On Construction of Multivariate Wavelets with Vanishing Moments

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Abstract

Wavelets with matrix dilation are studied. An explicit formula for masks providing vanishing moments is found. The class of interpolatory masks providing vanishing moments is also described. For an interpolatory mask, formulas for a dual mask which also provides vanishing moments of the same order and for wavelet masks are given explicitly. An example of construction of symmetric and antisymmetric wavelets for a concrete matrix dilation is presented.

1. Introduction

We discuss construction of compactly supported biorthogonal wavelets with a matrix dilation. For image compression and some other applications, it is very desirable to have wavelets possessing vanishing moment property. It is well known how to provide this property in the one-dimensional case with dyadic dilation: a generating mask $m_0$ should be represented in the form $m_0(x) = (1 + e^{2\pi i x})^k T(x)$ (see, e.g.,[1]). Situation is essentially different in the multi-dimensional case. Zero properties of masks can not be described by means of factorization because no Euclid algorithm for multivariate polynomials exists. It is known [2] how to describe vanishing moment property in terms of linear identities (so-called sum rule). The sum rule is appropriate to check vanishing moment property for a given mask. However, finding masks by means of the sum rule is possible only numerically. One has to solve a linear systems which can contain a large enough number of unknowns. An explicit formula for masks providing vanishing moments would be more preferable in many situations. Some other descriptions of masks satisfying

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the sum rule are known, in particular, in terms of zero-conditions [2] and in terms of containment in a quotient ideal [3]. All these characterizations give methods for construction of required masks but do not allow to find an explicit general form. The goal of this paper is to present such a general form.

Let $N$ be the set of positive integers, $\mathbb{R}^d$ denotes the $d$-dimensional Euclidean space, $x = (x_1, \ldots, x_d)$, $y = (y_1, \ldots, y_d)$ are its elements (vectors), $(x, y) = x_1 y_1 + \ldots + x_d y_d$, $|x| = \sqrt{(x, x)}$, $e_j = (0, \ldots, 1, \ldots, 0)$ is the $j$-th unit vector in $\mathbb{R}^d$, $0 = (0, \ldots, 0) \in \mathbb{R}^d$; $\mathbb{Z}^d$ is the integer lattice in $\mathbb{R}^d$. For $x, y \in \mathbb{R}^d$, we write $x > y$ if $x_j > y_j$, $j = 1, \ldots, d$; $\mathbb{Z}^d_+ = \{x \in \mathbb{Z}^d : x \geq 0\}$.

If $\alpha, \beta \in \mathbb{Z}^d_+$, $a, b \in \mathbb{R}^d$, we set

\[
\alpha! = \prod_{j=1}^d \alpha_j!, \quad \binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}, \quad a^b = \prod_{j=1}^d a_j^{b_j},
\]

\[
[\alpha] = \sum_{j=1}^d \alpha_j, \quad D^\alpha f = \frac{\partial^{[\alpha]} f}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}; \quad \delta_{ab} \text{ denotes Kronecker delta.}
\]

Let $M$ be a non-degenerate $d \times d$ integer matrix whose eigenvalues are bigger than 1 in module, $M^*$ is the conjugate matrix to $M$, $I_d$ denotes the unit $d \times d$ matrix. We say that numbers $k, n \in \mathbb{Z}^d$ are congruent modulo $M$ (write $k \equiv n \pmod{M}$) if $k - n = M \ell$, $\ell \in \mathbb{Z}^d$. The integer lattice $\mathbb{Z}^d$ is splitted into cosets with respect to the introduced relation of congruence. The number of cosets is equal to $|\det M|$ (see, e.g., [5, p. 107]). Let us take an arbitrary representative from each coset, call them digits and denote the set of digits by $D(M)$.

We will consider wavelets constructed in the framework of multiresolution analysis (see [5, Chapter 5], [7]). Let a MRA in $L_2(\mathbb{R}^d)$ be generated by a scaling function $\varphi$ which satisfies the refinement equation

\[
\vec{\varphi}(x) = m_0(M^{*-1}x)\vec{\varphi}(M^{*-1}x),
\]

where $m_0 \in L_2([0, 1]^d)$ is its mask (refinable mask). For any $m_\nu \in L_2([0, 1]^d)$, there exists a unique set of functions $\mu_{\nu k} \in L_2([0, 1]^d)$, $k = 0, \ldots, m-1$, (polyphase representatives of $m_\nu$) so that

\[
m_\nu(x) = \frac{1}{\sqrt{m}} \sum_{k=0}^{m-1} e^{2\pi i(s_k, x)} \mu_{\nu k}(M^*x).
\]
The functions $\mu_{\nu k}$ can be expressed by

$$\mu_{\nu k}(x) = \frac{1}{\sqrt{m}} \sum_{s \in D(M^*)} e^{-2\pi i (M^{-1} s_k x + s)} m_{\nu}(M^{* - 1}(x + s)).$$

It is clear from this formula that a function $m_{\nu}$ is differentiable ($n$ times) on $R(M^*)$ if and only if its polyphase representatives $\mu_{\nu k}, k = 0, \ldots, m - 1$, are differentiable ($n$ times) at the origin. If $m_{\nu}$ is a trigonometric polynomial, then the polyphase representatives, are also a trigonometric polynomials.

Now, let another MRA be generated by a scaling function $\tilde{\varphi}$ with a mask $\tilde{m}_0$ such that the integer shifts of $\varphi, \tilde{\varphi}$ are biorthogonal. According to Unitary Extension Principle (see[5], [6]), to construct biorthogonal wavelets we should find wavelet masks $m_{\nu}, \tilde{m}_{\nu}, \nu = 1, \ldots, m - 1$, so that the polyphase matrices

$$M := \{\mu_{\nu k}\}_{\nu, k=0}^{m-1}, \tilde{M} := \{\tilde{\mu}_{\nu k}\}_{\nu, k=0}^{m-1},$$

satisfy

$$M\tilde{M}^* = I_m, \quad (2)$$

and define wavelet functions by

$$\hat{\psi}^{(\nu)}(x) = m_{\nu}(M^{* - 1}x)\tilde{\varphi}(M^{* - 1}x),$$

$$\tilde{\hat{\psi}}^{(\nu)}(x) = \tilde{m}_{\nu}(M^{* - 1}x)\hat{\varphi}(M^{* - 1}x).$$

The corresponding dual systems consisting of the functions

$\psi^{(\nu)}_{jk} = \psi^{(\nu)}(M^j \cdot + k), \tilde{\psi}^{(\nu)}_{jk} = \tilde{\psi}^{(\nu)}(M^j \cdot + k), j, k \in \mathbb{Z}^d,$

are biorthogonal.

