Multivariate Periodic Wavelets [1]

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Abstract

A description of general construction of a multiresolution analysis of periodic functions with a matrix dilation and a method of finding wavelet biorthogonol bases are given. The convergence of expansions with respect to these bases is studied.

1. Introduction

Wavelet bases play an important role both for solving a number of applied problems and as a tool for approximation theory. In the late eighties, a method of construction of wavelet bases for $L_2(\mathbb{R})$ based on the structure of multiresolution analysis (MRA in the sequel) has been proposed in the papers of S. Mallat [2] and Y. Meyer [3]. The essence of the method looks as follows. MRA is generated by a function (called a scaling function) with a number of special properties. One creates another function (called a wavelet function) depending on the scaling function and such that its shifts and dilations constitute a wavelet basis for $L_2(\mathbb{R})$ (see, for example, [1], Chapter 5). There are different approaches for construction of multivariate wavelet bases. First, it is possible to take the tensor product of several one-dimensional wavelet bases. Such a way is simple but the obtained multivariate basis does not inherit all advantages of the generating univariate bases. In particular, the localisation property which is of great value for applied problems is not preserved. It is easy to see this on the Haar basis. In the one-dimensional case, a basis function with a large number has a small support. A basis function of the tensor product of two Haar systems (enumerated by the double index in a natural way) with an arbitrary large modulo of its number may have a support which is large in one of the directions. Secondly, to construct a d-dimensional wavelet basis, we can consider the tensor product of d one-dimensional MRA. Such a structure is similar to the one-dimensional MRA and generated by the tensor product of one-dimensional scaling functions. In this case, we have sevral wavelet functions whose shifts and dilations constitute a basis for $L_2(\mathbb{R}^d)$. A more general definition of multivariate multiresolution analysis has been given by Y. Meyer [3]. By this definition, MRA of $L_2(\mathbb{R}^d)$ is a collection of closed subspaces V_j, $j \in \mathbb{Z}$, of the space $L_2(\mathbb{R}^d)$ satisfying to following properties.

1. $V_j \subset V_{j+1}$, for any $j \in \mathbb{Z}$;
2. $\bigcup\limits_{j\in\mathbb{Z}} V_j$ is dense in $L_2(\mathbb{R}^d)$;
3. $\bigcap\limits_{j\in\mathbb{Z}} V_j = \{0\}$;
4. $f(x_1, \dots, x_d) \in V_0 \iff f(2^j x_1, \dots, 2^j x_d) \in V_j$;
5. there exists $\varphi \in V_0$ (scaling function) such that the functions $\varphi(\cdot + k)$, $k \in \mathbb{Z}$, constitute an orthonormal basis for the space V_0.

It turned out, that the problem of finding wavelet functions is much more complicated in the multi-dimensional case. Under different assumptions on the scaling function, this problem was considered by C. de Boor, R. DeVore and A. Ron [4], R.Q. Jia and C.A. Micchelli [5], [6], S.D. Riemenschneider and Z. Shen [7], [8]. In a most general case, an explicit description of the method for construction of wavelet functions has been obtained by R.Q. Jia and Z. Shen [11]. In the Meyer's definition of MRA, the scale factor is the diagonal matrix with twos on the diagonal, i.e. dilations over all direction are the same. Other scale factors are also of interest for some applied problems. A more general approach to multivariate MRA has been given, for example, in the book of P. Wojtaszczyk [18]. An integer matrixes satisfying some natural requirements is

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considered as the scale factor. A general scheme for construction of wavelet functions is presented. The problem is reduced to finding unitary matrix, whose elements are functions, from a given first line. Similarly, to find a biorthogonol pair of wavelet bases we need to construct two matrixes from a given pair of orthogonal function vectors being their first lines. In [9], [10], R.Q. Jia, S.D. Riemenschneider and Z. Shen have been given a description of an algorithm for constructions of orthogonal and biorthogonol compactly supported wavelet bases ,

Univariate periodic wavelets are usually defined as periodized wavelets in $L_2(\mathbb{R})$. Such an approach to periodic objects is not natural enough, the more so that it is possible to find in the literature periodic wavelets (for example, in the paper of C.K. Chui and H.N. Mhaskar [12]) which do not fit to such a definition. A definition of the multiresolution analysis of periodic functions (PMRA in the sequel) have been offered by number of authors (C.K. Chui and J. Wang [13]; V.A. Zheludev [14]; S.S. Goh, S.Z. Lee, Z. Shen and W.S. Tang [16]; A.P. Petukhov [15] etc.). A most general definition of PMRA of the spaces L_p, $1 < p < \infty$, and C has been offered by the author [19], where also a description of periodic wavelet bases and a method of their construction are given. Moreover, some conditions for convergences of Fourier series with respect to a wavelet system are found. All descriptions are given in terms of Fourier coefficients. This has an advetage over the non-periodic case in applied aspect because the problems become discrete.

The present work is devoted to description of multivariate PMRAs, construction of orthogonal and biorthogonol wavelet bases and expansions with respect to these bases. We consider $d \times d$-matrixes, where d is the dimension of the space, as the scale factor. Let M be an integer matrix such that the modules of all its eigenvalues are bigger than 1. We note that such a matrix, applied many times, provides dilation in all directions because

$$\lim_{n \to +\infty} \|M^{-n}\| = 0. \tag{1}$$

It follows from the facts that all spectrum of an operator (in the finite-dimensional space the spectrum coincides with the set of eigenvalues) is located in the circle $|\lambda| \le r(M^{-1})$, where $r(M^{-1}) := lim_{n\to\infty} \|M^{-n}\|^{1/n}$ is the spectral radius M^{-1}, and there exists at least one point of the spectrum on the boundary of the circle (see, for example, [17, page 267]). Since the modules of all eigenvalues of the matrix M^{-1} are strictly less than 1 and the set of the eigenvalues is finite, $r(M^{-1}) < 1$. It follows that the sequence $\|M^{-n}\|$ decreases faster than the geometric progression.

2. Notations and Preliminary Information

Let $\mathbb{N}$ be the set of positive integers, $\mathbb{R}^d$ denotes the d-dimensional Euclidean space, $x = (x_1, \dots, x_d)$, $y = (y_1, \dots, y_d)$ are its elements (vectors), $(x, y) = x_1y_1 + \ldots + x_dy_d$, $|x| = \sqrt{(x, x)}$, $\mathbf{0} = (0..., 0) \in \mathbb{R}^d$, $\mathbb{Z}^d$ is the integer lattice in $\mathbb{R}^d$, $\mathbb{Z}_+ = \{x \in Z^1 : \ x \ge 0\}$, $\mathbb{T}^d = [0, 1)^d$ is the d-dimensional single torus, δ_{lk} denotes the Kronecker delta. The space X is either $C(\mathbb{T}^d)$ or $L_p(\mathbb{T}^d)$, $1 \le p < \infty$; $\widehat{f}(k) = \int\limits_{\mathbb{T}^d} f(t)e^{-2\pi i(k,t)}\, dt$ is the k-th Fourier coefficient of $f \in X$; $\langle\, f, g\,\rangle = \int\limits_{\mathbb{T}^d} f\overline{g}$. We shall identify functions defined on $\mathbb{T}^d = [0, 1)^d$ with their periodic extensions defined on $\mathbb{R}^d$.

Let A be a non-degenerate integer $d \times d$-matrix, $\|A\|$ denotes its operator norm from $\mathbb{R}^d$ to $\mathbb{R}^d$, A^* is the conjugate matrix to A , $\det A$ is the determinant of A, E_d is the unit $d \times d$-matrix. We shal say that numbers $k, n \in \mathbb{Z}^d$ are congruent modulo A (write $k \equiv n \pmod A$) if $k - n = A\ell$, $\ell \in \mathbb{Z}^d$. The integer lattice $\mathbb{Z}^d$ is splited into cosets with respect to the introduced relation of congruence. The number of cosets is equal to $|\det A|$ (see, for example, [18, p. 107]). Let us take an arbitrary representative from each coset, call them digits and denote the set of digits by $D(A)$. A set K is said to be congruent to a set L modulo A (modulo $\mathbb{Z}^d$ as $A = E_d$) if the set K can be splited on a finite number of disjoint subsets: $K = \bigcup\limits_{n=1}^{N} K_n$, there exist integer vectors $\ell_1..., \ell_N$ so that $L = \bigcup\limits_{n=1}^{N} (K_n + A\ell_n)$ and the sets $(K_n + A\ell_n)$ are mutually disjoint. Evidently, if a set K

is congruent to a set L then L is congruent to K. Congruence modulo $\mathbb{Z}^d$ means that shifting parts of K to integer vectors we can "aggregate" L from them. Congruence modulo A means that shifting parts of K to vectors $A\ell$ with integer ℓ we can "to aggregate" L from them. Obviously, if two measurable sets are congruent to each other then they have equal measures.

Lemma 1 *Let A be a non-degenerate integer $d \times d$-matrix, then the set $K := \bigcup\limits_{r\in D(A)} (A^{-1}[0,1)^d + A^{-1}r)$ is congruent to $[0,1)^d$ modulo $\mathbb{Z}^d$ and*

$$\int\limits_{\mathbb{T}^d} f = \sum_{r\in D(A)} \int\limits_{A^{-1}[0,1)^d+A^{-1}r} f$$

for any $f \in L_1(\mathbb{T}^d)$.

`Proof.` Set $K_n := [n,\ n+1)^d \cap K$, $n \in \mathbb{Z}^d$. It is clear that $K_n \cap K_{n_1} = \emptyset$ as $n \neq n_1$, and $K = \bigcup\limits_n K_n$. Since the set K is bounded, the number of non-empty sets K_n is finite. Let us show that $K_n \cap (K_{n_1} - \ell) = \emptyset$ for any $\ell \in \mathbb{Z}^d$. Let $u \in K_n \subset K$, this means that $u = A^{-1}v + A^{-1}r$, where $v \in [0,1)^d$, $r \in D(A)$. Assume that there exists a vector $u_1 \in K_{n_1}$, $n \neq n_1$ such that $u = u_1 - \ell$, $\ell \in \mathbb{Z}^d$ or, what is the same, the difference $u - u_1$ is integer. Let $u_1 = A^{-1}v_1 + A^{-1}r_1$, $v_1 \in [0,1)^d$, $r_1 \in D(A)$, then $A^{-1}(v - v_1) + A^{-1}(r - r_1) = \ell$, $\ell \in \mathbb{Z}^d$. Multiplying this equality by A from the right, we get $v - v_1 + r - r_1 = A\ell$, $\ell \in \mathbb{Z}^d$. Since the vectors r, r_1 and $A\ell$ are integer and the vectors v and v_1 are in $[0,1)^d$, the equality can hold only if $v = v_1$. However the vectors r, r_1 are in $D(A)$, hence they can not be congruent to each other modulo A. Therefore, $r = r_1$, i.e. we obtained $u = u_1$, what contradicts to the assumption $n \neq n_1$. Now we show that $[0,1)^d = \bigcup\limits_n (K_n - n)$. Note that $K_n - n \subset [0,1)^d$ for any n by the definition of K_n. Moreover, $(K_n - n) \cap (K_{n_1} - n_1) = \emptyset$. Let us check that, for any $u \in [0,1)^d$, there exist $\in \mathbb{Z}^d$ and $w \in K_n$ such that $u = w - n$. Multiplying the vector u by the matrix A from the left, we represent Au in the form $Au = p + v$, where $p \in \mathbb{Z}^d$, $v \in [0,1)^d$. The vector p is congruent to one of digits, i.e. there are $r \in D(A)$ and $l \in \mathbb{Z}^d$ such that $p = r + Al$. Expressing u, we obtain $u = A^{-1}r + l + A^{-1}v$, hence $u - l \in K$. Therefore, $u - l \in K_n$ for some $n \in \mathbb{Z}^d$, and $n = -l$. By setting $w = u - l = u + n$, we get $w \in K_n$.

