\infty} \|Q_j(\varphi, \tilde{f}, f)\|_p = 0$.

**Proof.** Let $\Omega$ be a finite subset of $\mathbb{Z}^d$. By the Riesz representation theorem,
\[
\left\| \sum_{k \in \Omega} (f, \tilde{\varphi}_{jk}) \varphi_{jk} \right\|_p = \left\| \sum_{k \in \Omega} (f, \tilde{\varphi}_{jk}) \varphi_{jk}, g \right\|,
\]
where $g = g(\Omega) \in L_q(\mathbb{R}^d)$, $\|g\|_q \leq 1$. Using Hölder’s inequality and Lemma 4, we obtain
\[
\left\| \sum_{k \in \Omega} (f, \tilde{\varphi}_{jk}) \varphi_{jk}, g \right\| \leq C_{q, \varphi} m^{\frac{1}{2} - \frac{1}{q}} \|g\| \left( \sum_{k \in \Omega} |(f, \tilde{\varphi}_{jk})|^p \right)^{\frac{1}{p}} \leq C_{q, \varphi} m^{\frac{1}{2} - \frac{1}{q}} \left( \sum_{k \in \Omega} |(f, \tilde{\varphi}_{jk})|^p \right)^{\frac{1}{p}}.
\]
This implies (a) because, due to Lemma 4, the series $\sum_{k \in \mathbb{Z}^d} |(f, \tilde{\varphi}_{jk})|^p$ is convergent. Similarly,
\[
\left\| \sum_{k \in \mathbb{Z}^d} (f, \tilde{\varphi}_{jk}) \varphi_{jk} \right\|_p \leq C_{q, \varphi} m^{\frac{1}{2} - \frac{1}{q}} \left( \sum_{k \in \mathbb{Z}^d} |(f, \tilde{\varphi}_{jk})|^p \right)^{\frac{1}{p}}.
\]
To prove (b) it remains to apply Lemma 4; to prove (c) it remains to apply Lemma 6. □

Now we show that condition (c) from Theorem 7 does not hold for $p = 1$. Let $f, \varphi \in L_1(\mathbb{R}^d)$, $f, \varphi \geq 0$, $\text{supp} f \subset B_r$, $\varphi = \chi_{B_r}$. Choose the numbers $R$ and $r$ so that $M^j B_r \subset B_R$ for all $j < 0$. Then for every $j < 0$
\[
\left\| \sum_{k \in \mathbb{Z}^d} (f, \tilde{\varphi}_{jk}) \varphi_{jk} \right\|_1 \geq \int_{\mathbb{R}^d} (f, \tilde{\varphi}_{j0}) \varphi_{j0}(x) \, dx \geq m^j \int_{B_r} \psi(M^j x) \, dx \int_{B_r} f(t) \tilde{\varphi}(M^j t) \, dt = \|\varphi\|_1 \|f\|_1.
\]
Condition (a) from Theorem 7 does not hold for $p = \infty$. Indeed, let $\varphi = \tilde{\varphi} = \chi_{[-1, 1]^d}$, $f = 1$. Then $(f, \varphi_{0k}) = 1$ for all $k \in \mathbb{Z}^d$, and, evidently, the series $\sum_{k \in \mathbb{Z}^d} \langle f, \varphi_{0k}\rangle \varphi_{0k} = \sum_{k \in \mathbb{Z}^d} \varphi_{0k}$ does not converge in $L_\infty(\mathbb{R}^d)$. A fortiori, we cannot discuss
if conditions (b) and (c) are satisfied in this case. This loss may be fixed by restricting the class of functions \( f \). Radial decay of \( f \) changes the situation. Almost the same arguments as for the proofs of Lemma 6 and Theorem 7 with replacing \( L_p \) by \( L_p^\alpha \) for \( p = \infty \) and using Lemma 5 instead of Lemma 4 give the following statements.

**Lemma 8.** Let \( \varphi \in L_1(\mathbb{R}^d) \), \( f \in L_{\infty}(\mathbb{R}^d) \) and \( \varphi \) be compactly supported. Then

\[
\lim_{j \to -\infty} \sup_{k \in \mathbb{Z}^d} |\langle f, \varphi_{jk} \rangle| = 0.
\]

**Theorem 9.** Let \( \tilde{\varphi} \in L_1(\mathbb{R}^d) \) and \( \varphi \in L_{\infty}(\mathbb{R}^d) \) be compactly supported functions, \( f \in L_{\infty}(\mathbb{R}^d) \). Then

(a) for every \( j \in \mathbb{Z} \), the series \( \sum_{k \in \mathbb{Z}^d} \langle f, \varphi_{jk} \rangle \varphi_{jk} \) converges unconditionally in \( L_{\infty}(\mathbb{R}^d) \), in particular, \( Q_j(\varphi, \tilde{\varphi}, f) \in L_{\infty}(\mathbb{R}^d) \);

(b) \( \| Q_j(\varphi, \tilde{\varphi}, f) \|_\infty \leq C_{\infty, \varphi, \tilde{\varphi}, f} \| f \|_\infty \);

(c) \( \lim_{j \to -\infty} \| Q_j(\varphi, \tilde{\varphi}, f) \|_\infty = 0 \).

A function/distribution \( \varphi \) whose Fourier transform is continuous is said to satisfy the Strang–Fix condition of order \( n \), \( n \in \mathbb{N} \), if \( D^d \hat{\varphi}(0) = 0 \) for all \( l \in \mathbb{Z}^d \), \( l \neq 0 \), \( |\omega| \leq n - 1 \).

**Theorem 10.** Let \( \varphi \in L_2(\mathbb{R}^d) \), \( \tilde{\varphi} \in S' \), \( \varphi, \tilde{\varphi} \) be compactly supported, \( f \in S \). Then

(a) for every \( j \in \mathbb{Z} \), the series \( \sum_{k \in \mathbb{Z}^d} \langle f, \varphi_{jk} \rangle \varphi_{jk} \) converges unconditionally in \( L_2(\mathbb{R}^d) \), in particular, \( Q_j(\varphi, \tilde{\varphi}, f) \in L_2(\mathbb{R}^d) \);

(b) if \( \hat{\varphi}(0) = \hat{\tilde{\varphi}}(0) = 1 \), then the Strang–Fix condition for \( \varphi \) is necessary and sufficient for the convergence of \( Q_j(\varphi, \tilde{\varphi}, f) \) to \( f \) in \( L_2 \)-norm as \( j \to +\infty \).

**Proof.** Let \( \Omega \) be a finite subset of \( \mathbb{Z}^d \). By the Riesz representation theorem,

\[
\left\| \sum_{k \in \Omega} \langle f, \varphi_{jk} \rangle \varphi_{jk} \right\|_2 = \left\| \sum_{k \in \Omega} \langle f, \varphi_{jk} \rangle \langle \varphi_{jk}, g \rangle \right\|_2,
\]

where \( g = g(\Omega) \in L_2(\mathbb{R}^d) \), \( \|g\|_2 \leq 1 \). Using the Cauchy inequality and Lemma 4, we obtain

\[
\left\| \sum_{k \in \Omega} \langle f, \varphi_{jk} \rangle \varphi_{jk} \right\|_2 \leq C_{2, \varphi} \|g\|_2 \left( \sum_{k \in \Omega} |\langle f, \varphi_{jk} \rangle|^2 \right)^{\frac{1}{2}} \leq C_{2, \varphi} \left( \sum_{k \in \Omega} |\langle f, \varphi_{jk} \rangle|^2 \right)^{\frac{1}{2}}.
\]

This proves (a) because, due to Lemma 1, the series \( \sum_{k \in \mathbb{Z}^d} |\langle f, \varphi_{jk} \rangle|^2 \) is convergent.