Throughout the paper we will consider that wavelet systems $\{\psi^{(\nu)}_{jk}\}, \{\tilde{\psi}^{(\nu)}_{jk}\}$ are constructed by means of Unitary Extension Principle from generating scaling functions $\varphi, \tilde{\varphi}$ whose masks $m_0, \tilde{m}_0$ are continuous at the origin and $m_0(0) = \tilde{m}_0(0) = 1.$

2. Polyphase criterion for vanishing moments

It is well known that the order of vanishing moments is one of the most important factors for success of wavelets in various applications. In particular, vanishing moments are necessary for smoothness of wavelets (see [4, Th. 3.3]) and guarantee the approximation order (see [2, Th. 2.1]).
Definition 1 We say that a wavelet system \( \{ \psi_{jk}^{(\nu)} \} \) has vanishing moments up to order \( \alpha, \alpha \in \mathbb{Z}_+^d \), (has \( VM_\alpha \) property in the sequel), if \( D^\beta \hat{\psi}^{(\nu)}(0) = 0 \), \( \nu = 1, \ldots, m-1 \), for all \( \beta \in \mathbb{Z}_+^d \), \( \beta \leq \alpha \).

Theorem 2 Let \( \alpha \in \mathbb{Z}_+^d \), the functions \( \hat{\varphi}, \tilde{m}_0, m_1, \ldots, m_{m-1} \) have continuous derivatives up to order \( \alpha \) on the set \( R(M^*) \). The following conditions are equivalent:

(i) \( VM_\alpha \) property is valid for \( \{ \psi_{jk}^{(\nu)} \} \);

(ii) \( D^\beta (m_{\nu}(M^*-1)x)|_{x=0} = 0 \), \( \nu = 1, \ldots, m-1 \), for all \( \beta \in \mathbb{Z}_+^d \), \( \beta \leq \alpha \);

(iii) there exist complex numbers \( \lambda_\gamma \), \( \gamma \in \mathbb{Z}_+^d \), \( \gamma \leq \alpha \), such that

\[
D^\beta \tilde{\mu}_{0k}(0) = \frac{1}{\sqrt{m}} \sum_{0 \leq \gamma \leq \beta} \lambda_\gamma \left( \begin{array}{c} \beta \\ \gamma \end{array} \right) (-2\pi i r_k)^{\beta-\gamma} \tag{3}
\]

for all \( \beta \in \mathbb{Z}_+^d \), \( \beta \leq \alpha \).

We will prove this theorem after the following statement.

Proposition 3 If \( \tilde{m}_0 \) is so that condition (iii) of Theorem 2 is valid, then

\[
\lambda_\beta = D^\beta \left( \tilde{m}_0(M^*-1)x) \right)|_{x=0} \tag{4}
\]

for all \( \beta \in \mathbb{Z}_+^d \), \( \beta \leq \alpha \).

Proof. By Leibniz formula and (3),

\[
D^\beta \left( e^{2\pi i (r_k,x)} \tilde{\mu}_{0k}(x) \right)|_{x=0} = \sum_{0 \leq \gamma \leq \beta} \left( \begin{array}{c} \beta \\ \gamma \end{array} \right) D^\gamma (e^{2\pi i (r_k,x)}) \left|_{x=0} \right. D^{\beta-\gamma} \tilde{\mu}_{0k}(0) = \sum_{0 \leq \gamma \leq \beta} \left( \begin{array}{c} \beta \\ \gamma \end{array} \right) (2\pi i r_k)^{\gamma} D^{\beta-\gamma} \tilde{\mu}_{0k}(0) = \frac{1}{\sqrt{m}} \sum_{0 \leq \gamma \leq \beta} \left( \begin{array}{c} \beta \\ \gamma \end{array} \right) (2\pi i r_k)^{\gamma} \sum_{0 \leq \epsilon \leq \beta-\gamma} \lambda_\epsilon \left( \begin{array}{c} \beta-\gamma \\ \epsilon \end{array} \right) (-2\pi i r_k)^{\beta-\gamma-\epsilon} = \frac{1}{\sqrt{m}} \sum_{0 \leq \gamma \leq \beta} \sum_{0 \leq \epsilon \leq \beta-\gamma} \lambda_\epsilon \left( \begin{array}{c} \beta-\gamma \\ \epsilon \end{array} \right) \left( \begin{array}{c} \beta \\ \gamma \end{array} \right) (-2\pi i r_k)^{\beta-\epsilon} \prod_{j=1}^d (-1)^{-\gamma_j} = \frac{1}{\sqrt{m}} \sum_{0 \leq \epsilon \leq \beta} \lambda_\epsilon (-2\pi i r_k)^{\beta-\epsilon} \left( \begin{array}{c} \beta \\ \epsilon \end{array} \right) \sum_{0 \leq \gamma \leq \beta-\epsilon} \left( \begin{array}{c} \beta-\epsilon \\ \gamma \end{array} \right) \prod_{j=1}^d (-1)^{-\gamma_j}.
\]
Since
\[ \sum_{0 \leq \gamma \leq \beta - \varepsilon} \left( \frac{\beta - \varepsilon}{\gamma} \right) \prod_{j=1}^{d} (-1)^{-\gamma_j} = \prod_{j=1}^{d} \sum_{0 \leq \gamma_j \leq \beta_j - \varepsilon_j} \left( \frac{\beta_j - \varepsilon_j}{\gamma_j} \right) (-1)^{-\gamma_j} = \prod_{j=1}^{d} (1 - 1)^{\beta_j - \varepsilon_j} = \begin{cases} 0, & \beta \neq \varepsilon, \\ 1, & \beta = \varepsilon, \end{cases} \]
we have
\[ D^\beta \left( e^{2\pi i (r_k, x)} \tilde{\mu}_{0k}(x) \right) \bigg|_{x=0} = \frac{\lambda_\beta}{\sqrt{m}}, \quad k = 0, \ldots, m - 1. \]

It follows from (1) that
\[ D^\beta (\tilde{\mu}_0(M^{*-1}x)) \bigg|_{x=0} = \frac{1}{\sqrt{m}} \sum_{k=0}^{m-1} D^\beta \left( e^{2\pi i (r_k, x)} \tilde{\mu}_{0k}(x) \right) \bigg|_{x=0} = \lambda_\beta. \quad \diamond \]

So we established that given \( \alpha \), the set of parameters \( \lambda_\beta, \beta \in \mathbb{Z}_+^d, \beta \leq \alpha \), in (3) is unique, and \( \lambda_\beta \) does not depend on \( \alpha \).

**Proof of Theorem 2.** By Leibniz formula,
\[ D^\alpha \varphi(0) = \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} D^\beta m_\nu(M^{*-1}x) \bigg|_{x=0} D^{\alpha-\beta} \varphi(M^{*-1}x) \bigg|_{x=0}. \]

Since \( \varphi(0) = 1 \), this yields \( (i) \iff (ii) \).