The second statement of Lemma is evidently follows from 1-periodicity of the function f. ■

Lemma 2 *([22]). Let A be a non-degenerate integer $d \times d$-matrix, $|\det A| > 1$. Then*

$$\sum_{s\in D(A^*)} e^{2\pi i(A^{-1}r,\ s)} = \begin{cases} |\det A|, & \text{if } r \equiv \mathbf{0} \pmod{A}, \\ 0, & \text{if } r \not\equiv \mathbf{0} \pmod{A}. \end{cases} \tag{2}$$

For completeness of the presentation, we give the proof of this lemma.

`Proof.` Set $m := |\det A| = |\det A^*|$. The cosets with respect to A^* form a group in regard to addition which consists of m elements. If $r \equiv \mathbf{0} \pmod{A}$, i.e. $r = Al$, $l \in \mathbb{Z}^d$, then (2) is obvious. If $r \not\equiv \mathbf{0} \pmod{A}$, we take any vector $a \in \mathbb{Z}^d$ so that $(A^{-1}r, a) \notin \mathbb{Z}$, then, in particular, $a \not\equiv \mathbf{0} \pmod{A^*}$. Consider the vectors a, $2a$, $3a$.... Let m_1 be the minimal positive integer such that $m_1 a \equiv 0 \pmod{A^*}$, then $1 < m_1 \leq m$ (if all elements ka, $k = 1, \dots, m$ are not congruent to zero, then they are mutually non-congruent to each other, but in the group there are only m elements). Thus, the vectors a, $2a..., m_1 a$ form a subgroup consisting of m_1 elements. Hence, by Lagrange theorem, m is divisible by m_1. Let $m = m_1 n$ and let $a_1 = 0$; $a_2..., a_n$ be representatives of elements from the corresponding factor group. Then $a_k + ja$, $k = 1...n$, $j = 1...m_1$ runs over the elements of the group, and

$$\sum_{k=1}^{m} e^{2\pi i(A^{-1}r, s_k)} = \sum_{k=1}^{n}\sum_{j=1}^{m_1} e^{2\pi i(A^{-1}r, a_k + ja)} =$$

$$\sum_{k=1}^{n} e^{2\pi i(A^{-1}r,a_k)} \sum_{j=1}^{m_1} e^{2\pi i(A^{-1}r,ja)} = C\frac{1-e^{2\pi i m_1(A^{-1}r,a)}}{1-e^{2\pi i(A^{-1}r,a)}}$$

Due to $m_1 a = A^* l,\ l \in \mathbb{Z}^d$ and the assumption $(A^{-1}r, a) \notin \mathbb{Z}$ it follows that the fraction in the right hand side is equal to zero. ■

Corollary 3 *Under the assumptions of Lemma 2, the matrix* $\{e^{2\pi i(A^{-1}n,r)}\}_{n\in D(A), r\in D(A^*)}$ *is unitary up to the factor* $\sqrt{|\det A|}$*.*

Indeed, the inner product of columns with the numbers n_1, n_2 is equal to

$$\sum_{r\in D(A^*)} e^{2\pi i(A^{-1}(n_1-n_2),r)}.$$

By Lemma 2, this sum is equal to $|\det A|$ as $n_1 = n_2$, otherwise it is equal to zero because $n_1 \not\equiv n_2 \pmod{A}$. ■

Lemma 4 *Let A be a non-degenerate integer $d\times d$-matrix, $|\det A| > 1$. Then the set $\{r + A^j p\}$, $r \in D(A^j)$, $p \in D(A)$, is the set of digits of the matrix A^{j+1}.*

`Proof.` The number of all possible pairs $(r,\ p)$ with $r \in D(A^j)$ and $p \in D(A)$ is equal to $|detA|^{j+1}$. So it suffices to prove that the vectors from different pairs can not be congruent modulo A^{j+1}. Let $r,\ r_1 \in D(A^j)$ and $p,\ p_1 \in D(A)$. Assume that $r + A^j p$ and $r_1 + A^j p_1$ are congruent to each other modulo A^{j+1}, hence $(r - r_1) + A^j(p - p_1) = A^{j+1}n$ for some $n \in \mathbb{Z}^d$. Multiplying both the sides of the equality by A^{-j} form the left, we get $A^{-j}(r - r_1) = -(p - p_1) + An \in \mathbb{Z}^d$, i.e. $r \equiv r_1 \pmod{A^j}$. However, r and r_1 belong to $D(A^j)$, where there is only one representative of each coset, so $r = r_1$. It follows from the equality $(p - p_1) = An$ that $p = p_1$. Thus, $r + A^j p$ and $r_1 + A^j p_1$ can be congruent to each other modulo A^{j+1} if and only if $r = r_1$ and $p = p_1$. ■

Throughout the paper M denotes a fixed integer $d \times d$-matrix such that the modules of all its eigenvalues are bigger than 1, $m := |\det M|$. For such a matrix, obviously, $m \in \mathbb{Z}$, $m > 1$ and, as it has been noted above,

$$\lim_{n\to\infty} |M^n x| = \infty \tag{3}$$

for all $x \in \mathbb{R}^d$, $x \neq \mathbf{0}$.

Define a shift operator S_p^j, $p \in \mathbb{Z}^d$, $j \in \mathbb{Z}_+$, on X by

$$S_p^j f(x) := f(x + M^{-j}p). \tag{4}$$

3. PMRA and Scaling Sequence

Definition 5 *Let $V_j \subset X,\ j \in \mathbb{Z}_+$. The collection $\{V_j\}_{j=0}^{\infty}$ is said to be a PMRA of X if the following properties (axioms) hold.*

$MR1.$ $V_j \subset V_{j+1}$,

$MR2.$ $\overline{\bigcup_{j=0}^{\infty} V_j} = X$,

$MR3.$ $dim V_j = m^j$,

$MR4.$ $dim\{f \in V_j : S_n^j f = \lambda_n f\ \forall n \in \mathbb{Z}^d\} \leq 1 \quad \forall \{\lambda_n\}_{n\in\mathbb{Z}^d}$,

$MR5.$ $f \in V_j \Leftrightarrow S_n^j f \in V_j\ \forall n \in \mathbb{Z}^d$,

$MR6.$ $a) f \in V_j \Rightarrow f(M\cdot) \in V_{j+1}$

$b) f \in V_{j+1} \Rightarrow \sum_{s\in D(M)} f(M^{-1}\cdot + M^{-1}s) \in V_j$

Remark. Because of the periodicity of f, it is possible to consider only the digits of the matrix M^j instead of all $n \in \mathbb{Z}^d$ in the conditions $MR4$ and $MR5$. So, these conditions can be replaced by

$MR4'$. $dim\{f \in V_j : S_r^j f = \lambda_r f \ \ \forall r \in D(M^j)\} \le 1 \quad \forall \{\lambda_r\}_{r \in D(M^j)}$.
$MR5'$. $f \in V_j \Leftrightarrow S_n^j f \in V_j \ \forall n \in D(M^j)$.

Definition 6 *Let V_j be a PMRA of X. A sequence of functions $\{\varphi_j\}_{j=0}^{\infty}$, $\varphi \in V_j$ is called a scaling sequence if $\{S_n^j \varphi_j\}_{n \in D(M^j)}$ is a basis for the space V_j.*

Theorem 7 *Functions $\varphi_j \in X$, $j \in \mathbb{Z}_+$ constitute a scaling sequence for a PMRA of X if and only if*

$\Phi1.$ $\widehat{\varphi}_0(k) = 0, \ \ \forall k \neq 0$;
$\Phi2.$ $\forall j \in \mathbb{Z}_+ \quad \forall n \in \mathbb{Z}^d \ \ \exists\, k \equiv n \pmod{M^{*j}} : \widehat{\varphi}_j(k) \neq 0$;
$\Phi3.$ $\forall k \in \mathbb{Z}^d \ \ \exists j \in \mathbb{Z}_+ : \ \ \widehat{\varphi}_j(k) \neq 0$;
$\Phi4.$ $\forall j \in \mathbb{N} \quad \forall n \in \mathbb{Z}^d \ \ \exists \mu_n^j : \widehat{\varphi}_{j-1}(k) = \mu_n^j \widehat{\varphi}_j(k) \ \ \forall k \equiv n \pmod{M^{*j}}$;
$\Phi5.$ $\forall j \in \mathbb{Z}_+ \quad \forall n \in \mathbb{Z}^d \ \ \exists \gamma_n^j \neq 0 : \ \gamma_n^j \widehat{\varphi}_j(k) = \widehat{\varphi}_{j+1}(M^* k) \ \ \forall k \equiv n \pmod{M^{*j}}$.

We preface a number of auxiliary statements with the proof of the theorem.

Lemma 8 *Let $V_j \subset X$, $j \in \mathbb{Z}_+$, and the axioms $MR1$, $MR2$, $MR3$, $MR5$, $MR6$ of Definitions 5 are valid, then $V_0 = \{const\}$.*

`Proof.` By $MR3$, the space V_0 is one-dimensional. Let $f \in V_0$, $\|f\| \neq 0$. Show that $\widehat{f}(\mathbf{0}) \neq 0$. Assume that $\widehat{f}(\mathbf{0}) = 0$. Introduce the following operator $A: \ \ Af = \sum\limits_{s \in D(M)} f(M^{-1} \cdot + M^{-1}s)$, and set $g = Af$. By Lemma 1,

$$\widehat{g}(\mathbf{0}) = \sum_{s \in D(M)} \int_{\mathbb{T}^d} f(M^{-1}x + M^{-1}s)dx = m \sum_{s \in D(M)} \int_{M^{-1}\mathbb{T}^d + M^{-1}s} f(z)dz = m \int_{\mathbb{T}^d} f(z)dz = m\widehat{f}(0).$$

Let $g_0 \in V_j$, then $g_1 := Ag_0 \in V_{j-1} ..., g_j := Ag_{j-1} \in V_0$. If $\widehat{g}_j(\mathbf{0}) \neq 0$, then we have a contradiction to the one-dimensionality of V_0 because $\widehat{f}(\mathbf{0}) = 0$. If $\widehat{g}_0(\mathbf{0}) = 0$, then any function from V_j has zero mean value, what contradicts the axiom $MR2$.