Assume now that \( \hat{\varphi}(0) = \hat{\tilde{\varphi}}(0) = 1 \). To prove (b), first of all we observe that \( \sum_{k \in \mathbb{Z}^d} |\hat{\varphi}(\xi + l)|^2 \) is a trigonometric polynomial, and hence a continuous and bounded function on \( \mathbb{R}^d \). Indeed, it is easy to check that \( \hat{\varphi}(\xi + l) \), \( \xi \in \mathbb{R}^d \), \( l \in \mathbb{Z}^d \), is the \( l \)-th Fourier coefficient of the function

\[
\Phi_\xi(x) := \sum_{k \in \mathbb{Z}^d} \varphi(x + k)e^{-2\pi i \langle x, k, \xi \rangle}.
\]

It follows that

\[
\sum_{k \in \mathbb{Z}^d} |\hat{\varphi}(\xi + l)|^2 = \int_{[-\frac{1}{2}, \frac{1}{2}]^d} |\Phi_\xi(x)|^2 dx = \int_{[-\frac{1}{2}, \frac{1}{2}]^d} \hat{\Phi}_\xi(x) \sum_{k \in \mathbb{Z}^d} \varphi(x + k)e^{-2\pi i \langle x, k, \xi \rangle} dx
\]

\[
= \int_{[-\frac{1}{2}, \frac{1}{2}]^d} \varphi(x)e^{-2\pi i \langle x, \xi \rangle} \sum_{k \in \mathbb{Z}^d} \varphi(x + k)e^{-2\pi i \langle x, k, \xi \rangle} dx = \sum_{k \in \mathbb{Z}^d} e^{2\pi i \langle k, \xi \rangle} \int_{[-\frac{1}{2}, \frac{1}{2}]^d} \varphi(x)\varphi(x + k) dx.
\]

Note that all sums in these equalities are finite because \( \varphi \) is compactly supported, which approves changes of integration and summation and yields the statement.

Using notations of Lemma 1, set \( G_j(\xi) = G_j(\varphi, f, \xi) \). By the Plancherel theorem and Lemma 1,

\[
\left\| f - \sum_{k \in \mathbb{Z}^d} \langle f, \varphi_{jk} \rangle \varphi_{jk} \right\|_2 = \left( \int_{\mathbb{R}^d} |\hat{f}(\xi) - \sum_{k \in \mathbb{Z}^d} G_j(k)e^{2\pi i \langle k, M^{-j} \xi \rangle}\hat{\varphi}(M^{-j} \xi) |^2 d\xi \right)^{\frac{1}{2}}.
\]
3. Wavelet approximation

Let \( j \geq 0 \). Since \( f \in L_1(\mathbb{R}^d) \cap L_2(\mathbb{R}^d) \), taking into account the boundedness of \( \hat{\psi} \) and (11), we have

\[
\int_{\mathbb{R}^d} \left( \left| \hat{f}(M^{j}\xi) \right|^2 - \hat{f}(M^{j}\xi) \hat{G}_j(\xi) \hat{\psi}(\xi) - \hat{f}(M^{j}\xi) G_j(\xi) \hat{\psi}(\xi) + \left| G_j(\xi) \hat{\psi}(\xi) \right|^2 \right) d\xi \to 0.
\]

Combining this with (18), using (12) and the boundedness of \( \hat{f} \), \( \hat{\psi} \), \( \sum_{l \in \mathbb{Z}^d} |\hat{\psi}(\cdot + l)|^2 \) on \([-\frac{1}{2}, \frac{1}{2}]^d \), we obtain

\[
\left\| f - \sum_{k \in \mathbb{Z}^d} \langle f, \hat{\psi}_j k \rangle \hat{\psi}_j k \right\|^2_{L^2} = \int_{[-\frac{1}{2}, \frac{1}{2}]^d} \left| \hat{f}(M^{j}\xi) \right|^2 - 2 \text{Re} \hat{f}(M^{j}\xi) \hat{G}_j(\xi) \hat{\psi}(\xi) + |G_j(\xi)|^2 \sum_{l \in \mathbb{Z}^d} |\hat{\psi}(\xi + l)|^2 \right| d\xi + o(1) \tag{19}
\]

After the change of variable the latter integral is reduced to

\[
\int_{\mathbb{R}^d} \chi_{M^{j}\mathbb{I}_{[-\frac{1}{2}, \frac{1}{2}]^d}} |\hat{f}(\xi)|^2 \left( 2 \text{Re} \hat{\psi}(M^{j}\xi) \hat{\psi}(M^{j}\xi) + |\hat{\psi}(M^{j}\xi)|^2 \sum_{l \in \mathbb{Z}^d} |\hat{\psi}(\xi + l)|^2 \right) d\xi.
\]

The integrand tends to \( |\hat{f}(\xi)|^2 \sum_{l \in \mathbb{Z}^d, l \neq 0} |\hat{\psi}(l)|^2 \) as \( j \to +\infty \) for each \( \xi \in \mathbb{R}^d \). Repeating the arguments of the proof of (15) and taking into account the boundedness of \( \sum_{l \in \mathbb{Z}^d} |\hat{\psi}(\cdot + l)|^2 \) on \( \mathbb{R}^d \), it is not difficult to see that the integrand has a summable majorant. Thus, by Lebesgue’s dominated convergence theorem,

\[
m^j \int_{[-\frac{1}{2}, \frac{1}{2}]^d} \int_{\mathbb{R}^d} \chi_{M^{j}\mathbb{I}_{[-\frac{1}{2}, \frac{1}{2}]^d}} |\hat{f}(\xi)|^2 \left( 1 - 2 \text{Re} \hat{\psi}(\xi) \hat{\psi}(\xi) + |\hat{\psi}(\xi)|^2 \sum_{l \in \mathbb{Z}^d} |\hat{\psi}(\xi + l)|^2 \right) d\xi \to 0 \quad \text{as } j \to +\infty, \notag
\]

Combining this with (19) completes the proof of (b). \( \square \)

It is well known that the shift-invariant space generated by a function \( \varphi \) has approximation order \( n \) if and only if the Strang–Fix condition of order \( n \) is satisfied for \( \varphi \). This fact appeared in the literature in different forms many times (see [2–4, 29]). Analyzing the proofs one can see that, in fact, Theorem 10 is not a new result. However we could not find an appropriate formulation implying either necessity or sufficiency of (b), so we decided to give a direct proof of this theorem.

3. Wavelet approximation

Above we studied the scaling operators \( Q_j : f \to \sum_{k \in \mathbb{Z}^d} \langle f, \hat{\psi}_j k \rangle \psi_j k \) with arbitrary functions or distributions \( \varphi, \hat{\psi} \). If \( \varphi, \hat{\psi} \) are refinable, then there exists a pair of MRA-based dual wavelet systems generated by associated wavelet functions/distributions \( \psi^{(v)}, \hat{\psi}^{(v)}, v = 1, \ldots, r \). We are interested if this wavelet system is frame-like, i.e. if (9) holds for every function \( f \) (from an appropriate class) or almost frame-like, i.e. (10) holds for every \( f \). Because of next lemma, (10) holds if and only if \( \lim_{j \to +\infty} Q_j(f) = f \), and (9) holds if and only if \( \lim_{j \to -\infty} Q_j(f) = 0 \), where the convergence is in the same sense as the convergence of the series in (9) and (10).