We will prove \( (ii) \implies (iii) \) by induction on \( \alpha \). Check the initial step for \( \alpha = 0 \). Assume that \( m_\nu(0) = 0, \nu = 1, \ldots, m - 1 \). It follows form (1) that
\[ \sum_{k=0}^{m-1} \mu_{0k}(0) = 0, \quad \nu = 1, \ldots, m - 1. \quad (5) \]

On the other hand, by (2),
\[ \sum_{k=0}^{m-1} \mu_{0k}(0) \mu_{\nu k}(0) = 0, \quad \nu = 1, \ldots, m - 1. \]

Because of linear independence of the vectors \( (\mu_{\nu 0}(0), \ldots, \mu_{\nu m-1}(0)) \in \mathbb{R}^m, \nu = 1, \ldots, m - 1 \), there exists \( \lambda \) so that
\[ \tilde{\mu}_{00}(0) = \ldots = \tilde{\mu}_{0,m-1}(0) = \lambda. \]
Taking into account the condition \( \tilde{m}_0(0) = 1 \) which is equivalent to

\[
\frac{1}{\sqrt{m}} (\tilde{\mu}_0(0) + \ldots + \tilde{\mu}_{m-1}(0)) = 1,
\]

we obtain \( \lambda = \frac{1}{\sqrt{m}} \).

For the inductive step we assume that \((ii)\) is valid for \( \alpha > 0 \) and \((iii)\) holds for all \( \alpha' \in \mathbb{Z}_+^d, \alpha' < \alpha \). So, due to Proposition 3, there exist constants \( \lambda_\gamma, \gamma \in \mathbb{Z}_+^d, \gamma < \alpha \) such that (3) holds for all \( \beta < \alpha \). If \( \gamma \in \mathbb{Z}_+^d, \gamma < \alpha \), due to (1) and Leibniz formula, we have

\[
\frac{1}{\sqrt{m}} \sum_{0 \leq \beta \leq \alpha - \gamma} \left( \frac{\alpha - \gamma}{\beta} \right) \sum_{k=0}^{m-1} \mu_0^k \mu_\nu = 0.
\]

It follows from (2) that

\[
\sum_{k=0}^{m-1} \mu_0^k \mu_\nu = 0, \quad \nu = 1, \ldots, m - 1.
\]

Differentiating this equality \( \alpha \) times gives

\[
\sum_{0 \leq \beta \leq \alpha} \left( \frac{\alpha}{\beta} \right) \sum_{k=0}^{m-1} \frac{D^{\alpha - \beta} \mu_0^k(0)}{D^{\beta} \mu_\nu(0)} = 0.
\]

Multiply (6) by \( \left( \frac{-\alpha}{\alpha - \gamma} \right) \lambda_\gamma \) and subtract from (7). After the same manipulation with each \( \gamma \in \mathbb{Z}_+^d, \gamma < \alpha \), we obtain

\[
0 = \sum_{0 \leq \beta \leq \alpha} \left( \frac{\alpha}{\beta} \right) \sum_{k=0}^{m-1} \frac{D^{\alpha - \beta} \mu_0^k(0)}{D^{\beta} \mu_\nu(0)} - \sum_{0 \leq \beta \leq \alpha} \left( \frac{\alpha}{\beta} \right) \sum_{k=0}^{m-1} \frac{\mu_0^k}{D^{\beta - \gamma} \mu_\nu(0)} - \sum_{0 \leq \beta \leq \alpha} \left( \frac{\alpha - \gamma}{\beta} \right) \left( \frac{\alpha}{\alpha - \gamma} \right) \left( \frac{\alpha}{\beta} \right)^{-1} \lambda_\gamma \left( -2\pi ir_k \right)^{\alpha - \beta - \gamma} D^{\beta} \mu_\nu(0).
\]
From this, taking into account that
\[
\begin{pmatrix}
\alpha - \gamma \\
\beta
\end{pmatrix}
\begin{pmatrix}
\alpha \\
\alpha - \gamma
\end{pmatrix}^{-1}
= \frac{(\alpha - \beta)!}{\gamma!(\alpha - \beta - \gamma)!}
= \begin{pmatrix}
\alpha - \beta \\
\gamma
\end{pmatrix},
\]
and using the inductive hypotheses, we have
\[
0 = \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} \sum_{k=0}^{m-1} \left(D^{\alpha-\beta} \tilde{\mu}_{0k}(0) - \frac{1}{\sqrt{m}} \sum_{0 \leq \gamma < \alpha} \binom{\alpha}{\gamma} \lambda_{\gamma}(-2\pi ir_k)^{\alpha-\beta} \right) D^\beta \mu_{\nu k}(0) =
\]
\[
\sum_{k=0}^{m-1} \left(D^\alpha \tilde{\mu}_{0k}(0) - \frac{1}{\sqrt{m}} \sum_{0 \leq \gamma < \alpha} \binom{\alpha}{\gamma} \lambda_{\gamma}(-2\pi ir_k)^{\alpha-\gamma} \right) \mu_{\nu k}(0).
\]
Similarly to the arguments for the initial step, it follows from (5) that there exists \(\lambda_{\alpha}\) such that
\[
D^\alpha \tilde{\mu}_{0k}(0) - \frac{1}{\sqrt{m}} \sum_{0 \leq \gamma < \alpha} \binom{\alpha}{\gamma} \lambda_{\gamma}(-2\pi ir_k)^{\alpha-\gamma} = \lambda_{\alpha} \sqrt{m}.
\]
Thus, (3) is valid for \(\beta = \alpha\) as was to be proved.

The implication \((iii) \Rightarrow (ii)\) will be also proved by induction on \(\alpha\). If (3) is valid for \(\alpha = 0\), then \(\tilde{\mu}_{0k}(0) = 1/\sqrt{m}, k = 0, \ldots, m-1\). It follows from (2) that
\[
\mu_{\nu 0}(0) + \ldots + \mu_{\nu,m-1}(0) = 0, \quad \nu = 1, \ldots, m-1.
\]
Hence, on the basis of (1), \(m_{\nu}(0) = 0, \nu = 1, \ldots, m-1\), what proves the initial step.