Now we assume that $\widehat{f}(n) \neq 0$ for some $n \neq \mathbf{0}$. Set $f_1 := Af$. Since $f \in V_0$, by $MR1$, $f \in V_1$. Hence, by $MR6(b)$, $f_1 \in V_0$. Since V_0 is one-dimensional, $f_1 = \lambda f$ This implies $\widehat{f}_1(n) = \lambda \widehat{f}(n)$. However, $\lambda = m$ because $\widehat{f}_1(0) = m\widehat{f}(0)$. On the other hand, direct computation with using Lemma 1 gives

$$\widehat{f}_1(n) = \sum_{s \in D(M)} \int_{\mathbb{T}^d} f(M^{-1}x + M^{-1}s) e^{2\pi i (x,n)} dx =$$

$$m \sum_{s \in D(M)} \int_{M^{-1}\mathbb{T}^d + M^{-1}s} f(z) e^{2\pi i (Mz - s, n)} dz = m \int_{\mathbb{T}^d} f(z) e^{2\pi i (Mz, n)} dz = m\widehat{f}(M^* n).$$

Therefore,

$$\widehat{f}(n) = \widehat{f}(M^* n) = ... = \widehat{f}(M^{l*} n) = ...,$$

However, since the Fourier coefficients of $f \in X$ tend to zero as the modules of their numbers increase, taking into account (3), we get a contradiction. ■

By recursion, define operators ω_n^j, $j \in \mathbb{Z}_+$ $n \in \mathbb{Z}^d$ acting on X:

$$\begin{aligned} \omega_n^0 f &:= f, \\ \omega_n^{j+1} f(x) &:= \frac{1}{m} \sum_{s \in D(M)} e^{-2\pi i (M^{-j-1}s, n)} \omega_n^j f(x + M^{-j-1}s). \end{aligned}$$

Lemma 9 *Let $V_j \subset X$, $j \in \mathbb{Z}_+$, and the axiom $MR5$ of Definitions 5 holds. If $f \in V_{j_0}$, then $\omega_n^j f \in V_{j_0}$ for all $j = 0, \dots, j_0$, $n \in \mathbb{Z}^d$, and*

$$\omega_n^j f \sim \sum_{m \in \mathbb{Z}^d} \widehat{f}(M^{*j} m + n) e^{2\pi i (M^{*j} m + n, \cdot)}, \tag{5}$$

i.e. $\widehat{\omega_n^j f}(k) = \widehat{f}(k)$ *as* $k \equiv n \pmod{M^{*j}}$, *and* $\widehat{\omega_n^j f}(k) = 0$ *as* $k \not\equiv n \pmod{M^{*j}}$.

`Proof.` We shall prove by induction on j. The inductive base for $j = 0$ is obvious. Let $\omega_n^j f \in V_{j_0}$, $0 \le j < j_0$, and (5) holds for all $n \in \mathbb{Z}^d$. It follows from $MR5$ that $\omega_n^j f(\cdot + M^{-j-1} s) = \omega_n^j f(\cdot + M^{-j_0}(M^{j_0 - j - 1} s) \in V_{j_0}$, This implies $\omega_n^{j+1} f \in V_{j_0}$. Further, we have

$$\widehat{\omega_n^{j+1} f}(k) = \frac{1}{m} \int_{\mathbb{T}^d} \sum_{s \in D(M)} e^{-2\pi i (M^{-j-1} s, n)} \omega_n^j f(x + M^{-j-1} s) e^{-2\pi i (x, k)} dx =$$

$$\frac{1}{m} \sum_{s \in D(M)} e^{-2\pi i (M^{-j-1} s, n)} \int_{\mathbb{T}^d} \omega_n^j f(t) e^{-2\pi i (t - M^{-j-1} s, k)} dt = \frac{1}{m} \sum_{s \in D(M)} e^{-2\pi i (M^{-j-1} s, n-k)} \widehat{\omega_n^j f}(k)$$

If $k \equiv n \pmod{M^{*j+1}}$, then the sum on the right hand side is, obviously, equal to m. So, $\widehat{\omega_n^{j+1} f}(k) = \widehat{\omega_n^j f}(k)$. If $k \not\equiv n \pmod{M^{*j+1}}$ and $n \equiv k \pmod{M^{*j}}$, i.e. $n - k = M^{*j} l$, where the vector l is not congruent to zero modulo M^*, then, by Lemma 2, the latter sum is equal to zero. Finally, if $k \not\equiv n \pmod{M^{*j}}$, then, by the inductive hypothesis, $\widehat{\omega_n^j f}(k) = 0$, so and $\widehat{\omega_n^{j+1} f}(k) = 0$. ■

Lemma 10 *Let V_j be a PMRA of X, then, in each space V_j, there exists a basis $\{v_n^j\}_{n \in D(M^{*j})}$, satisfying the following conditions:*

V1. $\widehat{v}_n^j(k) = 0 \quad \forall k \not\equiv n \pmod{M^{*j}}$;
V2. if $\widehat{v}_n^j(k) \ne 0$, *then* $\widehat{v}_n^j(\ell) = \widehat{v}_n^{j+1}(\ell) \quad \forall \ell \equiv k \pmod{M^{*j+1}}$;
V3. $\widehat{v}_n^j(k) = \widehat{v}_{M^* n}^{j+1}(M^* k) \quad \forall k \in \mathbb{Z}^d$.

For ease, we set $\widehat{v}_l^j := \widehat{v}_n^j$ for $l \equiv n \pmod{M^j}$.

`Proof.` We prove by induction on j. The inductive base for $j = 0$ is obvious because all integer vectors are congruent to each other as j=0. Assume that in the spaces V_j, $j = 0, \dots, j_0$ there exist bases satisfying $V1$, $V2$, $V3$. Introduce the following spaces. $V_j^{(n)} := \{f \in V_j : \widehat{f}(k) = 0 \ \ \forall k \not\equiv n \pmod{M^{*j}}\}$. Let $F \in V_j$, then

$$F = \sum_{n \in D(M^{*j})} \omega_n^j F = \sum_{n \in D(M^{*j})} F_n, \quad F_n \in V_j^{(n)}.$$

This means that $V_j = \sum_{n \in D(M^{*j})} V_j^{(n)}$. Therefore,

$$m^j = dim V_j \le \sum_{n \in D(M^{*j})} dim V_j^{(n)}. \tag{6}$$

Let us find the dimension of $V_j^{(n)}$. If $f \in V_j^{(n)}$, then

$$f(x) \sim \sum_{m \in \mathbb{Z}^d} \widehat{f}(M^{*j} m + n) e^{2\pi i (M^{*j} m + n, x)}.$$

Applying the shift operator, we get

$$(S_p^j f)(x) \sim \sum_{m \in \mathbb{Z}^d} \widehat{f}(M^{*j} m + n) e^{2\pi i (M^{*j} m + n, x + M^{-j} p)} \sim e^{2\pi i (n, M^{-j} p)} f(x),$$

i.e. $S_p^j f(x) = e^{2\pi i(n, M^{-j}p)} f(x)$ for all $p \in \mathbb{Z}^d$. From this, by $MR4$, taking into account (6), we have $dim V_j^{(n)} = 1$.

Construct the basis $\{v_n^{j_0+1}\}$. If $\widehat{v}_k^{j_0}(k) \neq 0$, we set $v_k^{j_0+1} := \omega_k^{j_0+1} v_k^{j_0}$. The properties $V1$, $V2$ are valid by Lemma 9. Let us check $V3$. If $k \equiv \mathbf{0} \pmod{M^*}$, $n \equiv k \pmod{{M^*}^{j_0+1}}$, then, by the inductive hypothesis,

$$\widehat{v}_k^{j_0+1}(n) = \widehat{v}_k^{j_0}(n) = \widehat{v}_{{M^*}^{-1}k}^{j_0-1}({M^*}^{-1}n) = \widehat{v}_{{M^*}^{-1}k}^{j_0}({M^*}^{-1}n).$$

So we defined the basis functions with the numbers k for which there exists $n \equiv k \pmod{{M^*}^{j_0+1}}$, such that $\widehat{v}_k^{j_0}(n) \neq 0$. Now suppose that $\widehat{v}_k^{j_0}(n) = 0$ for all $n \equiv k \pmod{{M^*}^{j_0+1}}$ and $k \equiv \mathbf{0} \pmod{M^*}$. In theis case we set $v_k^{j_0+1}(x) := v_{{M^*}^{-1}k}^{j_0}(Mx)$. Using Lemma 1, we have

$$\begin{aligned}
\widehat{v}_k^{j_0+1}(n) = \int\limits_{\mathbb{T}^d} v_k^{j_0+1}(x) e^{-2\pi i(x,n)} dx = \int\limits_{\mathbb{T}^d} v_{{M^*}^{-1}k}^{j_0}(Mx) e^{-2\pi i(x,n)} dx &= \\
\sum_{s \in D(M)} \int\limits_{M^{-1}\mathbb{T}^d + M^{-1}s} v_{{M^*}^{-1}k}^{j_0}(Mx) e^{-2\pi i(x,n)} dx &= \\
\frac{1}{m} \sum_{s \in D(M)} \int\limits_{\mathbb{T}^d + s} v_{{M^*}^{-1}k}^{j_0}(t) e^{-2\pi i(t, {M^*}^{-1}n)} dt &= \widehat{v}_{{M^*}^{-1}k}^{j_0}({M^*}^{-1}n).
\end{aligned}$$

It is clear that the property $V3$ is fulfilled, $V1$ is valid due to the inductive hypothesis, Finally, let $\widehat{v}_k^{j_0}(n) = 0$ for all $n \equiv k \pmod{{M^*}^{j_0+1}}$ and $k \not\equiv \mathbf{0} \pmod{M^*}$. In this case we can use any non-zero element of the space $V_{j_0+1}^{(k)}$ as $v_k^{j_0+1}$. The property $V1$ follows from the definition of $V_{j_0+1}^{(k)}$, there is nothing to prove for $V2$ and $V3$. ∎

Remark. If only the axioms $MR1$, $MR3$, $MR4$, $MR5$ of Definitions 5 hold for a sequence of subspaces $V_j \subset X$, $j \in \mathbb{Z}_+$, then, in each space V_j, there exist a basis $\{v_n^j\}_{n \in D({M^*}^j)}$ satisfying the conditions $V1$, $V2$. The proof can be realized by the same scheme using any non-zero element of the space V_0 as v_0^0, and any non-zero element of the space $V_{j_0+1}^{(k)}$ as $v_k^{j_0+1}$, in the case when $\widehat{v}_k^{j_0}(n) = 0$ for all $n \equiv k \pmod{{M^*}^{j_0+1}}$ and $k \equiv \mathbf{0} \pmod{M^*}$.

Lemma 11 *If in each space $V_j \subset X$, $j \in \mathbb{Z}_+$, there exists a basis $\{v_n^j\}_{n \in D({M^*}^j)}$, satisfying the condition $V1$ of Lemma 10, then the axiom $MR4$ of Definitions 5 is fulfilled.*

`Proof`. Taking into account the remark to Definition 5, it suffices to check that $MR4'$ is valid. Let $r \in D(M^j)$, f the eigenvector of the operator S_r^j, i.e. $S_r^j f = \lambda_r f$, and let $f = \sum_{n \in D({M^*}^j)} \alpha_n^j v_n^j$. Since $V1$ holds, the operator S_r^j acts on v_n^j as an operator of multiplication by $e^{2\pi i({M^*}^{-j}n, r)}$. Applying S_r^j to the function f we get

$$S_r^j f(x) = \sum_{n \in D({M^*}^j)} \alpha_n^j S_n^j v_n^j(x) = \sum_{n \in D({M^*}^j)} \alpha_n^j e^{2\pi i({M^*}^{-j}n, r)} v_n^j(x).$$

Taking into account that the function f is an eigenvector of S_r^j, we have

$$0 = S_r^j f(x) - \lambda_r f(x) = \sum_{n \in D({M^*}^j)} \alpha_n^j [e^{2\pi i({M^*}^{-j}n, r)} - \lambda_r] v_n^j(x).$$