Lemma 11. Let \( \varphi, \hat{\psi} \in \mathcal{S}' \) be compactly supported refinable distributions, \( \psi^{(v)}, \hat{\psi}^{(v)}, v = 1, \ldots, r \), be associated wavelet functions, \( f \in \mathcal{S}, j, j' \in \mathbb{Z}, j' > j \). Then

\[
\sum_{k \in \mathbb{Z}^d} \langle f, \hat{\psi}_j k \rangle \psi_j k - \sum_{k \in \mathbb{Z}^d} \langle f, \hat{\psi}_{j'} k \rangle \psi_{j'} k = \sum_{l = j}^{j'-1} \sum_{v = 1}^{r} \sum_{k \in \mathbb{Z}^d} \langle f, \hat{\psi}_l k \rangle \psi_l k, \notag \tag{20}
\]

If, moreover, \( \hat{\psi} \in L_q(\mathbb{R}^d), 1 \leq q < \infty \), then (20) holds for any \( f \in L_p(\mathbb{R}^d), \frac{1}{p} + \frac{1}{q} = 1 \).
Proof. Evidently, it suffices to prove (20) for the case \( j' = j + 1 \), i.e. to check that

\[
\sum_{k \in \mathbb{Z}^d} (f \cdot \tilde{\varphi}_{j+1,k}) \tilde{\varphi}_{j+1,k} = \sum_{k \in \mathbb{Z}^d} (f \cdot \tilde{\varphi}_{j,k}) \tilde{\varphi}_{j,k} + \sum_{v=1}^{r} \sum_{k \in \mathbb{Z}^d} (f \cdot \tilde{\psi}_{j,k}^{(v)}) \tilde{\psi}_{j,k}^{(v)}. \tag{21}
\]

It follows from the refinable equations for \( \varphi, \tilde{\varphi} \) and (7) that

\[
\begin{align*}
\varphi_{j,k} &= \sum_{l \in \mathbb{Z}^d} h_{l-Mk}^{(0)} \varphi_{j+1,l}, & \psi_{j,k}^{(v)} &= \sum_{l \in \mathbb{Z}^d} h_{l-Mk}^{(v)} \varphi_{j+1,l}, \\
\tilde{\varphi}_{j,k} &= \sum_{l \in \mathbb{Z}^d} \tilde{h}_{l-Mk}^{(0)} \tilde{\varphi}_{j+1,l}, & \tilde{\psi}_{j,k}^{(v)} &= \sum_{l \in \mathbb{Z}^d} \tilde{h}_{l-Mk}^{(v)} \tilde{\varphi}_{j+1,l},
\end{align*}
\tag{22}
\]

where \( h_{l}^{(v)} \) and \( \tilde{h}_{l}^{(v)} \) are the coefficients of the masks

\[ m_{l}(\xi) = \frac{1}{\sqrt{M}} \sum_{n \in \mathbb{Z}^d} h_{n}^{(v)} e^{2\pi i (n, \xi)}, \quad \tilde{m}_{l}(\xi) = \frac{1}{\sqrt{M}} \sum_{n \in \mathbb{Z}^d} \tilde{h}_{n}^{(v)} e^{2\pi i (n, \xi)}, \quad v = 0, \ldots, r. \]

Substituting (22) into the right-hand side of (21), we obtain

\[
\sum_{k \in \mathbb{Z}^d} (f \cdot \tilde{\varphi}_{j,k}) \sum_{l \in \mathbb{Z}^d} h_{l-Mk}^{(0)} \tilde{\varphi}_{j+1,l} + \sum_{v=1}^{r} \sum_{k \in \mathbb{Z}^d} (f \cdot \tilde{\psi}_{j,k}^{(v)}) \sum_{l \in \mathbb{Z}^d} h_{l-Mk}^{(v)} \varphi_{j+1,l}.
\]

Denote by \( A_{l} \) the coefficient at \( \varphi_{j+1,l} \). Using (23), we have

\[
A_{l} = \sum_{k \in \mathbb{Z}^d} h_{l-Mk}^{(0)} (f \cdot \tilde{\varphi}_{j,k}) + \sum_{v=1}^{r} \sum_{k \in \mathbb{Z}^d} h_{l-Mk}^{(v)} (f \cdot \tilde{\psi}_{j,k}^{(v)}) = \sum_{v=0}^{r} \sum_{k \in \mathbb{Z}^d} h_{l-Mk}^{(v)} (f \cdot \tilde{\varphi}_{j+1,k}).
\]

It follows from (6) that

\[
\sum_{v=0}^{r} \sum_{k \in \mathbb{Z}^d} h_{l-Mk}^{(v)} \tilde{h}_{l-Mk}^{(v)} = \delta_{l,0}.
\]

Hence, we obtain \( A_{l} = (f \cdot \tilde{\varphi}_{j+1,l}) \) and this yields (21). \( \square \)

Lemma 11 together with Theorem 2 imply immediately the following statement.

**Theorem 12.** Let \( f \in S, \varphi, \tilde{\varphi} \in S', \varphi, \tilde{\varphi} \) be compactly supported and refinable, \( \tilde{\varphi}(0) = \hat{\varphi}(0) = 1, \varphi^{(v)}, \tilde{\varphi}^{(v)}, v = 1, \ldots, r, \) be associated wavelet functions. Then (10) holds with the series converging in \( S' \).

In a recent papers [23,24] B. Han introduced a notion of frequency-based dual framelets in the distribution space (which are not necessary dual frames even if all generating wavelet functions are in \( L_{2}(\mathbb{R}^{d}) \)). Using his terminology, this theorem may be reformulated as follows: if \( \varphi, \tilde{\varphi}, \psi^{(v)}, \tilde{\psi}^{(v)}, v = 1, \ldots, r, \) are as in Theorem 12, then these functions generate nonhomogeneous frequency-based dual framelets in the distribution space.

Due to Lemma 11, under the assumptions of Theorem 12, expansion (10) can be replaced by

\[
f = \sum_{k \in \mathbb{Z}^d} (f \cdot \tilde{\varphi}_{j,k}) \varphi_{j,k} + \sum_{i=1}^{\infty} \sum_{v=1}^{r} \sum_{k \in \mathbb{Z}^d} (f \cdot \tilde{\psi}_{j,k}^{(v)}) \psi_{j,k}^{(v)} \quad \forall j \in \mathbb{Z}, \tag{24}
\]

but it cannot be replaced by (9) because, generally speaking, \( \sum_{k \in \mathbb{Z}^d} (f \cdot \tilde{\varphi}_{j,k}) \langle \varphi_{j,k}, g \rangle, f, g \in S, \) does not tend to zero as \( j \to -\infty \). The following example illustrates this. Let \( d = 1, M = 2, \varphi = \varphi_{j} \) be the \( \delta \)-function. It is not difficult to see that the \( \delta \)-function is a compactly supported refinable distribution. If \( f \in S, f \geq 0, f(0) \neq 0, g = f \), then

\[
\sum_{k \in \mathbb{Z}} (f \cdot \tilde{\varphi}_{j,k}) \langle \varphi_{j,k}, g \rangle = \sum_{k \in \mathbb{Z}} \left| \langle \varphi_{j,k}, g \rangle \right|^{2} \geq \left| \langle f \cdot \tilde{\varphi}_{j,0}, g \rangle \right|^{2} = 2^{-j} |f(0)|^{2} \xrightarrow{j \to -\infty} \infty.
\]

**Lemma 13.** Let \( \varphi \in L_{2}(\mathbb{R}^{d}) \) be compactly supported refinable function, \( \tilde{\varphi}(0) = 1, \) Then \( \varphi \) satisfies the Strang–Fix condition of order 1.

This simple statement is well known (see, e.g., [25, p. 1178]).
Lemma 14. Let \( \varphi, \tilde{\varphi} \) be compactly supported refinable distributions, \( \mu_{0k}, \tilde{\mu}_{0k}, k = 0, \ldots, m - 1 \), be polyphase components of their masks \( m_0, \tilde{m}_0 \) respectively, \( n \in \mathbb{N} \). If

\[
D^\beta \mu_{0k}(0) = \frac{1}{\sqrt{M}} \sum_{0 \leq r \leq M} \lambda_r \left( \begin{array}{c} \beta \\ y \end{array} \right) \left( -2\pi i M^{-1}s \right) y \forall \beta \in \mathbb{Z}^d_+, [\beta] < n, \ k = 0, \ldots, m - 1;
\]

for some complex numbers \( \lambda_y, y \in \mathbb{Z}^d_+, [y] < n, \lambda_0 = 1 \), and

\[
D^\beta \left( 1 - \sum_{k=0}^{m-1} \mu_{0k}(\xi) \bar{\mu}_{0k}(\xi) \right) |_{\xi = 0} = 0 \quad \forall \beta \in \mathbb{Z}^d_+, [\beta] < n.
\]

Then there exist associated wavelet functions \( \psi^{(v)}, \tilde{\psi}^{(v)}, v = 1, \ldots, r \), such that the wavelet system \( \{\tilde{\psi}^{(v)}_{jk}\} \) has \( VM^{n-1} \) property.