For the inductive step, we assume that \((iii)\) is valid for \(\alpha > 0\) and \((ii)\) holds for all \(\alpha' \in \mathbb{Z}_+^d, \alpha' < \alpha\), i.e.
\[
D^{\alpha-\gamma} \mu_{\nu}(M^{*-1} x) \bigg|_{x=0} = 0, \quad \gamma \in \mathbb{Z}_+^d, \quad \gamma \neq 0, \quad \gamma \leq \alpha.
\]
This yields (6) for \(\gamma \neq 0\). Multiply (6) by \(\binom{\alpha}{\alpha-\gamma} \lambda_{\gamma}\) and add to (1) differentiated \(\alpha\) times. After the same manipulation with each \(\gamma \in \mathbb{Z}_+^d, \gamma < \alpha\), we
Due to (8) and (3), this yields

$$D^\alpha m_\nu(M^*-1x)\big|_{x=0} = \frac{1}{\sqrt{m}} \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} \sum_{k=0}^{m-1} (2\pi ir_k)^{\alpha-\beta} D^\beta \mu_{\nu k}(0) +$$

$$\frac{1}{\sqrt{m}} \sum_{0 \leq \gamma \leq \alpha} \binom{\alpha}{\alpha-\gamma} \lambda_\gamma \sum_{0 \leq \beta \leq \alpha-\gamma} \binom{\alpha-\gamma}{\beta} \sum_{k=0}^{m-1} (2\pi ir_k)^{\alpha-\beta-\gamma} D^\beta \mu_{\nu k}(0) =$$

$$\frac{1}{\sqrt{m}} \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} \sum_{0 \leq \gamma \leq \alpha-\beta} \lambda_\gamma \binom{\alpha}{\alpha-\gamma} \binom{\alpha-\gamma}{\beta} (-2\pi ir_k)^{\alpha-\beta-\gamma} D^\beta \mu_{\nu k}(0).$$

Due to (8) and (3), this yields

$$D^\alpha m_\nu(M^*-1x)\big|_{x=0} = \frac{1}{\sqrt{m}} \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} \sum_{k=0}^{m-1} (2\pi ir_k)^{\alpha-\beta} D^\beta \mu_{\nu k}(0) =$$

$$\sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} \sum_{k=0}^{m-1} D^{\alpha-\beta} \mu_{\nu k}(0) D^\beta \mu_{\nu k}(0) = D^\alpha \left( \sum_{k=0}^{m-1} \mu_{\nu k}(x) \mu_{\nu k}(x) \right) \big|_{x=0}.$$

It follows from (2) that $D^\alpha m_\nu(M^*-1x)\big|_{x=0} = 0$ as was to be proved. ◊

Usually it is more useful to control univariate order of vanishing moment property (for example, to apply Taylor formula).

**Definition 4** We say that a wavelet system $\{\psi_\nu(jk)\}$ has vanishing moments up to order $n$, $n \in \mathbb{Z}_+$, (has $VM^n$ property in the sequel) if $D^\beta \tilde{\psi}_\nu(0) = 0$, $\nu = 1, \ldots, m-1$, for all $\beta \in \mathbb{Z}_d^+$, $[\beta] \leq n$.

**Theorem 5** Let $n \in \mathbb{Z}_+$, $\tilde{\varphi}, \tilde{m}_0, m_1, \ldots, m_{m-1}$ have continuous derivatives up to order $n$ on the set $R(M^*)$. The following conditions are equivalent:

(i) $VM^n$ property is valid for $\{\psi_\nu(jk)\}$;

(ii) $D^\beta(m_\nu(M^*-1x))\big|_{x=0} = 0$, $\nu = 1, \ldots, m-1$, for all $\beta \in \mathbb{Z}_d^+$, $[\beta] \leq n$;
(iii) there exist complex numbers \( \lambda, \gamma \in \mathbb{Z}_+^d \), \( [\gamma] \leq n \), such that \( \lambda_0 = 1 \),
\[
D^\beta \hat{\mu}_{0k}(0) = \frac{1}{\sqrt{m}} \sum_{0 \leq \gamma \leq \beta} \lambda_\gamma \left( \frac{\beta}{\gamma} \right) (-2\pi ir_k)^{\beta - \gamma} \tag{9}
\]
for all \( \beta \in \mathbb{Z}_+^d \), \( [\beta] \leq n \).

Proof of this theorem follows immediately from Theorem 2 and Proposition 3.

3. General forms for masks providing \( VM_\alpha \) and \( VM^n \)

Theorems 2 and 5 allow to find concrete masks \( \hat{m} \) providing \( VM_\alpha \) and \( VM^n \) properties for \( \{ \psi_{jk} \} \). Given sets of parameters
\[
\Lambda_\alpha := \{ \lambda_\gamma, \gamma \in \mathbb{Z}_+^d, \gamma \leq \alpha, \lambda_0 = 1 \}, \alpha \in \mathbb{Z}_+^d,
\]
\[
\Lambda^n := \{ \lambda_\gamma, \gamma \in \mathbb{Z}_+^d, [\gamma] \leq n, \lambda_0 = 1 \}, n \in \mathbb{Z}_+^d,
\]
we put
\[
\hat{m}^*(x, \Lambda_\alpha) := \frac{1}{m} \sum_{k=0}^{m-1} e^{2\pi i (s_k, x)} \sum_{0 \leq \beta \leq \alpha} g_\beta (M^* x) \sum_{0 \leq \gamma \leq \beta} \left( \frac{\beta}{\gamma} \right) \lambda_\gamma (-2\pi ir_k)^{\beta - \gamma}, \tag{10}
\]
\[
\hat{m}^*(x, \Lambda^n) := \frac{1}{m} \sum_{k=0}^{m-1} e^{2\pi i (s_k, x)} \sum_{0 \leq \beta \leq \gamma} g_\beta (M^* x) \sum_{0 \leq \gamma \leq \beta} \left( \frac{\beta}{\gamma} \right) \lambda_\gamma (-2\pi ir_k)^{\beta - \gamma}, \tag{11}
\]
where \( g_\beta \) is a trigonometric polynomial such that \( D^\gamma g_\beta(0) = \delta_{\beta \gamma} \), for all \( \gamma \in \mathbb{Z}_+^d \), \( \gamma \neq \beta \), \( 0 \leq \gamma \leq \alpha \) (\( 0 \leq [\gamma] \leq n \) for (11)). It is easy to check that \( \hat{m}^*_\alpha(\cdot, \Lambda_\alpha) \) and \( \hat{m}^*_n(\cdot, \Lambda^n) \) satisfy condition (iii) respectively of Theorem 2 and of Theorem 5. Functions \( g_\beta \) can be found, for example, by

\[
g_\beta(x) = \prod_{j=1}^d g_{\beta_j}(x_j),
\]
\[
g_{\beta_j}(u) = \frac{1}{\beta_j! (-2\pi i)^{\beta_j}} \left( 1 - e^{2\pi i u} \right)^{\beta_j} - \sum_{l=1}^{a_{\beta_j} - \beta_j} a_l \left( 1 - e^{2\pi i u} \right)^{\beta_j + l}, \tag{12}
\]
\[
a_l = \frac{(-2\pi i)^{-\beta_j - l}}{(\beta_j + l)!} \frac{d^{\beta_j + l}}{du^{\beta_j + l}} \left( 1 - e^{2\pi i u} \right)^{\beta_j} - \sum_{r=1}^{l-1} a_r \left( 1 - e^{2\pi i u} \right)^{\beta_j + r} \bigg|_{u=0}
\]
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(α_j = n in (12) for (11)). By this formula we have

\[\alpha_j = 1 : \quad g_1(u) = -\frac{1}{2\pi i} (1 - e^{2\pi i u})\]
\[g_2(u) = -\frac{1}{8\pi^2} \left((1 - e^{2\pi i u})^2 + (1 - e^{2\pi i u})^3\right)\]
\[g_3(u) = \frac{1}{48\pi^3} \left((1 - e^{2\pi i u})^3 + \frac{3}{2} (1 - e^{2\pi i u})^4 + \frac{7}{4} (1 - e^{2\pi i u})^5\right)\]