It follows from the linear independence of v_n^j that $\alpha_n^j [e^{2\pi i({M^*}^{-j}n, r)} - \lambda_r] = 0$ for all $n \in D({M^*}^j)$. Assume that $\alpha_{n_0}^j \neq 0$, $\alpha_{n_1}^j \neq 0$ for two various numbers $n_0 \neq n_1$. Subtracting, we get

$$(e^{2\pi i({M^*}^{-j}n_0, r)} - \lambda_r) - (e^{2\pi i({M^*}^{-j}n_1, r)} - \lambda_r) = 0$$

or $e^{2\pi i(M^{*-j}(n_0-n_1),r)}=1$. Now let f is an eigenvector of all operators S_r^j, $r\in D(M^j)$. Summing these equalities over all r, we have

$$\sum_{r\in D(M^j)} e^{2\pi i(M^{*-j}(n_0-n_1),r)} = m^j. \tag{7}$$

On the other hand, since n_0 is not congruent to n_1 modulo M^{*j}, by Lemma 2,

$$\sum_{r\in D(M^{*j})} e^{2\pi i(M^{*-j}(n_0-n_1),r)} = 0, \tag{8}$$

what contradicts (7). Thus α_n^j differs from zero only if $n=n_0$. Hence f is proportional to $v_{n_0}^j$. Moreover, it follows from the above arguments that

$$\lambda_r = e^{2\pi i(M^{*-j}n_0,r)}, \quad r\in D(M^j). \tag{9}$$

Let $g=\sum\limits_n \beta_n^j v_n^j$ be another eigenvector of all operators S_r^j, $r\in D(M^j)$. For this vector, an only β_n^j differs from zero as well. Suppose that $\beta_{n_1}^j\neq 0$, $n_1\neq n_0$. Similarly to (9), we have $\lambda_r = e^{2\pi i(M^{*-j}n_1,r)}$, $r\in D(M^j)$. From this and (9) we get $e^{2\pi i(M^{*-j}(n_0-n_1),r)}=1$ for any $r\in D(M^j)$. Summing these equalities over all r, we obtain (7), what contradicts (8). Therefore g is also proportional to $v_{n_0}^j$, and the dimension of the subspace consisting of all such functions does not exceed 1. ■

Proposition 12 *Let V_j be a PMRA of X. For a sequence $\{\varphi_j\}_{j=0}^\infty\subset X$ to be a scaling sequence it is necessary and sufficient to have*

$$\varphi_j = \sum_{n\in D(M^{*j})} \alpha_n^j v_n^j, \quad \alpha_n^j\neq 0 \quad \forall n\in D(M^{*j}), \tag{10}$$

where $\{v_n^j\}_{n\in D(M^{*j})}$ is the basis for V_j, defined in Lemma 10.

`Proof.` Necessity. Let $\{\varphi_j\}_{j=0}^\infty$ be a scaling sequence and let α_n^j, $n\in D(M^{*j})$ be a coefficient of expansion φ_j with respect to the basis $\{v_n^j\}_{n\in D(M^{*j})}$. Applying the shift operator S_r^j, $r\in D(M^j)$, to φ_j we have

$$S_r^j\varphi_j = \sum_{n\in D(M^{*j})} \alpha_n^j e^{2\pi i(M^{*-j}n,r)} v_n^j. \tag{11}$$

Assume that $\alpha_{n_0}^j=0$ for some n_0, then

$$V_j = span\{S_r^j\varphi_j,\ \ r\in D(M^j)\} = span\{v_n^j,\ \ n\in D(M^{*j}),\ n\neq n_0\},$$

what contradicts minimality of the basis $\{v_n^j\}_{n\in D(M^{*j})}$.

Sufficiency. Let φ_j be defined by the formula (10). As above, we have

$$S_r^j\varphi_j = \sum_{n\in D(M^{*j})} \alpha_n^j e^{2\pi i(M^{*-j}n,r)} v_n^j, \quad r\in D(M^j).$$

Consider these equalities as a system of equations with unknowns $\alpha_n^j v_n^j$. By Corollary 3, the matrix of this system is unitary (up to a factor). Since the function $\alpha_n^j v_n^j$ constitute a basis, and an unitary transformation takes a basis to a basis, the functions $S_r^j\varphi_j$, $r\in D(M^j)$ constitute a basis for the space V_j. ■

Corollary 13 *If $\{\varphi_j\}_{j=0}^\infty$ is a scaling sequence, then $\omega_n^j\varphi_j=\alpha_n^j v_n^j$, where $\alpha_n^j\neq 0$.*

The statement follows from the fact that $dim V_j^{(n)} = 1$ ($V_j^{(n)}$ are the spaces defined in Lemma 10), and (5).

Corollary 14 *There exists a scaling sequence in any PMRA .*

For the proof it suffices to set $\alpha_n^j = 1$ in (10).

`Proof of Theorem 7.` Necessity. The property $\Phi1$ follows from Lemma 8. To prove $\Phi2$ we use Corollary 13. From the equality $\omega_n^j \varphi_j = \alpha_n^j v_n^j$ we have $\widehat{\varphi}_j(k) = \widehat{\omega_k^j \varphi_j}(k) = \alpha_n^j \widehat{v}_n^j(k)$ for any $k \equiv n \pmod{M^{*j}}$. Existence of $k \equiv n \pmod{M^{*j}}$ such that $\widehat{v}_n^j(k) \neq 0$ follows from the relations $v_n^j \not\equiv 0$ (because v_n^j is a basis vector) and $\widehat{v}_n^j(\ell) = 0$ for all $\ell \not\equiv n \pmod{M^{*j}}$. Taking into account that $\alpha_n^j \neq 0$, we get $\widehat{\varphi}_j(k) \neq 0$. Prove the property $\Phi3$ by contradiction. Assume that $\widehat{\varphi}_j(k) = 0$ for all $j \in \mathbb{Z}_+$. This means that all functions φ_j do not have the harmonic $e^{2\pi i(k,x)}$. In this case all shifts of $S_k^j \varphi_j$ do not have the same harmonic, i.e. the inner product $< f,\ e^{2\pi i(k,x)} >$ is equal to zero for any function $f \in \bigcup_{j=0}^{\infty} V_j$, what contradicts completeness of the union of the spaces V_j (axiom $MR4$). For the proof $\Phi4$, we take an arbitrary $n \in \mathbb{Z}^d$. First consider the case when there exists $k \equiv n \pmod{M^{*j}}$ such that $\widehat{\varphi}_{j-1}(k) \neq 0$. By Corollary 13, $\omega_n^j \varphi_j = \alpha_n^j v_n^j$, $\omega_n^j \varphi_{j-1} = \omega_n^j \omega_n^{j-1} \varphi_{j-1} = \alpha_n^{j-1} \omega_n^j v_n^{j-1}$, $\alpha_n^j, \alpha_n^{j-1} \neq 0$. From this, taking into account the property $V2$ from Lemma 10, we have

$$\widehat{\varphi}_{j-1}(\ell) = \frac{\alpha_n^{j-1}}{\alpha_n^j} \widehat{\varphi}_j(\ell)$$

for all $\ell \equiv n \pmod{M^{*j}}$. It remains to set $\mu_n^j = \alpha_n^{j-1}/\alpha_n^j$. If $\widehat{\varphi}_{j-1}(k) = 0$ for any $k \equiv n \pmod{M^{*j}}$, we set $\mu_n^j = 0$. For the proof $\Phi5$, we again use Corollary 13. If $k \equiv n \pmod{M^{*j}}$, we have

$$\widehat{\varphi}_{j+1}(M^* k) = \widehat{\omega_{M^*n}^{j+1} \varphi_{j+1}}(M^* k) = \alpha_{M^*n}^{j+1} \widehat{v}_{M^*n}^{j+1}(M^* k),$$

where $\alpha_{M^*n}^{j+1} \neq 0$. On the other hand, $\widehat{\varphi}_j(n) = \alpha_k^j \widehat{v}_n^j(k)$, $\alpha_n^j \neq 0$. From this, taking into account the property $V3$ from Lemma 10, we get

$$\widehat{\varphi}_{j+1}(M^* k) = \frac{\alpha_{M^*n}^{j+1}}{\alpha_n^j} \widehat{\varphi}_j(k).$$

It remains to set $\gamma_n^j = \alpha_{M^*n}^{j+1}/\alpha_n^j$.

Sufficiency. Let the functions $\varphi_j \in X$ satisfy the properties $\Phi1-\Phi5$. Set $V_j = span\{S_n^j \varphi_j,\ \ n \in D(M^j)\}$, and show that $\{V_j\}_{j=0}^{\infty}$ is a PMRA of X and $\{\varphi_j\}_{j \in \mathbb{Z}_+}$ is a scaling sequence. First we check $MR5$. Let $f \in V_j$, then $f = \sum_{k \in D(M^j)} \alpha_k S_k^j \varphi_j$. Applying the shift operator S_p^j, $p \in D(M^j)$, to this equality, we have

$$S_p^j f = \sum_{k \in D(M^j)} \alpha_k S_p^j S_k^j \varphi_j. \tag{12}$$

It follows from the periodicity of f that $S_p^j S_n^j f = S_r^j f$, where $r \in D(M^j)$, $r \equiv (p+n) \pmod{M^j}$. Substituting these equalities to (12), by definition of V_j, we get $S_p^j f \in V_j$. If $S_p^j f \in V_j$, then, as has been proved already, $f = S_{-p}^j S_p^j f \in V_j$. Next we prove $MR3$. Similarly to (11), taking into account Lemma 9, we have

$$S_r^j \varphi_j = \sum_{n \in D(M^{*j})} e^{2\pi i (M^{*-j} n, r)} \omega_n^j \varphi_j, \tag{13}$$

It follows that $V_j = span\{\omega_n^j \varphi_j,\ \ n \in D(M^{*j})\}$. Show that the functions $\omega_n^j \varphi_j$, $n \in D(M^{*j})$ constitute a basis for V_j. Suppose that these functions are linearly dependent. In this case there exist numbers α_k, $k \in D(M^{*j})$, $\alpha_{k_0} \neq 0$ such that $\sum_{k \in D(M^{*j})} \alpha_k \omega_k^j \varphi_j = 0$. By Lemma 9, from this we obtain $\widehat{\omega_n^j\varphi_j}(k)=\widehat{\varphi}_j(k)=0$ for all $k\equiv k_0 \pmod{M^{*j}}$, what contradicts $\Phi2$. For the proof of $MR3$, it remains to note that the number of the functions $\omega_n^j\varphi_j$ is equal to m^j. Since the number of the functions $S_n^j\varphi_j, n\in D(M^j)$ is also equal to m^j, we have proved that $\{S_n^j\varphi_j\}_{n\in D(M^j)}$ is a basis for the space V_j. For the proof of $MR1$, it is necessary to check that $f\in V_j$ implies $f\in V_{j+1}$. It suffices to consider only the basis functions $\omega_n^j\varphi_j, n\in D(M^{*j})$. It follows from Lemmas 9 and 4 that

$$\omega_n^j\varphi_j=\sum_{p\in D(M^*)}\omega_{n+M^{*j}p}^{j+1}\omega_n^j\varphi_j. \tag{14}$$

Using Lemma 9 and $\Phi4$, taking into account the M^{*j+1}-periodicity of the sequence μ_n^{j+1} with respect to the subindex, we have

$$\omega_{n+M^{*j}p}^{j+1}\omega_n^j\varphi_j\sim\sum_{k\in\mathbb{Z}^d}\widehat{\omega_n^j\varphi_j}(M^{*j+1}k+n+M^{*j}p)e^{2\pi i(M^{*j+1}k+n+M^{*j}p,x)}=$$

$$\sum_{k\in\mathbb{Z}^d}\widehat{\varphi}_j(M^{*j+1}k+n+M^{*j}p)e^{2\pi i(M^{*j+1}k+n+M^{*j}p,x)}=$$

$$\sum_{k\in\mathbb{Z}^d}\mu_{M^{*j+1}k+n+M^{*j}p}^{j+1}\widehat{\varphi}_{j+1}(M^{*j+1}k+n+M^{*j}p)e^{2\pi i(M^{*j+1}k+n+M^{*j}p,x)}=$$

$$\mu_{n+M^{*j}p}^{j+1}\sum_{k\in\mathbb{Z}^d}\widehat{\varphi}_{j+1}(M^{*j+1}k+n+M^{*j}p)e^{2\pi i(M^{*j+1}k+n+M^{*j}p,x)}\sim\mu_{n+M^{*j}p}^{j+1}\omega_{n+M^{*j}p}^{j+1}\varphi_{j+1}.$$

It follows that

$$\omega_n^j\varphi_j=\sum_{p\in D(M^*)}\mu_{n+M^{*j}p}^{j+1}\omega_{n+M^{*j}p}^{j+1}\varphi_{j+1}. \tag{15}$$

It remains to note that, by Lemma 9, $\omega_{n+M^{*j}p}^{j+1}\varphi_{j+1}\in V_{j+1}$.