Proof. Mixed Extension Principle may be easily realized as follows. Set \( \mu_{0,m} = 1 - \sum_{k=0}^{m-1} \mu_{0k} \tilde{\mu}_{0k}, \tilde{\mu}_{0,m} = 1 \). Using a method described in [33, §2.6] (see also [28,35]) we can extend the rows \( \mu_{00}, \ldots, \mu_{0,m} \), \( \tilde{\mu}_{00}, \ldots, \tilde{\mu}_{0,m} \) to \( (m + 1) \times (m + 1) \) matrices \( N, \tilde{N} \) whose rows form a biorthonormal system. We obtain

\[
N := \begin{pmatrix} \mu_{00} & \mu_{01} & \cdots & \mu_{0,m-1} & \mu_{0,m} \\ 0 & 0 & \cdots & 1 & \mu_{0,m-1} \\ 0 & 0 & \cdots & 0 & \mu_{0,m-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & \mu_{01} \\ 1 & 0 & \cdots & 0 & \mu_{00} \end{pmatrix},
\]

\[
\tilde{N} := \begin{pmatrix} -\mu_{00} & -\mu_{01} & \cdots & -\mu_{0,m-1} & 1 \\ -\mu_{00} \tilde{\mu}_{0,m-1} & -\mu_{01} \tilde{\mu}_{0,m-1} & \cdots & -\mu_{0,m-1} \tilde{\mu}_{0,m-1} & -\mu_{0,m} \\ -\mu_{00} \tilde{\mu}_{0,m-2} & -\mu_{01} \tilde{\mu}_{0,m-2} & \cdots & -\mu_{0,m-1} \tilde{\mu}_{0,m-2} & -\mu_{0,m-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\mu_{00} \tilde{\mu}_{01} & 1 - \tilde{\mu}_{01} \tilde{\mu}_{01} & \cdots & -\mu_{0,m-1} \tilde{\mu}_{01} & -\mu_{01} \\ 1 - \mu_{00} \tilde{\mu}_{00} & -\mu_{01} \tilde{\mu}_{00} & \cdots & -\mu_{0,m-1} \tilde{\mu}_{00} & -\mu_{00} \end{pmatrix}.
\]

If now \( m_v, \tilde{m}_v, v = 1, \ldots, m \), are defined by (5) and \( \psi^{(v)}, \tilde{\psi}^{(v)} \) are defined by (7), then, due to Theorem A (with changing roles of \( m_0 \) and \( \tilde{m}_0 \)), the wavelet system \( \{\tilde{\psi}^{(v)}_{jk}\} \) has \( VM^{n-1} \) property. □

Remark 15. Due to Theorem B and the identity

\[
\sum_{k=0}^{m-1} \mu_{0k}(M^s \xi) \bar{\mu}_{0k}(M^s \xi) = \sum_{s \in D(M^s)} m_0(\xi + M^{s-1}s) \bar{m}_0(\xi + M^{s-1}s),
\]

conditions (25), (26) in Lemma 14 may be replaced by

\[
(i) \quad D^\beta m_0(M^{s-1} \xi) |_{\xi = 0} = 0, \quad \forall s \in D(M^s) \setminus \{0\}, \forall \beta \in \mathbb{Z}^d_+, [\beta] < n;
\]

\[
(ii) \quad D^\beta (1 - m_0(\xi) \bar{m}_0(\xi)) |_{\xi = 0} = 0 \quad \forall \beta \in \mathbb{Z}^d_+, [\beta] < n.
\]

Theorem 16. Let \( f \in S, \varphi, \tilde{\varphi} \in L_2(\mathbb{R}^d), \tilde{\varphi} \in S^*, \varphi, \tilde{\varphi} \) be compactly supported and refinable, \( \tilde{\varphi}(0) = \tilde{\varphi}(0) = 1 \), and let \( \psi^{(v)}, \tilde{\psi}^{(v)}, v = 1, \ldots, r \), be associated wavelet functions. Then (10) holds with the series converging in \( L_2 \)-norm. If, moreover, \( \varphi, \tilde{\varphi} \) are as in Lemma 14, then

\[
\left\| f - \sum_{k \in \mathbb{Z}^d} (f, \varphi_{0k}) \varphi_{0k} - \sum_{i=0}^{j-1} \sum_{v=1}^{r} \sum_{k \in \mathbb{Z}^d} (f, \tilde{\psi}^{(v)}_{ik}) \tilde{\psi}^{(v)}_{ik} \right\|_2 \leq \frac{C \|f\|_{W^{n^*}}} {(|\lambda| - \varepsilon) M^r},
\]

where \( \lambda \) is a minimal (in modulus) eigenvalue of \( M, \varepsilon > 0, |\lambda| - \varepsilon > 1, n^* \geq n, C \) and \( n^* \) do not depend on \( f \) and \( j \).

Proof. First of all we note that, due to Theorem 10, the series \( \sum_{k \in \mathbb{Z}^d} (f, \varphi_{0k}) \varphi_{0k} \) and \( \sum_{k \in \mathbb{Z}^d} (f, \tilde{\psi}^{(v)}_{ik}) \tilde{\psi}^{(v)}_{ik}, i \in \mathbb{Z}, v = 1, \ldots, r \), converge unconditionally in \( L_2 \)-norm. By Lemma 11,
Combining (33), (34), (35) with (32), we have

\[ \left\| f - \sum_{k \in \mathbb{Z}^d} (f \ast \tilde{\phi}_k) \psi_{ok} \right\|_2 \leq \sum_{i=0}^{j-1} \sum_{k \in \mathbb{Z}^d} \left\| f \ast \tilde{\phi}_k \right\|_2 \psi_{ok} + \sum_{i=0}^{j-1} \sum_{k \in \mathbb{Z}^d} \left\| f \ast \tilde{\phi}_k \right\|_2 \psi_{ok} \]

This relation, together with Theorem 10 and Lemma 13, yields (10).

Now we assume that \( \varphi, \tilde{\varphi} \) satisfy all assumptions of Lemma 14. Since the right-hand side of (30) does not depend of the choice of associated wavelet functions, it suffices to check (29) for at least one selection of \( \psi^{(v)}, \tilde{\psi}^{(v)}, v = 1, \ldots, r \). Due to Lemma 14, without loss of generality, we can consider that \( \{\tilde{\psi}^{(v)}_j\} \) has \( \mathcal{V} M^{n-1} \) property. It follows from (10) that

\[ \left\| f - \sum_{k \in \mathbb{Z}^d} (f \ast \tilde{\psi}^{(v)}_k) \psi_{ok} \right\|_2 \leq \sum_{i=0}^{j-1} \sum_{k \in \mathbb{Z}^d} \left\| f \ast \tilde{\psi}^{(v)}_k \right\|_2 \psi_{ok} + \sum_{i=0}^{j-1} \sum_{k \in \mathbb{Z}^d} \left\| f \ast \tilde{\psi}^{(v)}_k \right\|_2 \psi_{ok} \]