Similarly, real functions \(g_{\beta_j}\) can be found by

\[g_{\beta_j}(u) = \frac{1}{\beta_j!(2\pi)^{\beta_j}} \left(\sin^{\beta_j} 2\pi u - \sum_{l=1}^{\alpha_j-\beta_j} a_l \sin^{\beta_j+l} 2\pi u\right),\]
\[a_l = \frac{(2\pi)^{-\beta_j-l}}{(\beta_j + l)!} \left. \frac{d^{\beta_j+l}}{d\beta_j+l} \left(\sin^{\beta_j} 2\pi u - \sum_{r=1}^{l-1} a_r \sin^{\beta_j+r} 2\pi u\right)\right|_{u=0}.\]

Now it is not difficult to find a general form for polynomial masks \(\tilde{m}_0\) providing wavelets with vanishing moments. Let \(\tilde{m}_0\) satisfy condition (iii) of Theorem 2 with a set of parameters \(\Lambda_\alpha\). By (1),

\[\tilde{m}_0(M^*-1x) - \tilde{m}_0^*(M^*-1x,\Lambda_\alpha) = \frac{1}{\sqrt{m}} \sum_{k=0}^{m-1} e^{2\pi i (r_k, x)} (\tilde{\mu}_k(x) - \tilde{\mu}_k^*(x)),\]  \(\text{(13)}\)

where \(\tilde{\mu}_k^*\) is the k-th polyphase representative of the function \(\tilde{m}_0^*(\cdot, \Lambda_\alpha)\) defined by (10). There exist an algebraic polynomial \(P\) and \(N \in \mathbb{Z}^d\) such that

\[\tilde{\mu}_k(x) - \tilde{\mu}_k^*(x) = e^{2\pi i (N, x)} P(z),\]  \(\text{(14)}\)

where \(z = (z_1, \ldots, z_d), z_j = e^{2\pi i (x, e_j)}\). Since, due to Theorem 1,

\[D^\beta(\tilde{\mu}_k(x) - \tilde{\mu}_k^*(x))\big|_{x=0} = 0, \quad \beta \in \mathbb{Z}_+^d, \quad \beta \leq \alpha,
\]
it is clear that $P$ has vanishing derivatives up to order $\alpha$ at the point $(1, \ldots, 1)$. By the Taylor formula, there exist algebraic polynomials $Q_j$, $j = 1, \ldots, d$, so that

$$P(z) = \sum_{j=1}^{d} Q_j(z)(1 - z_j)^{\alpha_j+1}.$$ 

Combining this with (13) and (14), we obtain

$$\tilde{m}_0(x) = \tilde{m}_0^*(x, \Lambda_\alpha) + \sum_{j=1}^{d} T_j(x) (1 - e^{2\pi i (x, M e_j)})^{\alpha_j+1},$$

(15)

where $T_j$ are arbitrary trigonometric polynomials. Evidently, each function $\tilde{m}_0$ defined by (15) satisfies condition (iii) of Theorem 1 with a set of parameters $\Lambda_\alpha$.

Similarly, all trigonometric polynomials $\tilde{m}_0$ satisfying condition (iii) of Theorem 5 with a set of parameters $\Lambda^n$ can be described by

$$\tilde{m}_0(x) = \tilde{m}_0^*(x, \Lambda^n) + \sum_{[\alpha]=n+1}^{\alpha} T_\alpha(z) \prod_{j=1}^{d} (1 - e^{2\pi i (x, M e_j)})^{\alpha_j},$$

(16)

where $T_\alpha$ are an arbitrary trigonometric polynomials.

Other general forms for polynomial masks providing vanishing moments can be given by

$$\tilde{m}_0(x) = \frac{1}{m} \sum_{k=0}^{m-1} e^{2\pi i (s_k, x)} \left( f_k(M^*x) - \sum_{0 \leq \beta \leq \alpha} (D^\beta f_k(0) - \Delta_\beta) g_{\beta k}(M^*x) \right),$$

(17)

$$\tilde{m}_0(x) = \frac{1}{m} \sum_{k=0}^{m-1} e^{2\pi i (s_k, x)} \left( f_k(M^*x) - \sum_{0 \leq |\beta| \leq n} (D^\beta f_k(0) - \Delta_{\beta k}) g_{\beta}(M^*x) \right),$$

(18)

where

$$\Delta_{\beta k} = \frac{1}{\sqrt{m}} \sum_{0 \leq \gamma \leq \beta} \lambda_\gamma \binom{\beta}{\gamma} (-2\pi i r_k)^{\beta-\gamma},$$

the functions $f_k, g_\beta$ are trigonometric polynomials, $D^\gamma g_\beta(0) = \delta_\betax\gamma$ for all $\gamma \in \mathbb{Z}_+^d, \gamma \neq \beta, 0 \leq \gamma \leq \alpha (0 \leq \gamma \leq n$ for (18)). If $f_k, k = 0, \ldots, m - 1,$
are arbitrary functions from $L_2(0,1)$ with continuous derivatives at the origin, then (17), (18) give a general form for all masks providing $VM_\alpha, VM^n$ properties with the sets of parameters $\Lambda_\alpha, \Lambda^n$ respectively. This follows from the equalities

$$D^\gamma \left( f_k(M^*x) - \sum_{0 \leq \beta \leq \alpha} (D^\beta f_k(0) - \Delta_\beta) g_\beta(M^*x) \right) \bigg|_{x=0}^\gamma = \Delta_{\gamma k}, \ 0 \leq \gamma \leq \alpha$$

$$D^\gamma \left( f_k(M^*x) - \sum_{0 \leq [\beta] \leq n} (D^\beta f_k(0) - \Delta_\beta) g_\beta(M^*x) \right) \bigg|_{x=0}^\gamma = \Delta_{\gamma k}, \ 0 \leq [\gamma] \leq n.$$

### 4. Interpolatory masks

Finding a suitable refinable mask $\tilde{m}_0$ is the first step in construction of biorthogonal wavelets. After that, according to Unitary Extension Principle we should find a dual refinable mask $m_0$ and wavelet masks $m_\nu \tilde{m}_\nu$, $\nu = 1, \ldots, m - 1$, which is very complicated. This problem is closely related to the famous Serre conjecture stating that a unimodular line of algebraic polynomials can be extended to a unimodular matrix. The Serre conjecture was solved independently by Quillen and Suslin. Moreover, Suslin[10] proved an analog of this statement for a wider class of rings, in particular, for the ring of Laurent polynomials. On the basis of this Suslin’s result, an appropriate first line of the matrix $\tilde{M}$ can be extended to a matrix whose entries are trigonometric polynomials and the determinant equals 1. After this it is not difficult to find a required matrices $\tilde{M}, \tilde{M}$ (see [8], [9]). Though the problem is solved theoretically, implementable algorithms for matrix extensions are not known in general. Next we will consider a class of refinable masks for which construction of wavelets may be realized in practice.