Before the proof of remaining properties, we shall show that there exists a basis for V_j satisfying the properties of Lemma 10. Define the numbers α_n^j, $j\in\mathbb{Z}_+$; $n\in D(M^{*j})$ by recursion on j:
$\alpha_0^0:=1$;
if $\mu_n^j\neq 0$, then $\alpha_n^j:=\alpha_n^{j-1}/\mu_n^j$;
if $\mu_n^j=0$, then $n\equiv\mathbf{0}\pmod{M^*}$, $\alpha_n^j:=\alpha_{M^{*-1}n}^{j-1}\gamma_{M^{*-1}n}^{j-1}$;
if $\mu_n^j=0$, $n\not\equiv\mathbf{0}\pmod{M^*}$, then $\alpha_n^j:=1$.
By construction, it is clear that $\alpha_n^j\neq 0$. Set $v_n^j=\omega_n^j\varphi_j/\alpha_n^j$. As it has been already shown, $\omega_n^j\varphi_j$ constitute a basis for the space V_j. Therefore $\{v_n^j\}_{n\in D(M^{*j})}$ is also a basis, and it is not difficult to check that the properties $V1-V3$ from Lemma 10 are fulfilled. The property $MR4$ follows from Lemma 11 . To prove $MR6$ it suffices to show that this property is valid for the functions $\{v_n^j\}_{n\in D(M^{*j})}$. Using Lemma 9 and $V3$, we have

$$v_n^j(Mx)\sim\sum_{k\in\mathbb{Z}^d}\widehat{v}_n^j(k)e^{2\pi i(k,Mx)}=\sum_{k\in\mathbb{Z}^d}\widehat{v}_{M^*n}^{j+1}(M^*k)e^{2\pi i(M^*k,x)}=$$

$$\sum_{l\in\mathbb{Z}^d,\ l\equiv\mathbf{0}\pmod{M^*}}\widehat{v}_{M^*n}^{j+1}(l)e^{2\pi i(l,x)}\sim v_{M^*n}^{j+1}(x),$$

For the proof of $MR6(a)$, it remains to note that $v_{M^*n}^{j+1}\in V_{j+1}$. Let us check fulfillment of $MR6(b)$. First consider the case $n\equiv\mathbf{0}\pmod{M^*}$. Using $V3$, we get

$$\sum_{k\in D(M)}v_n^{j+1}(M^{-1}x+M^{-1}k)=\sum_{k\in D(M)}v_{M^{*-1}n}^j(x+k)\in V_j.$$

Now let $n \not\equiv \mathbf{0} \pmod{M^*}$. In this case

$$\sum_{k\in D(M)} v_n^{j+1}(M^{-1}x+M^{-1}k) \sim \sum_{l\in\mathbb{Z}^d}\sum_{k\in D(M)} \widehat{v}_n^{j+1}(M^{*j+1}l+n)e^{2\pi i(M^{*j+1}l+n,M^{-1}x+M^{-1}k)} =$$

$$\sum_{l\in\mathbb{Z}^d} \widehat{v}_n^{j+1}(M^{*j+1}l+n)e^{2\pi i(M^{*j+1}l+n,M^{-1}x)} \sum_{k\in D(M)} e^{2\pi i(M^{*j}l+M^{*-1}n,k)},$$

However the latter sum is equal to zero by Lemma 2, i.e. $MR6(b)$ is also fulfilled. It remains to prove $MR2$. Since the trigonometric polynomials are dense in X, it suffices to check that any trigonometric polynomial can be approximated by functions from $\bigcup\limits_{j=0}^{\infty} V_j$. In one's turn, it suffices to check this for one harmonic. Set $f_r(x) = e^{2\pi i(r,x)}$, $r \neq \mathbf{0}$. It follows from the property $\Phi 3$ that there exists j_0 so that $\widehat{\varphi}_{j_0}(r) \neq 0$. By $\Phi 4$, $\widehat{\varphi}_j(r) \neq 0$, therefore, $\widehat{v}_r^j(r) \neq 0$ for all $j \geq j_0$. Introduce the functions h_j for $j \geq j_0$ by

$$h_j(x) := 1 - \frac{v_r^j(x)}{\widehat{v}_r^j(r)}e^{-2\pi i(r,x)}.$$

We have

$$\widehat{h}_j(n) = \delta_{n0} - \frac{\widehat{v}_r^j(r-n)}{\widehat{v}_r^j(r)}, \quad n \in \mathbb{Z}^d.$$

Thus, $\widehat{h}_j(n)$ is not equal to zero only if $n \equiv \mathbf{0} \pmod{M^{*j}}$, $n \neq \mathbf{0}$. Hence $\widehat{h}_j(n) = 0$ for all integer n from $M^{*j}(-1,1)^d$. Select a subsequence of the embedded parallelepipeds $M^{*j_k}(-1,1)^d$. It follows from (1) that there exists n_0 such that $\|M^{*-n_0}\| \leq 1/2$. Setting $j_{k+1} = j_k + n_0$, we have $\|M^{*j_{k+1}}x\| \geq 2\|M^{*j_k}x\|$ for any $x \in \mathbb{R}^d$. For large enough k, a cube K_{j_k} with its center in the origin, whose edges of length a_k are parallel to coordinate axis so that a_k is integer and infinitely monotone increases as k increases, can be inscribed into each scalene parallelepiped $M^{*j_k}(-1,1)^d$. For ease, in the sequel we shall write K_j for the subsequence. As it has been already noted, $\widehat{h}_j(n) = 0$ for all integer n from K_j Hence the partial sums of Fourier series over K_j of h_j are equal to zero. Therefor the corresponding Fejer means $\sigma_{K_j}(h_j)$ over these cubes are also equal to zero. Further, for $j \geq j_0$, we need the following equality

$$h_j(x) = m^{j_0-j}\sum_{n\in D(M^{j-j_0})} h_{j_0}(x+M^{-j}n). \tag{16}$$

For the proof, applying Lemma 9, rearrange the right hand side.

$$m^{j_0-j}\sum_{n\in D(M^{j-j_0})} h_{j_0}(x+M^{-j}n) = m^{j_0-j}\sum_{n\in D(M^{j-j_0})}[1 - \frac{v_r^{j_0}(x+M^{-j}n)}{\widehat{v}_r^{j_0}(r)}e^{-2\pi i(r,\ x+M^{-j}n)}] \sim$$

$$-\frac{m^{j_0-j}}{\widehat{v}_r^{j_0}(r)}\sum_{n\in D(M^{j-j_0})} e^{-2\pi i(r,\ x+M^{-j}n)} \sum_{l\in\mathbb{Z}^d,\ l\neq\mathbf{0}} \widehat{v}_r^{j_0}(M^{*j_0}l+r)e^{2\pi i(M^{*j_0}l+r,\ x+M^{-j}n)} =$$

$$-\frac{m^{j_0-j}}{\widehat{v}_r^{j_0}(r)}\sum_{l\in\mathbb{Z}^d,\ l\neq\mathbf{0}} \widehat{v}_r^{j_0}(M^{*j_0}l+r)e^{2\pi i(M^{*j_0}l,\ x)} \sum_{n\in D(M^{j-j_0})} e^{-2\pi i(M^{*(j_0-j)}l,\ n)}.$$

By Lemma 2, the internal sum in the right hand side equals m^{j-j_0} as $l \equiv \mathbf{0} \pmod{M^{j-j_0}}$, and equals zero otherwise. This means that in the external sum, only terms with the numbers $l \equiv \mathbf{0}$ $\pmod{M^{j-j_0}}$ are not equal to zero. This sum can be rewritten as follows.

$$-\frac{m^{j_0-j}}{\widehat{v}_r^{j_0}(r)}\sum_{k\in\mathbb{Z}^d,\ k\neq\mathbf{0}} \widehat{v}_r^{j_0}(M^{*j}k+r)e^{2\pi i(M^{*j}k+r,\ x)}e^{-2\pi i(r,\ x)} \sim 1 - \frac{v_r^j(x)}{\widehat{v}_r^{j_0}(r)}e^{-2\pi i(r,\ x)}.$$

From this, taking into account that, by the property $V2$, $\widehat{v}_r^{j_0}(r) = \widehat{v}_r^j(r)$ as $j \geq j_0$, and $\widehat{v}_r^{j_0}(r) \neq 0$, we obtain (16). This equality and the linearity of the Fejer means yield

$$\|h_j\| = \|h_j - \sigma_{K_j}(h_j)\| = \|m^{j_0-j} \sum_{n\in D(M^{j-j_0})} [h_{j_0}(x + M^{-j}n) - \sigma_{K_j}(h_{j_0})(x + M^{-j}n)]\|$$

$$\leq m^{j_0-j} \sum_{n\in D(M^{j-j_0})} \|h_{j_0}(x + M^{-j}n) - \sigma_{K_j}(h_{j_0})(x + M^{-j}n)\| =$$

$$m^{j_0-j} \sum_{n\in D(M^{j-j_0})} \|h_{j_0} - \sigma_{K_j}(h_{j_0})\|.$$

The last expression tends to zero as $j \to \infty$ because the Fejer means of a function from X converge to the function in norm (see, for example, [23, Chapter 17, §1]). Thus we proved that

$$\lim_{j\to\infty} \left\| f_r(x) - \frac{v_r^j(x)}{\widehat{v}_r^j(r)} \right\| = \lim_{j\to\infty} \|h_j(x)\| = 0,$$

i.e. f_r is approximated by the functions $\frac{v_r^j(x)}{\widehat{v}_r^j(r)} \in V_j$. ■

It is clear from the proof of the theorem and the remark to Lemma 10 that if we exclude the axiom $MR6$ from the definition of PMRA then a scaling sequence will be characterized by the properties $\Phi2 - \Phi4$, i.e. the following statement holds.