Due to the Paley–Wiener theorem for tempered distributions, there exist \( N \in \mathbb{N} \) and \( C_\varphi > 0 \) so that \( |\hat{\varphi}(\xi)| \leq C_\varphi |\xi|^N \) for all \( \xi \notin [-\frac{1}{2}, \frac{1}{2}]^d \). Let \( n^* \geq n \) and \( n^* = N + d/2 \). Using notations of Lemma 1, set \( G_1(\xi) = G_1(\psi^{(v)}, f, \xi) \). Similarly to (18), using the Plancherel theorem and Lemma 1, we have

\[ \left\| \sum_{j \in \mathbb{Z}^d} \hat{\psi}^{(v)}(\xi) \right\|_2 = \left( \int_{\mathbb{R}^d} |G_1(\xi) \hat{\psi}^{(v)}(\xi)|^2 d\xi \right)^{\frac{1}{2}} \leq \left( \int_{\mathbb{R}^d} |G_1(\xi)|^2 \left\| \hat{\psi}^{(v)}(\xi + l) \right\|^2 d\xi \right)^{\frac{1}{2}} \]

Since \( \psi^{(v)} \) is in \( L_2(\mathbb{R}^d) \) and compactly supported,

\[ \sum_{j \in \mathbb{Z}^d} |\hat{\psi}^{(v)}(\xi + l)|^2 \leq C_1. \]

Since

\[ |\hat{\psi}^{(v)}(\xi)| \leq C_2 |\xi|^n, \quad \xi \in \left[ -\frac{1}{2}, \frac{1}{2} \right]^d, \quad |\hat{\psi}^{(v)}(\xi)| \leq C_3 |\xi|^N, \quad \xi \notin \left[ -\frac{1}{2}, \frac{1}{2} \right]^d, \]

we obtain

\[ m^l \int_{\left[ -\frac{1}{2}, \frac{1}{2} \right]^d} \left| \hat{\xi} \left( M^*(\xi) \hat{\psi}^{(v)}(\xi) \right) \right|^2 d\xi \leq C_2 m^l \int_{\left[ -\frac{1}{2}, \frac{1}{2} \right]^d} |\xi|^{2n} |\hat{\xi} (M^*(\xi))|^2 d\xi \]

\[ = C_2 \int_{M^*(\left[ -\frac{1}{2}, \frac{1}{2} \right]^d)} |\xi|^{2n} |\hat{\xi} (\xi)|^2 d\xi \leq C_2 \| \psi^{(v)} \|_{L^2} M^{-n} \int_{\mathbb{R}^d} |\xi|^{2n} |\hat{\xi} (\xi)|^2 d\xi \]

\[ \leq C_2 \| M^{-n} \| \int_{B_1} |\hat{\xi} (\xi)|^2 d\xi + \int_{\mathbb{R}^d \setminus B_1} |\xi|^{2n} |\hat{\xi} (\xi)|^2 d\xi \]

\[ \leq C_2 \| f \|_{L^2}^2 \| M^{-n} \| \int_{\mathbb{R}^d} |\hat{\xi} (\xi)|^2 d\xi \]

Combining (33), (34), (35) with (32), we have

\[ \left\| \sum_{k \in \mathbb{Z}^d} (f \ast \tilde{\psi}^{(v)}_k) \psi_{ok} \right\|_2 \leq C_6 \| f \|_{L^2}^2 \| M^{-n} \| \]
which together with (1) yields that
\[
\left\| \sum_{i=1}^{\infty} \sum_{j=1}^{r} \sum_{k \in \mathbb{Z}^d} (f, \tilde{\psi}_{ik}^{(v)}) \psi_{ik}^{(v)} \right\|_{L^2} \leq C \left\| f \right\|_{W_p^{n^*}} \frac{1}{(|\lambda| - \varepsilon)^{J^*}}. \tag{36}
\]

It remains to combine (36) with (31). □

**Remark 17.** A minor modification of the proof of Theorem 16 gives the following statement: let \( \varphi, \tilde{\varphi}, \psi^{(v)}, \tilde{\psi}^{(v)}, v = 1, \ldots, r \), generate a pair of frequency-based nonhomogeneous dual wavelet frames in the distribution space (see [23, 24]), \( \varphi, \psi^{(v)} \in L_2(\mathbb{R}^d) \) and \( \{\tilde{\psi}_{ik}^{(v)}\} \) has \( V M^{n-1} \) property, then (29) holds. Similarly, if \( \psi^{(v)}, \tilde{\psi}^{(v)}, v = 1, \ldots, r \), generate a pair of frequency-based homogeneous dual wavelet frames in the distribution space, \( \psi^{(v)} \in L_2(\mathbb{R}^d) \), the series \( \sum_{i=-\infty}^{\infty} \sum_{j=1}^{r} \sum_{k \in \mathbb{Z}^d} (f, \tilde{\psi}_{ik}^{(v)}) \psi_{ik}^{(v)} \) converges in \( L_2(\mathbb{R}^d) \) and \( \{\tilde{\psi}_{ik}^{(v)}\} \) has \( V M^{n-1} \) property, then
\[
\left\| \sum_{i=-\infty}^{\infty} \sum_{j=1}^{r} \sum_{k \in \mathbb{Z}^d} (f, \tilde{\psi}_{ik}^{(v)}) \psi_{ik}^{(v)} \right\|_{L^2} \leq C \left\| f \right\|_{W_p^{n^*}} \frac{1}{(|\lambda| - \varepsilon)^{J^*}},
\]
where all parameters are as in (29). In particular, this holds true for any pair of dual wavelet frames \( \{\psi_{ik}^{(v)}\}, \{\tilde{\psi}_{ik}^{(v)}\} \) in \( L_2(\mathbb{R}^d) \) (not necessary MRA-based) whenever the functions \( \psi^{(v)}, \tilde{\psi}^{(v)} \) are compactly supported and \( \{\tilde{\psi}_{ik}^{(v)}\} \) has \( V M^{n-1} \) property. It is possible to show that in this case \( n^* \) may be replaced by \( n \).

Due to Lemma 11, under the assumptions of Theorem 16, (10) can be replaced by (24), but it cannot be replaced by (9) because, generally speaking, \( \|Q_j(\varphi, \tilde{\varphi}, f)\|_2 \) does not tend to zero as \( j \to -\infty \). The following example illustrates this. Let \( d = 1, M = 2, \tilde{\varphi} \) be the \( \delta \)-function, \( \varphi = \chi_{[0,1]} \). It is not difficult to see that \( \tilde{\varphi} \) and \( \varphi \) are refinable. If \( f \in L^2 \), \( f \geq 0 \), \( f(0) \neq 0 \), then
\[
\left\| \sum_{k \in \mathbb{Z}} (f, \tilde{\varphi}_{jk}) \tilde{\varphi}_{jk} \right\|_{L^2}^2 = \int \left| \sum_{k \in \mathbb{Z}} f(-2^{-j}k) \varphi(2^j k + x) \right|^2 dx \\
\geq \int \left| f(0) \varphi(2^j x) \right|^2 dx = 2^{-j} |f(0)|^2 \to \infty, j \to -\infty.
\]

Now we compare Theorem 16 with some other results. Let \( \varphi, \tilde{\varphi}, \psi^{(v)}, \tilde{\psi}^{(v)}, v = 1, \ldots, r \), be as in the first part of Theorem 16. Due to [19, Theorem 2.2] by B. Han, there exists \( s > 0 \) such that \( \varphi \in W_s^1(\mathbb{R}^d) \). On the other hand, B. Han and Z. Shen proved in [18] that if \( \tilde{\varphi} \in W_2^2(\mathbb{R}^d) \) and \( \{\psi^{(v)}\} \) has \( V M^0 \) property, then the systems \( \{\psi_{ik}^{(v)}\}, \{\tilde{\psi}_{ik}^{(v)}\} \) are dual frames in \( W_2^2(\mathbb{R}^d) \), \( W_2^{-2}(\mathbb{R}^d) \) (it was proved only for the dyadic case). In particular, this yields (10) for any \( f \in W_2^2(\mathbb{R}^d) \) with the series converging in \( W_2^2 \)-norm unconditionally.