A mask $\tilde{m}_0$ is said to be interpolatory if $\tilde{\mu}_{00} \equiv \text{const}$. Different methods for construction of some families of interpolatory masks providing vanishing moments were suggested in [11], [12]. We will describe the class of all such masks.

Let $n \in \mathbb{Z}_+$, $\Lambda^n_0 := \{\lambda_0 = 1, \lambda_\gamma = 0, \gamma \in \mathbb{Z}^d_+, [\gamma] \leq n, \gamma \neq 0\}$. For this set of parameters, the function $\tilde{\mu}_{00} \equiv \frac{1}{\sqrt{m}}$ evidently satisfies (9). If we choose $\tilde{\mu}_{0\nu}, \nu = 1, \ldots, m - 1$, so that

$$D^\beta \tilde{\mu}_{0k}(0) = \frac{1}{\sqrt{m}} (-2\pi ir_k)^\beta, \ \beta \in \mathbb{Z}^d_+, [\beta] \leq n, \quad (19)$$
then condition (iii) of Theorem 5 is valid. Inversely, for each interpolatory
mask providing property $VM^n$, the set of parameters in (iii) is $\Lambda^n_0$. Using above arguments we obtain the following general form for polynomial interpolatory masks providing $VM^n$ property:

$$\tilde{m}_0(x) = \tilde{m}_0^*(x, \Lambda^n_0) + \sum_{|\alpha|=n+1}^{d} T_{\alpha}(z) \prod_{j=1}^{d} (1 - e^{2\pi i(x,Me_j)})^\alpha_j,$$

(20)

where

$$\tilde{m}_0^*(x, \Lambda^n_0) = \frac{1}{m} \sum_{k=0}^{m-1} e^{2\pi i(s_k,x)} \sum_{0 \leq |\beta| \leq n} g_\beta(M^*x)(-2\pi ir_k)^\beta,$$

(21)

$T_{\alpha}$ are trigonometric polynomials with vanishing Fourier coefficients whose numbers are congruent to $0$ modulo $M$.

A trivial dual mask of a given interpolatory mask $\tilde{m}_0$ is $m_0 \equiv m$. This mask does not provide vanishing moment. Numerical methods for finding dual masks providing vanishing moments (CBC algorithms, convolution method) were suggested in [13], [14]. We consider a dual mask $m_0$ defined explicitly in polyphase form by

$$\mu_{00} = \sqrt{m} \left(1 - \sum_{k=1}^{m-1} |\tilde{\mu}_{0k}|^2 \right), \quad \mu_{0k} = \tilde{\mu}_{0k}, \quad k = 1, \ldots, m - 1.$$  

(22)

Such a mask $m_0$ is not interpolatory in general.

**Proposition 6** If an interpolatory refinable mask $\tilde{m}_0$ provides property $VM^n$, then its dual mask $m_0$ defined by (22) satisfies condition (iii) of Theorem 5 with the set of parameters $\Lambda^n_0$, i.e. $m_0$ also provides $VM^n$ property.

**Proof.** Since each function $\mu_{0k}, k = 1, \ldots, m - 1$, coincides with $\tilde{\mu}_{0k}$, it satisfies (19). It remains to check that (19) is valid for $\mu_{00}$. Evidently, $\mu_{00}(0) = \frac{1}{\sqrt{m}}$. Let $\beta \in \mathbb{Z}_d^+$, $0 < [\beta] \leq n$. For any $k = 1, \ldots, m - 1$, using Leibniz formula and (19), we have

$$D^\beta |\tilde{\mu}_{0k}(x)|^2 \bigg|_{x=0} = D^\beta \left(\tilde{\mu}_{0k}(x)\overline{\tilde{\mu}_{0k}(x)}\right) \bigg|_{x=0} =$$

$$\sum_{0 \leq \gamma \leq \beta} \binom{\beta}{\gamma} D^\gamma \tilde{\mu}_{0k}(0) \overline{D^{\beta-\gamma} \tilde{\mu}_{0k}(0)} =$$
It follows that
\[ D^\beta \mu_{00}(0) = 0, \ \beta \in \mathbb{Z}_+^d, \ 0 < [\beta] \leq n. \]

So, (19) holds true for \( k = 0 \). ♦

Since extension of the line \( \sqrt{m}, \tilde{\mu}_0, \ldots, \tilde{\mu}_{0,m-1} \), to an unimodular matrix is trivial, using the method suggested in [9], we can easily find matrices \( M, \tilde{M} \):

\[
M = \begin{pmatrix}
\sqrt{m} & 1 - \sum_{k=1}^{m-1} |\tilde{\mu}_{0k}|^2 & \mu_{01} & \mu_{02} & \cdots & \mu_{0,m-1} \\
-\tilde{\mu}_{01} & 1/\sqrt{m} & 0 & \cdots & 0 \\
-\tilde{\mu}_{02} & 0 & 1/\sqrt{m} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-\mu_{0,m-1} & 0 & 0 & \cdots & 1/\sqrt{m}
\end{pmatrix}
\]  \( (23) \)

\[
\tilde{M} = \begin{pmatrix}
\frac{1}{\sqrt{m}} & \mu_{01} & \mu_{02} & \cdots & \mu_{0,m-1} \\
-\tilde{\mu}_{01} & \sqrt{m}(1 - |\mu_{01}|^2) & -\sqrt{m}\mu_{01}\mu_{02} & \cdots & -\sqrt{m}\mu_{01}\mu_{0,m-1} \\
-\tilde{\mu}_{02} & -\sqrt{m}\mu_{02}\mu_{01} & \sqrt{m}(1 - |\mu_{02}|^2) & \cdots & -\sqrt{m}\mu_{02}\mu_{0,m-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-\mu_{0,m-1} & -\sqrt{m}\mu_{0,m-1}\mu_{01} & -\sqrt{m}\mu_{0,m-1}\mu_{02} & \cdots & \sqrt{m}(1 - |\mu_{0,m-1}|^2)
\end{pmatrix}
\]  \( (24) \)

If it turned out that the dual refinable mask \( m_0 \), is also interpolatory, i.e.

\[
\frac{1}{m} + \sum_{k=1}^{m-1} |\mu_{0k}|^2 = 1,
\]

and hence the first lines of \( M \) and \( \tilde{M} \) coincide, then it is possible to extend this line to a unitary matrix \( M = \tilde{M} \) which can be found by means of
Householder transform:

$$M = \begin{pmatrix}
\frac{1}{\sqrt{m}} & \mu_{01} & \mu_{02} & \cdots & \mu_{0,m-1} \\
\mu_{01} & 1 - |\mu_{01}|^2 & -\mu_{01}\mu_{02} & \cdots & -\mu_{01}\mu_{0,m-1} \\
\mu_{02} & -\mu_{01}\mu_{02} & 1 - |\mu_{02}|^2 & \cdots & -\mu_{02}\mu_{0,m-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\mu_{0,m-1} & -\mu_{0,m-1}\mu_{01} & -\mu_{0,m-1}\mu_{02} & \cdots & 1 - |\mu_{0,m-1}|^2.
\end{pmatrix}$$

An orthogonal wavelet basis (or a tight frame) may be constructed in this way.