Theorem 15 *Let $\varphi_j \in X$, $j \in \mathbb{Z}_+$, $V_j = span\{S_n^j\varphi_j, \quad n \in D(M^{*j})\}$. The axioms $MR1 - MR5$ of Definitions 5 are fulfilled for a collection of spaces $\{V_j\}_{j=0}^{\infty}$ if and only if the functions φ_j satisfy the conditions $\Phi2 - \Phi4$ of Theorem 7.*

A wide class of PMRA of $L_2(\mathbb{T}^d)$ can be constructed by the standard scheme. A scaling sequence is obtained by periodization of a function $\varphi \in L_2(\mathbb{R}^d)$ by the formulas

$$\varphi_j(x) = \sum_{k\in\mathbb{Z}^d} \varphi\left(M^j x + M^j k\right) \tag{17}$$

(We shall say that such a PMRA is generated by the function φ). Let $\varphi \in L_2(\mathbb{R}^d)$ be a scaling function of a non-periodic MRA, i.e. the following conditions are fulfilled.
(i) there exist positive constants A, B such that

$$A \leq \sum_{m\in\mathbb{Z}^d} |\widehat{\varphi}(\xi + m)|^2 < B \quad \text{for a. a. } \xi \in \mathbb{R}^d;$$

(ii) there exists a function $m_0 \in L_2(\mathbb{T}^d)$ such that

$$\widehat{\varphi}(M^*\xi) = m_0(\xi)\widehat{\varphi}(\xi) \quad \text{for a.a. } \xi \in \mathbb{R}^d;$$

(iii) the function $\widehat{\varphi}$ is continuous in zero and $\widehat{\varphi}(\mathbf{0}) \neq 0$.
If, in addition, we assume that φ decays fast enough, for example,

$$\varphi(x) = O\left(\frac{1}{(1 + |x|)^{d+\epsilon}}\right), \quad \epsilon > 0$$

(what usually holds for known MRA), then $\varphi_j \in L_2(\mathbb{T}^d)$ and, by the Poisson summation formula,

$$\varphi_j(x) = m^{-j} \sum_{m\in\mathbb{Z}} \widehat{\varphi}(M^{*-j}m)e^{2\pi i(m,x)}.$$

Check that the functions φ_j satisfy the conditions $\Phi1-\Phi5$ of Theorem 7. To prove $\Phi1$ we note that, due to (ii) and (iii), $m_0(n) = m_0(0) = 1$ for all $n \in \mathbb{Z}^d$. Therefore, if $\widehat{\varphi}_j(k) = m^{-j}\widehat{\varphi}(M^{*-j}k) \neq 0$ for some $k \in \mathbb{Z}^d, k \neq \mathbf{0}$, then $\widehat{\varphi}_j(M^{*\ell}k) = \widehat{\varphi}_j(k) \neq 0$ for all $\ell \in \mathbb{Z}_+$, what can not hold for a function from $L_2(\mathbb{T}^d)$. The properties $\Phi2$ and $\Phi3$ follow respectively from (i) and (iii). It is not difficult to check the validity $\Phi4, \Phi5$ if we set $\mu_k^j = m_0(M^{*-j-1}k)$, $\gamma_k^j = m$.

In [19] a condition under which a PMRA of $L_p(\mathbb{T}^1)$ or $C(\mathbb{T}^1)$ is generated by a summable function, is found . However a PMRA can be generated by a non-summable function. For example, the function

$$\varphi(x) = \frac{\sin \pi x}{\pi x} \tag{18}$$

is scaling, and though $\varphi \notin L(\mathbb{R})$, its periodization is possible because the series (17) converges in the sense of principal value. A. P. Petukhov [24] found a PMRA which is not generated by a function in such sense. Really, the problem of generatability has not been properly investigated even in the one-dimensional case.

4. Wavelet Spaces

In this section, following standard idea for construction of wavelet bases, we shall define wavelet spaces and function whose shifts constitute bases in these spaces. In the orthogonal case (which can hold only if $X = L_2(\mathbb{T}^d)$), the wavelet space is the orthogonal projection of the spaces V_{j+1} to V_j. In the biorthogonol case we deal with two multiresolution analyses. The space V_{j+1} of one of the analyses is projected orthogonal to the corresponding component $\widetilde{V}_j$ of the other analysis.

We shall consider pairs of PMRA with the first component $\{V_j\}_{j=0}^{\infty}$ in $L_p(\mathbb{T}^d)$, $1 \leq p \leq \infty$, ($C(\mathbb{T}^d)$ for $p = \infty$) and the second component $\{\widetilde{V}_j\}_{j=0}^{\infty}$ in $L_q(\mathbb{T}^d)$, $1/p + 1/q = 1$, ($C(\mathbb{T}^d)$ for $p = 1$). Such a pair is called a (p, q)-pair.

Proposition 16 *Let $\{V_j\}_{j=0}^{\infty}$ and $\{\widetilde{V}_j\}_{j=0}^{\infty}$ form a (p, q)-pair, $\varphi \in V_j$, $\widetilde{\varphi} \in \widetilde{V}_j$. For the systems of functions $\{S_n^j\varphi\}_{n\in D(M^j)}$ and $\{S_k^j\widetilde{\varphi}\}_{k\in D(M^j)}$ to be biorthonormal it is necessary and sufficient to have*

$$\langle\, \omega_r^j\varphi,\ \omega_r^j\widetilde{\varphi}\,\rangle = m^{-j} \tag{19}$$

for all $r \in \mathbb{Z}^d$

`Proof.` By Lemma (9), we have

$$\langle\, S_n^j\varphi,\ S_k^j\widetilde{\varphi}\,\rangle = \left\langle \sum_{r\in D(M^{*j})} e^{2\pi i(M^{*-j}r,\ n)}\omega_r^j\varphi, \sum_{s\in D(M^{*j})} e^{2\pi i(M^{*-j}s,\ k)}\omega_s^j\widetilde{\varphi} \right\rangle =$$

$$\sum_{r\in D(M^{*j})}\ \sum_{s\in D(M^{*j})} e^{2\pi i(M^{*-j}r,\ n)}e^{-2\pi i(M^{*-j}s,\ k)}\langle\, \omega_r^j\varphi,\ \omega_s^j\widetilde{\varphi}\,\rangle\,.$$

Since the spectra of the functions $\omega_r^j\varphi,\ \omega_s^j\widetilde{\varphi}$ are disjoint as $r \neq s$, the corresponding terms in the latter sum are equal to zero. Thus,

$$\langle\, S_n^j\varphi,\ S_k^j\widetilde{\varphi}\,\rangle = \sum_{r\in D(M^{*j})} e^{2\pi i(M^{*-j}r,\ n-k)}c_r, \tag{20}$$

where $c_r := \langle\, \omega_r^j\varphi,\ \omega_r^j\widetilde{\varphi}\,\rangle$. This and Lemma 2 yield sufficiency. Now we assume that $\{S_n^j\varphi_j\}_{n\in D(M^j)}$ and $\{S_k^j\widetilde{\varphi}_j\}_{k\in D(M^j)}$ are biorthonormal systems. Consider the equalities (20) with a fixed $n \in D(M^j)$ for all $k \in D(M^j)$ as a system of equations with the unknowns c_r. Due to Corollary 3, the solution $c_p = m^{-j}$, $p \in D(M^{*j})$ is unique. It remains to note that $c_p = c_{p+M^{*j}}$, i.e. $c_p = m^{-j}$ for all $p \in \mathbb{Z}^d$, what proves necessity. ■

Corollary 17 *Let $\{V_j\}_{j=0}^{\infty}$ and $\{\widetilde{V}_j\}_{j=0}^{\infty}$ form a (p,q)-pair with scaling sequences respectively $\{\varphi_j\}_{j=0}^{\infty}$ and $\{\widetilde{\varphi}_j\}_{j=0}^{\infty}$, and let μ_n^j and $\widetilde{\mu}_k^j$ be the factors from the property $\Phi 4$ of Theorem 7. If the systems $\{S_n^j\varphi_j\}_{n\in D(M^j)}$ and $\{S_k^j\widetilde{\varphi}_j\}_{n\in D(M^j)}$ are biorthogonal, then*

$$\sum_{k\in D(M^*)} \mu^j_{p+M^{*j-1}k}\overline{\widetilde{\mu}^j_{p+M^{*j-1}k}} = m. \tag{21}$$

for all $p\in\mathbb{Z}^d$ and for all $j\in\mathbb{Z}_+$

`Proof`. Similarly to (15), we have

$$m^{-j+1} = \langle\, \omega_p^{j-1}\varphi_{j-1},\ \omega_p^{j-1}\widetilde{\varphi}_{j-1}\,\rangle =$$

$$\left\langle \sum_{k\in D(M^*)} \mu^j_{p+M^{*j-1}k}\omega^j_{p+M^{*j-1}k}\varphi_{j-1},\ \sum_{l\in D(M^*)} \widetilde{\mu}^j_{p+M^{*j-1}l}\omega^j_{p+M^{*j-1}l}\widetilde{\varphi}_{j-1} \right\rangle =$$

$$\sum_{k\in D(M^*)} \mu^j_{p+M^{*j-1}k}\overline{\widetilde{\mu}^j_{p+M^{*j-1}k}}\langle\, \omega^j_{p+M^{*j-1}k}\varphi_j,\ \omega^j_{p+M^{*j-1}k}\widetilde{\varphi}_j\,\rangle = m^{-j}\sum_{k\in D(M^*)} \mu^j_{p+M^{*j-1}k}\overline{\widetilde{\mu}^j_{p+M^{*j-1}k}},$$

It remains to multiply both the sides of the equality by m^{-j}. ∎

Now we begin construction of wavelet spaces and bases. First consider the orthogonal case. Let $\{V_j\}_{j=0}^{\infty}$ be a PMRA of $L_2(\mathbb{T}^d)$ with a scaling sequence $\{\varphi_j\}_{j=0}^{\infty}$ generating an orthonormal shift basis $\{S_n^j\varphi_j\}_{n\in D(M^j)}$ for the spaces V_j. Restrict ourself to consider the case when all μ_k^j (the factors from the property $\Phi 4$ of Theorem 7) are real. Our goal is to find functions $\psi^{(\nu)}$, $\nu = 1,\dots,m-1$ in the space V_{j+1} such that the corresponding systems $\{S_n^j\psi_j^{(\nu)}\}_{n\in D(M^j)}$ are orthonormal, orthogonal to V_j and complement $\{S_n^j\varphi_j\}_{n\in D(M^j)}$ to a basis for the space V_{j+1}. For this we need to complete a matrix up to the unitary one basing on its first line. Let real numbers $a_{00},\dots,a_{0m}$ (the first line of a future unitary matrix A) satisfy to the following condition.

$$\sum_{r=1}^{m} a_{0r}^2 = 1. \tag{22}$$

If $a_{00}=1, a_{01},\dots,a_{0m}=0$, we define A as the unit matrix. If $a_{00}\neq 1$ then the remaining elements of A are given by Holsholder transform:

$$a_{lk} = a_{kl} = -\frac{a_{0k}a_{0l}}{1-a_{00}} \quad k\neq l, \qquad a_{kk} = 1-\frac{a_{0k}^2}{1-a_{00}}. \tag{23}$$

Let us fix $j\in\mathbb{Z}_+$ and $n\in D(M^{*j})$. Let $s_k,\ k=0...,m-1$ be enumerated elements of the sets $D(M^*)$. Set $a_{0k} = \mu^{j+1}_{n+M^{*j}s_k}/\sqrt{m}$, $k=0,\dots,m-1$. By Corollary 17, these numbers satisfy (22), Therefore we can complete this line up to a unitary matrix A. Set $\alpha^{\nu,j}_{n+M^{*j}s_k} = \sqrt{m}a_{\nu k}$. If n runs over all the set $D(M^{*j})$, by Lemma 4, the vectors $n+M^{*j}s_k,\ k=0...,m-1$ run over all $D(M^{*j+1})$, i.e. we defined the numbers $\alpha_s^{\nu,j}$ for all $s\in D(M^{*j+1})$. Extend this sequence (with respect to the subindex) to $\mathbb{Z}^d$ setting $\alpha_l^{\nu,j} = \alpha_s^{\nu,j}$ for all $l\equiv s \pmod{M^{*j+1}}$. For each $\nu=1,\dots,m-1$, define the wavelet functions $\psi^{(\nu)}$ setting their Fourier coefficients by $\widehat{\psi}_j^{(\nu)}(l) = \alpha_s^{\nu,j}\widehat{\varphi}_{j+1}(l)$, $l\in\mathbb{Z}^d$, and the wavelet spaces

$$W_j^{(\nu)} := span\{S_n^j\psi_j^{(\nu)}:\ \ n\in D(M^j)\}.$$

Theorem 18 *Let $\{V_j\}$ be a PMRA of $L^2(\mathbb{T}^d)$ with a scaling sequence $\{\varphi_j\}$, and let $S_n^j\varphi_j,\ n\in D(M^j)$ be an orthonormal system for any $j\in\mathbb{Z}_+$. Then*

$$V_{j+1} = V_j\oplus W_j^{(1)}\oplus ...\oplus W_j^{(m-1)} \qquad j\in\mathbb{Z}_+,$$

and $\{S_n^j\psi_j^{(\nu)}\}_{n\in D(M^j)}$ is an orthonormal basis for the space $W_j^{(\nu)}$, $\nu=1,\dots,m-1$.