Note that the proof of Theorem 16 is very close to the proof of the main result of K. Jetter and D.X. Zhou [27]. They consider operators \( Q_j \) as in Theorem 16 (for \( d = 1 \)) and estimate the deviation of \( f \in S \) from \( Q_0 \) in \( L_2 \)-norm via \( W_2^0 \)-norm of \( f \). Such estimation could be useful for our goals, however there is an additional assumption on \( \tilde{\varphi}, \varphi \) in [27]. They assume some restriction on the order of growth of the function \( \tilde{\varphi}, \varphi \) at infinity. And this is just what is not necessary fulfilled under our assumptions. Theorem 16 states that the approximation order of a MRA-based wavelet system is guaranteed by appropriate masks.

**Theorem 18.** Let \( 1 < p \leq \infty \), \( \frac{1}{p} + \frac{1}{q} = 1 \), \( \psi \in L_p(\mathbb{R}^d) \), \( \tilde{\psi} \in L_q(\mathbb{R}^d) \), \( \varphi, \tilde{\varphi} \) be compactly supported refinable functions satisfying all assumptions of Lemma 14, and let \( \psi^{(v)}, \tilde{\psi}^{(v)}, v = 1, \ldots, r \), be associated wavelet functions. If \( 1 < p < \infty \), \( f \in W_p^n(\mathbb{R}^d) \) or \( p = \infty \), \( f \in W_\infty^n(\mathbb{R}^d) \cap L_p^\infty(\mathbb{R}^d) \), then
\[
\left\| f - \sum_{i=-\infty}^{\infty} \sum_{j=1}^{r} \sum_{k \in \mathbb{Z}^d} (f, \tilde{\psi}_{ik}^{(v)}) \psi_{ik}^{(v)} \right\|_p \leq C \left\| f \right\|_{W_p^n} \frac{1}{(|\lambda| - \varepsilon)^{J^*}}, \tag{37}
\]
where \( \lambda \) is a minimal (in modulus) eigenvalue of \( M, \varepsilon > 0, |\lambda| - \varepsilon > 1, C \) does not depend on \( f \) and \( j \).

**Proof.** First of all we note that, due to Theorem 7(a) and Theorem 9(a) the series \( \sum_{k \in \mathbb{Z}^d} (f, \tilde{\psi}_{ik}^{(v)}) \psi_{ik}^{(v)}, i \in \mathbb{Z}, v = 1, \ldots, r \), converge unconditionally in \( L_p \)-norm. By Theorem 7(c), Theorem 9(c) and Lemma 11,
\[
\sum_{i=-\infty}^{\infty} \sum_{j=1}^{r} \sum_{k \in \mathbb{Z}^d} (f, \tilde{\psi}_{ik}^{(v)}) \psi_{ik}^{(v)} = \sum_{k \in \mathbb{Z}^d} (f, \tilde{\varphi}_k) \varphi_k. \tag{38}
\]
So, taking into account (1), we see that it suffices to check that
\[
\left\| f - \sum_{k \in \mathbb{Z}^d} (f, \tilde{\varphi}_{jk}) \varphi_{jk} \right\|_p \leq C_1 \left\| M^{-j} \right\|^n \| f \|_{W^p},
\]
where $C_1$ does not depend on $f$ and $j$.

Since under the assumptions of Lemma 14 conditions (i) and (ii) of Remark 15 are fulfilled, we have some additional properties of $\varphi, \tilde{\varphi}$. It is well known that (i) implies the Strang–Fix condition of order $n$ for $\varphi$. It follows from (ii) that
\[
D^\beta (1 - \tilde{\varphi}(\xi) \overline{\tilde{\varphi}(\xi)}) |_{\xi = 0} = 0 \quad \forall \beta \in \mathbb{Z}^d_+ \quad \beta < n.
\]
Indeed, to check this we observe that the sequence of the differentiated several times partial products of
\[
\prod_{j=1}^{\infty} m_0(M^{s-j} \xi)\tilde{m}_0(M^{s-j} \xi)
\]
converge uniformly on a compact. Because of these properties of $\varphi, \tilde{\varphi}$, combining Theorem 3.2 of [31] with Lemma 3.2 of [30], we have
\[
\left\| g - \sum_{k \in \mathbb{Z}^d} (g, \tilde{\varphi}_{ok}) \varphi_{ok} \right\|_p \leq C_2 \omega_n(g, 1)_p, \quad \forall g \in W^p_p(\mathbb{R}^d),
\]
where $C_2$ does not depend on $g$. To prove (39) it remains to substitute $g(x) = f(M^{-j}x)$, change of variable $y = M^{-j}x$ in all integrals and take into account that
\[
m^{-j/p} \omega_n(g, 1)_p = \sup_{|t| \leq 1} \left\| \nabla^n_{M^{-j}t} f \right\|_p = \omega_n(f, \left\| M^{-j} \right\|)_p,
\]
\[
\omega_n(f, h)_p \leq C_3 h^n \| f \|_{W^p_p}. \quad \Box
\]

Relation (37) means that expansion (9) provides approximation order $n$. This fact was known for some special cases. I. Daubechies, B. Han, A. Ron and Z. Shen proved this in [9] under the assumptions that $\{\psi_{jk}^{(v)}\}, \{\tilde{\psi}_{jk}^{(v)}\}$ are dual frames in $L_2(\mathbb{R}^d)$ and $M = \lambda I$. For arbitrary dilation matrix $M$, one of the authors proved (37) with $p = 2$ for the tight frames in [40] assuming that $\{\psi_{jk}^{(v)}\}$ has $V M^{n-1}$ property instead of (25), (26), and for the dual frames in [39] assuming that $\{\tilde{\psi}_{jk}^{(v)}\}$ has $V M^{n-1}$ property and all wavelet functions are bounded.

Theorem 18 is not true for $p = 1$ because (38) does not hold. Using
\[
\sum_{k \in \mathbb{Z}^d} (f, \tilde{\varphi}_{ok}) \varphi_{ok} + \sum_{i=0}^{j-1} \sum_{k \in \mathbb{Z}^d} (f, \tilde{\psi}_{ik}^{(v)}) \psi_{ik}^{(v)} = \sum_{k \in \mathbb{Z}^d} (f, \tilde{\varphi}_{jk}) \varphi_{jk}
\]
instead of (38) and repeating all other arguments of the proof of Theorem 18, we obtain

**Theorem 19.** Let $f \in W^n_p(\mathbb{R}^d), \varphi \in L_1(\mathbb{R}^d), \tilde{\varphi} \in L_\infty(\mathbb{R}^d)$ be compactly supported refinable functions satisfying all assumptions of Lemma 14, and let $\psi^{(v)}, \tilde{\psi}^{(v)}, v = 1, \ldots, r$, be associated wavelet functions. Then
\[
\left\| f - \sum_{k \in \mathbb{Z}^d} (f, \tilde{\varphi}_{ok}) \varphi_{ok} - \sum_{i=0}^{j-1} \sum_{k \in \mathbb{Z}^d} (f, \tilde{\psi}_{ik}^{(v)}) \psi_{ik}^{(v)} \right\|_1 \leq C \frac{\| f \|_{W^n_p}}{(|\lambda| - \varepsilon)^{jn}},
\]
where $\lambda$ is a minimal (in modulus) eigenvalue of $M$, $\varepsilon > 0$, $|\lambda| - \varepsilon > 1$, $C$ does not depend on $f$ and $j$.

4. Examples

In this section we shall give several examples of frame-like and almost frame-like wavelet systems to illustrate the main results of the paper. All examples are based on the construction of wavelet functions $\psi^{(v)}, \tilde{\psi}^{(v)}$ associated with $\varphi, \tilde{\varphi}$ according to the scheme described in Section 1.2.

1. Let $M = \left( \begin{smallmatrix} 1 & 1 \\ -1 & -1 \end{smallmatrix} \right)$ be the quincunx dilation matrix. For this matrix $m = 2$, $D(M) = \{s_0 = (0, 0), s_1 = (1, 0)\}$ is a set of digits.