5. Example

Using general forms presented in Sections 3, we can easily find a lot of concrete refinable masks providing vanishing moments. Due to the results of Section 4, we can also write wavelet masks for an interpolatory mask $\tilde{m}_0$ explicitly. However, only symmetric/antysymmetric wavelets are useful for some engineering application. So, we have two problems: first, we have to find an even refinable mask, and second, we need an appropriate matrix extension to get even/odd wavelet masks. For $d = 1$, the second problem was solved by Petukhov [15] for the orthogonal case $m_0 = \tilde{m}_0$. His algorithm is essentially not suitable for the multidimensional case. We will illustrate how it is possible to construct even or odd wavelet masks with vanishing moments for some matrix dilations with the help of above formulas.

Let $M = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$, for this matrix $m = 3$, $M^* = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. If we choose $s_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $s_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, then $r_1 = \begin{pmatrix} 1/3 \\ 1 \end{pmatrix}$, $r_2 = \begin{pmatrix} -1/3 \\ -1/3 \end{pmatrix}$. By (21),

$$\tilde{m}_0^*(x, \Lambda_0^n) = \frac{1}{3} \left[ 1 + e^{2\pi ix_1} \sum_{0 \leq |\beta| \leq n} g_{\beta}(M^*x)(-2\pi ir_1)^{\beta} +
+ e^{-2\pi ix_1} \sum_{0 \leq |\beta| \leq n} g_{\beta}(M^*x)(-2\pi ir_2)^{\beta} \right] =$$

$$\frac{1}{3} \left[ 1 + 2 \cos 2\pi x_1 \sum_{0 \leq |\beta| \leq n, [\beta] \text{ is even}} g_{\beta}(M^*x) \left(\frac{-2\pi i}{3}\right)^{[\beta]} + \right.$$
$+2i \sin 2\pi x_1 \sum_{0 \leq [\beta] \leq n \atop [\beta] \text{ is odd}} g_\beta(M^*x) \left( \frac{-2\pi i}{3} \right)^{[\beta]}$.

If we use real functions $g_\beta$, then $\tilde{m}^*$ is also real. Moreover, if we take an even function $g_\beta$ whenever $[\beta]$ is even and an odd function $g_\beta$ whenever $[\beta]$ is odd (this can be easily realized by the formula $\frac{1}{2} \left( g_\beta(x) + (-1)^{[\beta]} g_\beta(-x) \right)$), then the mask $\tilde{m}^*$ is even.

It is not difficult to see that all the functions

$$
\tilde{m}_0(x) = \tilde{m}_0^*(x, \Lambda_0^n) + \sum_{[\alpha]=n+1} T_\alpha(x) \prod_{j=1}^2 \sin^{\alpha_j} 2\pi (M^*x, e_j) =
\tilde{m}_n^*(x, \Lambda_0^n) + \sum_{[\alpha]=n+1} T_\alpha(x) \sin^{\alpha_1} 2\pi (2x_1 - x_2) \sin^{\alpha_2} 2\pi (x_1 + x_2),
$$

where $T_\alpha$ is an even trigonometric polynomial whenever $[\alpha]$ is even and an odd trigonometric polynomial $T_\alpha$ whenever $[\alpha]$ is odd, are also even masks providing $VM^n$ property.

For $n = 1$, choosing $g_0(u) = 1$, $g_1(u) = \frac{1}{2\pi} \sin 2\pi u$, we obtain the function

$$
\tilde{m}_0(x) = \tilde{m}_0^*(x, \Lambda_0^1) = \frac{1}{9} [3 + 6 \cos 2\pi x_1 + 4 \sin 2\pi x_1 \sin 3\pi x_1 \cos \pi (x_1 - 2x_2)],
$$

the table of its Fourier coefficients looks as follows

$$
\begin{pmatrix}
-1/18 & 0 & 1/18 & 1/18 & 0 & -1/18 & 0 \\
0 & 0 & 1/3 & 1/3 & 1/3 & 0 & 0 \\
0 & -1/18 & 0 & 1/18 & 1/18 & 0 & -1/18
\end{pmatrix},
$$

the polyphase representatives are

$$
\tilde{\mu}_{00}(x) = \frac{1}{\sqrt{3}},
\tilde{\mu}_{01}(x) = \frac{1}{\sqrt{3}} \left( 1 - \frac{i}{3} \sin 2\pi x_1 - \frac{i}{3} \sin 2\pi x_2 \right),
\tilde{\mu}_{02}(x) = \frac{1}{\sqrt{3}} \left( 1 + \frac{i}{3} \sin 2\pi x_1 + \frac{i}{3} \sin 2\pi x_2 \right).
$$

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Computing polyphase representatives of a dual mask $m_0$ by (22)

$$
\begin{align*}
\mu_{00}(x) &= \frac{1}{\sqrt{3}} - \frac{2}{9\sqrt{3}}(\sin 2\pi x_1 + \sin 2\pi x_2)^2 \\
\mu_{0k}(x) &= \tilde{\mu}_{0k}(x), \quad k = 1, 2,
\end{align*}
$$

we find

$$
m_0(x) = \frac{1}{27} \left( 9 + 18 \cos 2\pi x_1 + 12 \sin 2\pi x_1 \sin 3\pi x_1 \cos \pi(x_1 - 2x_2) - 8 \sin^2 3\pi x_1 \cos^2 \pi(x_1 - 2x_2) \right).
$$

This function is also even. To find wavelet masks use the matrixes (23), (24):

$$
\mathcal{M} = \begin{pmatrix}
\mu_{00} & \mu_{01} & \mu_{02} \\
-\mu_{01} & 1/\sqrt{3} & 0 \\
-\mu_{02} & 0 & 1/\sqrt{3}
\end{pmatrix} =: \begin{pmatrix}
Q_0 \\
Q_1 \\
Q_2
\end{pmatrix}
$$

$$
\tilde{\mathcal{M}} = \begin{pmatrix}
\frac{1}{\sqrt{m}} & \mu_{01} & \mu_{02} \\
-\mu_{01} & \sqrt{3}(1 - |\mu_{01}|^2) & -\sqrt{3}\mu_{01}\mu_{02} \\
-\mu_{02} & -\sqrt{3}\mu_{02}\mu_{01} & \sqrt{3}(1 - |\mu_{02}|^2)
\end{pmatrix} =: \begin{pmatrix}
\tilde{Q}_0 \\
\tilde{Q}_1 \\
\tilde{Q}_2
\end{pmatrix}
$$