The proof of this theorem will be presented in a more general situation (Theorem 19).

Now we consider the biorthogonol case. Let a (p,q)-pair satisfy the conditions of Corollary 17. To construct wavelet functions we need to complement two suitable lines up to two mutually inverse matrixes. Let numbers $a_{00},\cdots,a_{0m-1}$ and $\widetilde{a}_{00},\cdots,\widetilde{a}_{0m-1}$ (respectively the first lines of future matrixes A and $\widetilde{A}$), be such that

$$\sum_{r=0}^{m-1} a_{0r}\overline{\widetilde{a}_{0r}} = 1. \tag{24}$$

First we assume that $a_{00}=\overline{\widetilde{a}_{00}}\neq 1$. In this case, for $k,l=1...,m-1$, remaining elements can be defined by

$$a_{l0}=\overline{\widetilde{a}_{0l}}, \quad a_{lk}=\delta_{lk}-\frac{\overline{\widetilde{a}_{0l}}a_{0k}}{1-a_{00}}, \tag{25}$$

$$\widetilde{a}_{l0}=\overline{a_{0l}}, \quad \widetilde{a}_{lk}=\delta_{lk}-\frac{\overline{a_{0l}}\widetilde{a}_{0k}}{1-\overline{a_{00}}}. \tag{26}$$

It is easy to check that

$$A\widetilde{A}^{*}=E_m. \tag{27}$$

Now we assume that $a_{00}\widetilde{a}_{00}\neq 0$. Consider numbers $a'_{0k}=Ca_{0k}$, $\widetilde{a}'_{0k}=\widetilde{a}_{0k}/\overline{C}$, $k=0,\dots,m-1$, where C satisfy the condition $a'_{00}=\overline{\widetilde{a}'_{00}}\neq 1$. It is not difficult to realize this by setting $C=\sqrt{\overline{\widetilde{a}_{00}}/a_{00}}$ and choosing a complex value of the squire root for which $a'_{00}\neq 1$. The new lines satisfy all the requirements of the previous case, So, we can complete these lines up to matrixes A'. $\widetilde{A}'$ such that $A'\widetilde{A}'^{*}=E_m$. After replacement the first lines by the initial ones in these matrixes, we obtain required matrixes A, $\widetilde{A}$. Finally we suppose that $a_{00}\widetilde{a}_{00}=0$. It follows from (24) that there exists r_0 such that $a_{0r_0}\widetilde{a}_{0r_0}\neq 0$. Changing a_{0r_0} with a_{00} and $\widetilde{a}_{0r_0}$ with $\widetilde{a}_{00}$, we reduce to the previous case. Completing new lines up to mutually inverse matrixes and changing the columns number 0 and number r_0 in these matrixes we obtain required matrixes A, $\widetilde{A}$.

Fix $j\in\mathbb{Z}_+$ and $n\in D(M^{*j})$. As above, s_k, $k=0...,m-1$ are enumerated elements of the sets $D(M^*)$. Set $a_{0k}=\mu^{j+1}_{n+M^{*j}s_k}/\sqrt{m}$, $\widetilde{a}_{0k}=\widetilde{\mu}^{j+1}_{n+M^{*j}s_k}/\sqrt{m}$, $k=0,\dots,m-1$. By Corollary 17, these numbers satisfy (24), Therefore we can complete these lines up to matrixes A, $\widetilde{A}$, satisfying (27). Set $\alpha^{\nu,j}_{n+M^{*j}s_k}=\sqrt{m}a_{\nu k}$, $\widetilde{\alpha}^{\nu,j}_{n+M^{*j}s_k}=\sqrt{m}\widetilde{a}_{\nu k}$. It follows from (27) that

$$\sum_{k=0}^{m-1}\alpha^{\nu,j}_{n+M^{*j}s_k}\overline{\widetilde{\mu}^{j+1}_{n+M^{*j}s_k}}=0, \quad \sum_{k=0}^{m-1}\widetilde{\alpha}^{\nu,j}_{n+M^{*j}s_k}\overline{\mu^{j+1}_{n+M^{*j}s_k}}=0, \quad \forall\nu=1...,m-1. \tag{28}$$

$$\sum_{k=0}^{m-1}\alpha^{l,j}_{n+M^{*j}s_k}\overline{\widetilde{\alpha}^{p,j}_{n+M^{*j}s_k}}=m\delta_{\nu l} \quad \forall l,\nu=1...,m-1. \tag{29}$$

If n runs over all the set $D(M^{*j})$, by Lemma 4, the vectors $n+M^{*j}s_k$, $k=0...,m-1$ run over all $D(M^{*j+1})$, i.e. we defined the numbers $\alpha^{\nu,j}_s$, $\widetilde{\alpha}^{\nu,j}_s$ for all $s\in D(M^{*j+1})$. Extend these sequences (with respect to the subindex) to $\mathbb{Z}^d$ setting $\alpha^{\nu,j}_l=\alpha^{\nu,j}_s$, $\widetilde{\alpha}^{\nu,j}_l=\widetilde{\alpha}^{\nu,j}_s$, for all $l\equiv s \pmod{M^{*j+1}}$. For each $\nu=1,\dots,m-1$, define the wavelet functions $\psi^{(\nu)}$, $\widetilde{\psi}^{(\nu)}$ setting their Fourier coefficients by $\widehat{\psi}^{(\nu)}_j(l)=\alpha^{\nu,j}_s\widehat{\varphi}_{j+1}(l)$, $\widehat{\widetilde{\psi}}^{(\nu)}_j(l)=\widetilde{\alpha}^{\nu,j}_s\widehat{\widetilde{\varphi}}_{j+1}(l)$, $l\in\mathbb{Z}^d$, and the wavelet spaces

$$W^{(\nu)}_j := span\{S^j_n\psi^{(\nu)}_j : \ n\in D(M^j)\},$$

$$\widetilde{W}^{(\nu)}_j := span\{S^j_n\widetilde{\psi}^{(\nu)}_j : \ n\in D(M^j)\}.$$

Note that the orthogonal wavelets can be constructed by the general scheme, but this is more complicated.

Theorem 19 *Let $\{V_j\}_{j=0}^{\infty}$ and $\{\widetilde{V}_j\}_{j=0}^{\infty}$ form a (p,q)-pair with scaling sequences respectively $\{\varphi_j\}_{j=0}^{\infty}$ and $\{\widetilde{\varphi}_j\}_{j=0}^{\infty}$ such that $\{S_n^j\varphi_j\}_{n\in D(M^j)}$ and $\{S_k^j\widetilde{\varphi}_j\}_{n\in D(M^j)}$ are biorthonormal systems. Then*

1. $W_j^{(\nu)} \subset V_{j+1} \quad \forall\nu = 1...,m-1.$
2. $\forall f \in V_{j+1} \quad f = f_0 + \sum\limits_{\nu=1}^{m-1} f_\nu,$ *Where* $f_0 \in V_j,\ f_\nu \in W_j^{(\nu)}$.
3. $W_j^{(\nu)} \perp \widetilde{V}_j,\ \ \widetilde{W}_j^{(\nu)} \perp V_j,\ \forall\nu = 1...,m-1;$
4. $W_j^{(\nu)} \perp \widetilde{W}_j^{(\kappa)}, \quad \nu \neq \kappa \quad \nu,\ \kappa = 1...,m-1.$
5. $(S_n^j\psi_j^{(\nu)},\ S_k^j\widetilde{\psi}_j^{(\nu)}) = \delta_{nk} \ \ \forall\nu = 1...,m-1, \qquad \forall n,\ k \in D(M^j).$

`Proof.` For fixed n and j, it follows from (15) that

$$\omega_n^j\varphi_j(x) = \sum_{l\in D(M^*)} \mu_{n+M^{*j}l}^{j+1}\omega_{n+M^{*j}l}^{j+1}\varphi_{j+1}(x) \tag{30}$$

Similarly,

$$\omega_n^j\psi_j^{(\nu)}(x) = \sum_{l\in D(M^*)} \alpha_{n+M^{*j}l}^{\nu,j}\omega_{n+M^{*j}l}^{j+1}\varphi_{j+1}(x), \quad \nu = 1...,m-1. \tag{31}$$

Consider the equalities (30), (31) as a system of m equations with m unknowns $\{\omega_{n+M^{*j}l}^{j+1}\varphi_{j+1}\},\ l \in D(M^*)$. By construction of the numbers $\{a_{\nu k}\}$, the matrix of this system has an inverse one. Hence its determinant is not equal to zero. Therefore the unknowns $\omega_{n+M^{*j}l}^{j+1}\varphi_{j+1}$ can be expressed via $\omega_n^j\varphi_j \in V_j$ and $\omega_n^j\psi_j^{(\nu)} \in W_j^{(\nu)}$, what yields 1. On the other hand, similarly to (13), we have

$$S_r^j\psi_j^{(\nu)} = \sum_{n\in D(M^{*j})} e^{2\pi i(M^{*-j}n,r)}\omega_n^j\psi_j^{(\nu)}, \quad \nu = 1,\dots,m-1.$$

From this, taking into account Corollary 3, each $\omega_n^j\psi_j^{(\nu)}$ can be expressed via $S_r^j\psi_j^{(\nu)}$, $r \in D(M^j)$. For the proof 2, it remains to note that $\{\omega_{n+M^{*j}l}^{j+1}\varphi_{j+1}\}$ is a basis for the space V_{j+1}. Moreover, since the functions $\omega_n^j\psi_j^{(\nu)}$, $n \in D((M^{*j})$ are linearly independent, it follows from these arguments and the property $MR3$ of Definition 5 that $\dim W_j^{(\nu)} = m^j$, i.e both the systems $\{\omega_n^j\psi_j^{(\nu)}\}_{n\in D(M^{*j})}$ and $\{S_r^j\psi_j^{(\nu)}\}_{r\in D(M^j)}$ are bases for $W_j^{(\nu)}$. To prove proof 3 it suffices to check that the basis functions of the space $\widetilde{V}_j$ are orthogonal to the basis functions of the space $W_j^{(\nu)}$, $\nu = 1,\dots,m-1$. Using equality (28) and Proposition 16, we have

$$\langle\, \omega_n^j\psi_j^{(\nu)},\ \omega_k^j\widetilde{\varphi}_j\,\rangle = \left\langle \sum_{l\in D(M^*)} \alpha_{n+M^{*j}l}^{\nu,j}\omega_{n+M^{*j}l}^{j+1}\varphi_{j+1},\ \sum_{k\in D(M^*)} \widetilde{\mu}_{n+M^{*j}k}^{j+1}\omega_{n+M^{*j}k}^{j+1}\widetilde{\varphi}_{j+1} \right\rangle =$$

$$\sum_{l\in D(M^*)} \alpha_{n+M^{*j}l}^{\nu,j}\overline{\widetilde{\mu}_{n+M^{*j}l}^{j+1}}\langle\, \omega_{n+M^{*j}l}^{j+1}\varphi_{j+1},\ \omega_{n+M^{*j}l}^{j+1}\widetilde{\varphi}_{j+1}\,\rangle = m^{-j-1}\sum_{l\in D(M^*)} \alpha_{n+M^{*j}l}^{\nu,j}\overline{\widetilde{\mu}_{n+M^{*j}l}^{j+1}} = 0.$$