Let $\varphi$ be a box spline whose Fourier transform is defined by
\[
\hat{\varphi}(\xi) = \left( \frac{1 - e^{-2\pi i \xi_1}}{2\pi i \xi_1} \right)^2 \left( \frac{1 - e^{-2\pi i \xi_2}}{2\pi i \xi_2} \right)^2 \left( \frac{1 - e^{-2\pi i (\xi_1 + \xi_2)}}{2\pi i (\xi_1 + \xi_2)} \right)^2 \left( \frac{1 - e^{-2\pi i (\xi_1 - \xi_2)}}{2\pi i (\xi_1 - \xi_2)} \right)^2.
\]
This box spline is globally $C^2$ (see [38]) and satisfies the refinement equation with the mask
\[
m_0(\xi) = \left(1 + \frac{e^{-2\pi i \xi_1}}{2}\right) \left(1 + \frac{e^{-2\pi i \xi_2}}{2}\right)^2.
\]

The polyphase components
\[
\begin{align*}
\mu_{00} &= \frac{1}{\sqrt{2}} \left(\frac{1}{4} e^{2\pi i \xi_1} + \frac{1}{4} e^{-2\pi i \xi_1} + \frac{1}{2} e^{-2\pi i \xi_1}\right), \\
\mu_{01} &= \frac{1}{\sqrt{2}} \left(\frac{1}{4} e^{2\pi i \xi_1} + \frac{1}{4} e^{-2\pi i \xi_1} + \frac{e^{-4\pi i \xi_1}}{2}\right),
\end{align*}
\]
satisfy condition (25) in Lemma 14 with $n = 2$ and $\lambda_{00} = 1, \lambda_{10} = -\frac{3\pi i}{2}, \lambda_{01} = \frac{\pi i}{2}$. Since $e^{2\pi i (c, \xi)} m_0(\xi)$ with $c = (-\frac{1}{2}, -1)$ is real and even, the refinable function $\psi$ is real and symmetric with respect to the point $C = (M-I)^{-1} c = (-2, -\frac{1}{2})$.

Let $\tilde{\psi}$ be the $\delta$-function. Then $\tilde{m}_0 \equiv 1$ and the corresponding polyphase components are $\tilde{\mu}_{00} \equiv \sqrt{2}, \tilde{\mu}_{01} \equiv 0$. We construct associated wavelet functions $\psi^{(v)}(\xi), \tilde{\psi}^{(v)}(\xi), v = 1, 2$, using the method given in the proof of Lemma 14. The polyphase matrices $N, \tilde{N}$ defined by (27), (28) look as
\[
N = \begin{pmatrix}
\mu_{00} & \mu_{01} \\
0 & 1
\end{pmatrix}, \quad \tilde{N} = \begin{pmatrix}
\sqrt{2} & 0 & 1 \\
0 & -1 & -\mu_{00}
\end{pmatrix}.
\]

This leads to the wavelet masks $m_1(\xi) = \frac{1}{\sqrt{2}} e^{2\pi i \xi_1}, m_2(\xi) \equiv 1$ and the corresponding wavelet functions $\psi^{(1)}(\chi) = \sqrt{2} \psi(Mx + (1))$, $\psi^{(2)}(\chi) = \sqrt{2} \psi(Mx + (0))$ which are in $C^2$ and symmetric with respect to the points $(-\frac{7}{4}, -\frac{5}{4}), (-\frac{3}{4}, -\frac{3}{4})$ respectively.

The dual wavelet masks and the corresponding wavelet distributions are
\[
\begin{align*}
\tilde{m}_1(\xi) &= \frac{1}{\sqrt{2}} \left(\frac{1}{2} e^{2\pi i \xi_1} - \frac{1}{4} e^{4\pi i \xi_1} - \frac{1}{4} e^{2\pi i \xi_1} + e^{2\pi i \xi_1}\right), \\
\tilde{m}_2(\xi) &= \frac{1}{\sqrt{2}} \left(\frac{3}{4} - \frac{1}{4} e^{4\pi i \xi_2} - \frac{1}{2} e^{2\pi i \xi_1} + e^{2\pi i \xi_1}\right), \\
\tilde{\psi}^{(1)}(\chi) &= \sqrt{2} \left[-\frac{1}{2} \delta(Mx + (1)) - \frac{1}{4} \delta(Mx + (2)) - \frac{1}{4} \delta(Mx + (2)) + \delta(Mx + (1))\right], \\
\tilde{\psi}^{(2)}(\chi) &= \sqrt{2} \left[-\frac{1}{2} \delta(Mx + (0)) - \frac{1}{4} \delta(Mx + (0)) - \frac{1}{2} \delta(Mx + (1))\right].
\end{align*}
\]

By Theorem 16, for any $f \in S$, expansion (10) holds in $L_2$-norm and looks as follows
\[
f(x) = \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^2} \left[ \frac{1}{2} f(-M^{-j-1}(Mk + (1))) - \frac{1}{4} f(-M^{-j-1}(Mk + (2))) \right] \psi(M^{j+1}x + Mk + (1))
\]
\[
+ \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^2} \left[ \frac{3}{4} f(-M^{-j-1}k) + \frac{1}{4} (-M^{-j-1}(Mk + (0))) \right] \psi(M^{j+1}x + Mk + (0))
\]
\[
- \frac{1}{2} f(-M^{-j-1}(Mk + (1))) \psi(M^{j+1}x + Mk).
\]
Since all assumptions of Lemma 14 are fulfilled with $n = 1$, this expansion has approximation order 1.

Now, using the same refinable function $\psi$, we construct an almost frame-like wavelet system providing approximation order 2. Choosing
\[
\tilde{\mu}_{00} = \sqrt{2} - \frac{3i}{2\sqrt{2}} \sin 2\pi \xi_1 + \frac{i}{2\sqrt{2}} \sin 2\pi \xi_2, \quad \tilde{\mu}_{01} \equiv 0,
\]
we have (26) satisfied with $n = 2$. Again we construct $\psi^{(v)}(\chi), \tilde{\psi}^{(v)}(\chi), v = 1, 2$, using the method given in the proof of Lemma 14. The polyphase matrices $N, \tilde{N}$ defined by (27), (28) look as
The dual refinable mask is 
\[ N = \begin{pmatrix} \mu_{00} & \mu_{01} & \mu_{02} \\ 0 & 1 & 0 \\ 1 & 0 & -\mu_{00} \end{pmatrix}, \quad \tilde{N} = \begin{pmatrix} \tilde{\mu}_{00} & 0 & 1 \\ -\mu_{00}\tilde{\mu}_{01} & 1 & -\tilde{\mu}_{01} \\ 1 - \mu_{00}\tilde{\mu}_{00} & 0 & -\tilde{\mu}_{00} \end{pmatrix}, \]

where \( \mu_{02}(\xi) = 1 - \mu_{00}(\xi) \tilde{\mu}_{00}(\xi) \). This leads to the same wavelet masks \( m_1, m_2 \) and wavelet functions \( \psi^{(1)}, \psi^{(2)} \) as above. The dual refinable mask is
\[ \tilde{m}_0(\xi) = 1 - \frac{3}{8} e^{2\pi i (\xi_1 + \xi_2)} + \frac{3}{8} e^{-2\pi i (\xi_1 + \xi_2)} + \frac{1}{8} e^{2\pi i (\xi_1 - \xi_2)} - \frac{1}{8} e^{-2\pi i (\xi_1 - \xi_2)}. \]

Since \( \tilde{m}_0 \) is a trigonometric polynomial, \( \tilde{\psi} \) is a compactly supported tempered distribution. The dual wavelet masks \( \tilde{m}_1, \tilde{m}_2 \) are trigonometric polynomials whose Fourier coefficients are given by the following tables
\[
\frac{1}{32\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 2 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -18 & 0 & 2 \\ 0 & 0 & 0 & -6 & 32 & -10 & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},
\frac{1}{32\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & -6 & 0 & 6 \\ 0 & -2 & 0 & -14 & 0 \\ 0 & 0 & 18 & 0 & -2 \\ 0 & -3 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},
\]

the corresponding dual wavelet distributions \( \tilde{\psi}^{(1)}, \tilde{\psi}^{(2)} \) are defined by (7). By Theorem 16, for any \( f \in S \), expansion (10) holds in \( L_2 \)-norm and has approximation order 2.