These two matrices are mutually inverse. It is clear that the matrices

$$
\mathcal{M}' := \begin{pmatrix}
Q_0 \\
\frac{1}{2}(Q_1 + Q_2) \\
\frac{1}{2i}(Q_1 - Q_2)
\end{pmatrix} \quad \tilde{\mathcal{M}}' := \begin{pmatrix}
\tilde{Q}_0 \\
\frac{1}{2}(\tilde{Q}_1 + \tilde{Q}_2) \\
\frac{1}{2i}(\tilde{Q}_1 - \tilde{Q}_2)
\end{pmatrix}
$$

are also mutually inverse. Now, if we set

$$
\begin{align*}
(\mu_{10}, \mu_{11}, \mu_{12}) &:= \frac{1}{2}(Q_1 + Q_2) = \left( -\frac{1}{\sqrt{3}}, \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}} \right) \\
(\mu_{20}, \mu_{21}, \mu_{22}) &:= \frac{1}{2i}(Q_1 - Q_2) = \left( -\frac{\sin 2\pi x_1 + \sin 2\pi x_2}{3\sqrt{3}}, \frac{2\sqrt{3}}{2\sqrt{3}}, \frac{i}{2\sqrt{3}} \right),
\end{align*}
$$

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the corresponding wavelet masks are

\[ m_1(x) = \frac{1}{3}(\cos 2\pi x_1 - 1), \]
\[ m_2(x) = \frac{1}{9}(3 \sin 2\pi x_1 - 2 \sin 3\pi x_1 \cos \pi(x_1 - 2x_2)). \]

For \( n = 2 \), choosing \( g_0(u) = 1 \), \( g_1(u) = \frac{1}{2\pi} \sin 2\pi u \), \( g_2(u) = \frac{1}{8\pi^2} \sin^2 2\pi u \), we obtain a refinable mask

\[ \tilde{m}_0(x) = m_0^*(x, \Lambda_0^2) = \frac{1}{27}[9 + 18 \cos 2\pi x_1 + 12 \sin 2\pi x_1 \sin 3\pi x_1 \cos \pi(x_1 - 2x_2) - 4 \cos 2\pi x_1 \sin^2 3\pi x_1 \cos^2 \pi(x_1 - 2x_2)], \]

the table of its Fourier coefficients looks as follows

\[
\begin{pmatrix}
\frac{1}{27} & 0 & -\frac{1}{18} & 0 & -\frac{1}{18} & 0 & 0 & \frac{1}{116} & 0 & 0 \\
0 & 0 & -\frac{1}{18} & 0 & \frac{1}{18} & \frac{1}{18} & 0 & -\frac{1}{18} & 0 & 0 \\
0 & \frac{1}{108} & 0 & \frac{1}{108} & \frac{17}{54} & \frac{1}{3} & \frac{17}{54} & \frac{1}{108} & 0 & \frac{1}{108} \\
0 & 0 & 0 & -\frac{1}{18} & 0 & \frac{1}{18} & \frac{1}{18} & 0 & -\frac{1}{18} & 0 & 0 \\
0 & 0 & \frac{1}{216} & 0 & \frac{1}{216} & -\frac{1}{108} & 0 & -\frac{1}{108} & \frac{1}{216} & 0 & \frac{1}{216}
\end{pmatrix}
\]

Similarly to the case \( n = 1 \), we can find the following dual refinable mask \( m_0 \) and wavelet masks \( m_1, m_2 \):

\[ m_0(x) = \frac{1}{243}(81 + 162 \cos 2\pi x_1 + 106 \sin 2\pi x_1 \sigma(x) - 36 \cos 2\pi x_1 \sigma^2(x) - 8\sigma^4(x)), \]
\[ m_1(x) = \frac{1}{81}(-9 + 27 \cos 2\pi x_1 + 2\sigma^2(x)), \]
\[ m_2(x) = \frac{1}{9}(-3 \sin 2\pi x_1 + 2\sigma(x)), \]

where \( \sigma(x) = \sin 3\pi x_1 \cos \pi(x_1 - 2x_2) \).

By construction, we have

\[ \sum_{k=0}^{m-1} \mu_{0k} \mu_{0k} = 1. \]
This is a necessary condition for biorthogonality of the scaling functions \( \varphi, \tilde{\varphi} \) with the masks \( m_0, \tilde{m}_0 \). To prove a sufficient condition we will use Cohen criterion (which was extended to the multivariate biorthogonal case with an arbitrary matrix dilation in [16]). The set \( P := M^{-1}([-1/2, 1/2]^2) \) is the parallelogram depicted on Figure 1a. By cutting and removing the triangles \( \Delta_1 := P \cap \{ x \in \mathbb{R}^2 : x_1 \geq 1/5 \} \), \( \Delta_2 := P \cap \{ x \in \mathbb{R}^2 : x_1 \leq -1/5 \} \) as it is depicted on Figure 1b, we obtain a polygon \( \Omega \). It is clear that the set \( K := M^*\Omega \) is a Cohen compact. Since \( \cos 2\pi x_1 > 0.3 \) whenever \( |x_1| \leq 1/5 \), we have

\[
\tilde{m}_0(x) \geq \frac{1}{27} (9 + \cos 2\pi x_1 (18 - 4\sigma^2) + 12 \sin 2\pi x_1 \sigma) > (9 + 4 - 12) > 0,
\]

\[
m_0(x) \geq \frac{1}{243} (81 + \cos 2\pi x_1 (162 - 36\sigma^2) + 106 \sin 2\pi x_1 \sigma - 8\sigma^4) > (81 + 37 - 106 - 8) > 0
\]

for all \( x \in \{ x \in \mathbb{R}^2 : |x_1| \leq 1/5 \} =: S \). We proved that \( \Omega \subset S \). It is not difficult to see that \( M^{*-k}\Omega \subset S \) for \( k = 1, 2, \ldots \). Hence,

\[
\inf_{k \in \mathbb{N}} \inf_{x \in K} |m_0(M^{*-k}x)| \neq 0, \quad \inf_{k \in \mathbb{N}} \inf_{x \in K} |\tilde{m}_0(M^{*-k}x)| \neq 0,
\]

which means that Cohen criterion is fulfilled.

The same method for finding refinable and wavelet masks with symmetric properties may be used for a wide class of matrices with an odd determinant.

References


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