2. Let \( M = \left( \frac{1}{2}, \frac{1}{2} \right) \). For this matrix, \( D(M) = \{ s_0 = (0, 0), s_1 = (0, -1), s_2 = (0, 1) \} \) is a set of digits, \( m = 3 \). Let the mask \( m_0 \) be defined by the table of its Fourier coefficients
\[
\frac{1}{2187} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 7 & 4 & 0 & 4 & 7 \\ 0 & 0 & 0 & 4 & 0 & -32 & -32 & 0 & 4 \\ 0 & 0 & 0 & -32 & -20 & 0 & -20 & -32 & 0 \\ 0 & 4 & -32 & 0 & 312 & 312 & 0 & -32 & 4 \\ 7 & 0 & -20 & 312 & 729 & 312 & -20 & 0 & 7 \\ 4 & -32 & 0 & 312 & 312 & 0 & -32 & 4 & 0 \\ 0 & -32 & -20 & 0 & -20 & -32 & 0 & 0 & 0 \\ 4 & 0 & -32 & -32 & 0 & 4 & 0 & 0 & 0 \\ 7 & 4 & 0 & 4 & 7 & 0 & 0 & 0 & 0 \end{pmatrix},
\]

This mask was found and investigated by B. Han [21]. The mask is interpolatory, i.e. \( \mu_{00} \equiv \frac{1}{\sqrt{3}} \), the corresponding refinable function \( \psi \) is in \( C^2 \), supported on \([-4, 4]^2 \), symmetric with respect to the origin and with respect to the axes. Condition (25) from Lemma 14 is satisfied with \( n = 6 \) and \( \lambda_0 = 1, \lambda_k = 0, k \in \mathbb{Z}_+^2, 0 < |k| < 6 \).

Let \( \tilde{\psi} \) be the \( \delta \)-function. Then the corresponding polyphase components are \( \tilde{\mu}_{00} \equiv \sqrt{3}, \tilde{\mu}_{01} = 0, \tilde{\mu}_{02} = 0 \). So, \( \sum_{k=0}^{m-1} \mu_{0k} \mu_{0k} = 1 \), and Mixed Extension Principle may be realized with the minimal number of wavelet functions in this case. For such refinable masks, a method for matrix extension leading to symmetric/antisymmetric wavelet functions was developed in [32]. The method gives the following polyphase matrices
\[ N = \begin{pmatrix} \frac{1}{\sqrt{3}} & \mu_{01} & \mu_{02} \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}, \quad \tilde{N} = \begin{pmatrix} \frac{\sqrt{3}}{2} (\mu_{02} + \mu_{01}) & 0 & 0 \\ \frac{\sqrt{3}}{2} (\mu_{02} - \mu_{01}) & \frac{1}{2} & \frac{1}{2} \\ \frac{\sqrt{3}}{2} (\mu_{02} - \mu_{01}) & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}. \]

The corresponding wavelet functions are
\[ \psi^{(1)} = \sqrt{3} \left[ \psi \left( Mx + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) + \psi \left( Mx + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \right], \]
\[ \psi^{(2)} = \sqrt{3} \left[ \psi \left( Mx + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right) - \psi \left( Mx + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \right]. \]

The dual wavelet masks \( \tilde{m}_1, \tilde{m}_2 \) are trigonometric polynomials whose Fourier coefficients are given by the following tables.
The corresponding dual wavelet distributions $\tilde{\psi}(1)$, $\tilde{\psi}(2)$ defined by (7) are finite linear combinations of the $M$-scales and integer shifts of the $\delta$-function.

By Theorem 16, for any $f \in S$, expansion (10) holds in $L_2$-norm and has approximation order 6; the wavelet functions $\psi(1)$, $\psi(2)$ are in $C^2$, $\tilde{\psi}(1)$, $\tilde{\psi}(1)$ are symmetric with respect to the origin, $\psi(2)$, $\tilde{\psi}(2)$ are antisymmetric with respect to the origin.

The following example is intended to illustrate Theorems 18 and 19. Let $d = 1$, $M = 2$, $\varphi$ be the B-spline of order 3 whose Fourier transform is given by

$$\hat{\varphi}(\xi) = \left(\frac{\sin \pi \xi}{\pi \xi}\right)^4.$$

The function $\varphi$ is refinable with the mask $m_0(\xi) = \cos^4 \pi \xi$; $\varphi$ is in $C^2$ and supported on $[-2, 2]$. The polyphase components of the mask

$$\mu_{00}(\xi) = \sqrt{2} \left(\frac{1}{16} e^{2\pi i \xi} + \frac{1}{16} e^{-2\pi i \xi} + \frac{3}{8}\right), \quad \mu_{01}(\xi) = \sqrt{2} \left(\frac{1}{4} e^{-2\pi i \xi} + \frac{1}{4}\right)$$

satisfy condition (25) of Lemma 14 with $n = 4$ and $\lambda_0 = 1$, $\lambda_1 = 0$, $\lambda_2 = -\pi^2$, $\lambda_3 = 0$. To provide approximation order 2, we have to satisfy (26) from Lemma 14 with $n = 2$. This condition is satisfied for the B-spline of order 1 as $\tilde{\varphi}$. Its Fourier transform is

$$\tilde{\varphi}(\xi) = \left(\frac{\sin \pi \xi}{\pi \xi}\right)^2,$$

the function $\tilde{\varphi}$ is continuous, supported on $[-1, 1]$ and refinable with the mask $\tilde{m}_0(\xi) = \cos^2 \pi \xi$. The polyphase components of the mask are

$$\tilde{\mu}_{00}(\xi) = \frac{\sqrt{2}}{2}, \quad \tilde{\mu}_{01}(\xi) = \frac{\sqrt{2}}{2} \left(\frac{1}{4} e^{-2\pi i \xi} + \frac{1}{4}\right).$$

Next we construct polyphase matrices $N$, $\tilde{N}$ by (27), (28), which leads to the wavelet functions

$$\psi(1)(x) = \sqrt{2} \varphi(2x + 1),$$

$$\psi(2)(x) = \sqrt{2} \varphi(2x),$$

$$\tilde{\psi}(1)(x) = \frac{1}{2\sqrt{2}} \left(-\frac{1}{2} \varphi(2x - 1) - \varphi(2x) + 3\varphi(2x + 1) - \tilde{\varphi}(2x + 2) - \frac{1}{2} \varphi(2x + 3)\right),$$

$$\tilde{\psi}(2)(x) = \frac{1}{8\sqrt{2}} \left(-\frac{1}{2} \varphi(2x - 3) - \tilde{\varphi}(2x - 2) - \frac{7}{2} \varphi(2x - 1) + 10\varphi(2x) - \frac{7}{2} \tilde{\varphi}(2x + 1) - \tilde{\varphi}(2x + 2) - \frac{1}{2} \varphi(2x + 3)\right).$$
Since \( \varphi, \tilde{\varphi} \) are bounded functions, they are in \( L_p(\mathbb{R}) \) for any \( p \in [1, \infty] \). So all assumptions of Theorems 18 and 19 are fulfilled. Thus, \( \{\psi_{jk}^{(t)}\} \) is frame-like with approximation order 2 in \( W^2_p(\mathbb{R}) \) if \( 1 < p < \infty \) or in \( W^2_\infty(\mathbb{R}) \cap L^p_\infty(\mathbb{R}) \) if \( p = \infty \), and almost frame-like with approximation order 2 in \( W^2_1(\mathbb{R}) \).

References

[41] V.S. Vladimirov, Generalized Functions in Mathematical Physics, Mir, 1979 (translated from Russian).