Similarly, because of (29),

$$\langle\, \omega_n^j\psi_j^{(\nu)},\ \omega_n^j\widetilde{\psi}_j^{(\kappa)}\,\rangle = m^{-j-1}\sum_{l\in D(M^*)} \alpha_{n+M^{*j}l}^{\nu,j}\overline{\widetilde{\alpha}_{n+M^{*j}l}^{\kappa,j}} = m^{-j}\delta_{\nu\kappa}.$$

This yields 4. Taking into account Proposition 16, we obtain 5. ∎

5. Kotel'nikov-Shannon Wavelets

Construct an example of PMRA of $L^2(\mathbb{T}^2)$ with a scaling sequence, consisting from trigonometric polynomials with minimal possible symmetric spectra. A one-dimensional analog is the well known Kotel'nikov-Shannon PMRA for which the sequence of Dirichlet kernels is scaling, and the function (18) is generating. Expansions with respect to the one-dimensional Kotel'nikov-Shannon system for the first time were used for transmission of continuous information by communication channels.

We take the matrix $M = \begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix}$ as the scale factor. Note that $m = 4$, $M^* = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}$, $M^{*-1} = \frac{1}{4}\begin{pmatrix} 1 & -2 \\ 1 & 2 \end{pmatrix}$, and fix a set $D(M^*)$ consisting of the vectors $s_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $s_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, $s_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$, $s_3 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$. Let Ω_j denote the parallelogram $M^{*j}[-1, 1]^2$ with excluded vertices. Set $a_j := M^{*j}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $b_j := M^{*j}\begin{pmatrix} 0 \\ -1 \end{pmatrix}$, define a function φ_j setting its Fourier coefficients by $\widehat{\varphi}_0(k) := \delta_{k0}$ for all $k \in \mathbb{Z}^2$, for $j \in \mathbb{N}$

$$\widehat{\varphi}_j(k) := \begin{cases} 2^{-j}, and k \in \Omega_{j-1} \setminus \{a_{j-1}, b_{j-1}\}, \\ 2^{-j-1/2}, and k = a_{j-1} \text{ or } k = b_{j-1}, \\ 0, and k \notin \Omega_{j-1}. \end{cases} \tag{32}$$

To show that this sequence is scaling we need the following lemma.

Lemma 20 *The number of integer points in Ω_j is equal to $4^{j+1} + 1$, all integer points from Ω_j except the point a_j are in different cosets of M^{*j+1}, the point a_j is congruent to b_j modulo M^{*j+1}.*

`Proof.` First we show that no integer points on the medians of Ω_j , except boundary points and zero. The medians of Ω_j are segments connecting the point $M^{*j}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ with $M^{*j}\begin{pmatrix} 0 \\ -1 \end{pmatrix}$, and the point $M^{*j}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ with $M^{*j}\begin{pmatrix} -1 \\ 0 \end{pmatrix}$. These segments pass through the origin and are symmetric with respect to the origin. Therefore it suffices to prove for the semisegments. Let us prove for one of the semisegments (similarly for the other one), i.e. show, that no integer points on the interval $\left(\mathbf{0},\ M^{*j}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$ Set $\begin{pmatrix} x_j \\ y_j \end{pmatrix} := M^{*j}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = M^*\begin{pmatrix} x_{j-1} \\ y_{j-1} \end{pmatrix}$. For all $j > 0$, the first coordinate of this vector is even, the second one is odd. Hence, x_j and y_j have no common divisors divisible by 2. It is clear from the formulas

$$x_{j-1} = \frac{x_j - 2y_j}{4}, \quad y_{j-1} = \frac{x_j + 2y_j}{4}$$

that if x_j and y_j have a common divisor not divisible by 2, then x_{j-1} and y_{j-1} have the same common divisor. However, $x_0 = 1$ and $y_0 = 0$ have no common divisors. Hence, by induction, x_j and y_j have no common divisors for any $j \in \mathbb{Z}_+$. Fixed j, represent the segments $\left(\mathbf{0},\ M^{*j}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$ in the parametric form: $\begin{cases} x = tx_j \\ y = ty_j \end{cases}$, where $t \in (0, 1)$. Suppose that there exists an integer point $(x^0,\ y^0)$ belonging to this segment, i.e. there exists $t^0 \in (0, 1)$ such that $x^0 = t^0 x_j$, $y^0 = t^0 y_j$ and $x^0,\ y^0,\ x_j,\ y_j \in \mathbb{Z}$. Hence, t^0 can not be irrational. But t^0 can not also be rational because if we assume that $t^0 = p/q$, then x_j and y_j are divisible by q, i.e. they have a common divisor, what is impossible.

Note that the set of the integer points in $M^{*j}\mathbb{T}^2$ constitute a set of digits $D(M^{*j})$, because for each two elements from $M^{*j}\mathbb{T}^2 \cap \mathbb{Z}^2$, their difference $M^{*j}r_1 - M^{*j}r_2$, where r_1, $r_2 \in \mathbb{T}^2$, can be

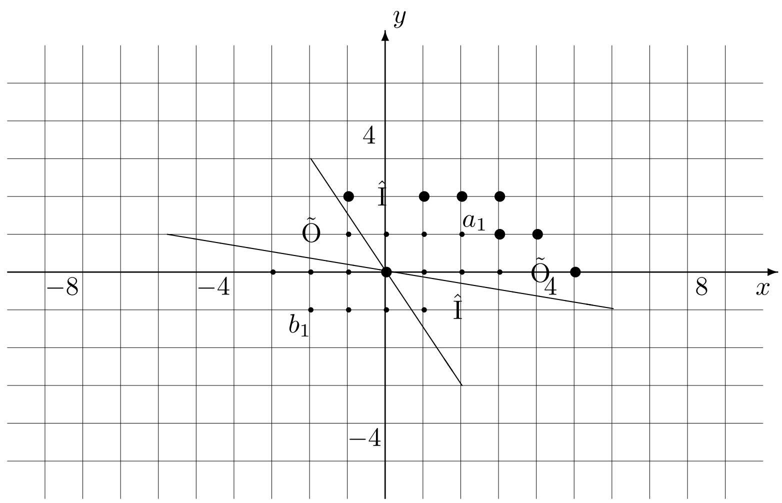

Figure 1: The internal and the external parallelograms correspond respecively to Ω_1 and Ω_2

congruent to zero modulo M^{*j} only if r_1 and r_2 coincide. However, the number of integer vectors in $M^{*j}\mathbb{T}^2$ is equal to m^j, i.e. the same number as in the set $D(M^{*j})$. Since no integer points on the medians of Ω_j except the boundary points and zero, the number of integer points in Ω_j is summed from integer internal points $M^{*j}\mathbb{T}^2$ (whose number is 4^j-1 because the digit corresponding to the zero coset is excluded from $M^{*j}[0,1)^2$), taken four times, zero and boundary points. The number of boundary points are equal to 4 because the boundaries of medians are integer, the vertices do not belong to Ω_j, and since each edge differs from the corresponding median by an integer shift, no other integer points on the boundary. Thus the number of integer points in Ω_j is equal to $4(4^j-1)+1+4=4^{j+1}+1$.

Check that all points from Ω_j except a_j belong to different cosets of the matrix M^{*j+1}, and $a_j \equiv b_j \pmod{M^{*j+1}}$. By Lemma 1, the set $\bigcup\limits_{k=0}^{3}\left(M^{*-1}\mathbb{T}^2+M^{*-1}s_k\right)$ is congruent to $\mathbb{T}^2$ modulo $\mathbb{Z}^2$. However, since

$$\bigcup_{k=0}^{3}\Big(M^{*-1}\mathbb{T}^2+M^{*-1}s_k\Big)=\bigcup_{k=0}^{3}\Big(M^{*-1}\left(\mathbb{T}^2+s_k\right)\Big)=M^{*-1}[-1,1)^2,$$

we obtain that $M^{*-1}[-1,1)^2$ is congruent to $\mathbb{T}^2$ modulo $\mathbb{Z}^2$. Fix $j\in\mathbb{Z}_+$ and apply the operator M^{*j+1} to each the set. Thus the set $M^{*j}[-1,1)^2$ is congruent to the set $M^{*j+1}\mathbb{T}^2$ modulo M^{*j+1}. Since all integer points from $M^{*j+1}[0,1)^2$ belong to different cosets of the matrix M^{*j+1}, all integer points from $M^{*j}[-1,1)^2$ also belong to different cosets of M^{*j+1}.

One can see on Figure 1 the areas $M^{*}[-1,1]^2$ (internal parallelogram), $M^{*2}[-1,1]^2$ (external parallelogram) and $M^{*2}[0,1]^2$(bold type parallelogram). The integer points from the set Ω_1 are emphasized non-brightly, the brightly emphasized points are that which are in the set $M^{*2}[0,1]^2$ but not in Ω_1, i.e. for which there exist points from $M^{*1}[-1,1]^2$ congruent to them modulo M^{*2}. Splitting, as it is shown on the figure, large parallelogram to four small ones and imposing each of them on the right upper one, we obtain that the points which coincide under such transition are congruent to each other modulo M^{*4}. One can see that the points a_3 and b_3 are congruent to each other modulo M^{*4}, and the points noted by crosses and ciphers are also respectively congruent.

The set of the integer points in $M^{*j}[-1,1)^2$ differs from the set of the integer points from Ω_j as follows Ω_j does not contain the points $M^{*j}\begin{pmatrix}-1\\-1\end{pmatrix}$, but contains the points $M^{*j}\begin{pmatrix}1\\0\end{pmatrix}$ and $a_j = M^{*j}\begin{pmatrix}0\\1\end{pmatrix}$. It remains to note that $M^{*j}\begin{pmatrix}-1\\-1\end{pmatrix} \equiv M^{*j}\begin{pmatrix}1\\0\end{pmatrix} \pmod{M^{*j+1}}$. And $a_j \equiv b_j \pmod{M^{*j+1}}$. ■

Let us check that the sequence of functions $\{\varphi_j\}_{j=0}^{\infty}$, defined by (32)á satisfies the conditions of Theorem 19. The item Φ1 follows from the definition, Φ2 is fulfilled by Lemma 20. The item Φ3 follows from the fact that the module of eigenvalues of M is equal to 2, what implies that the corresponding operator, applied many times, provides dilation in all directions (see Introduction). To check Φ4 find factors μ_k^j from the condition $\widehat{\varphi}_j(k) = \mu_k^{j+1}\widehat{\varphi}_{j+1}(k)$. Note that Ω_j is strictly contained in Ω_{j+1} for any $j \in \mathbb{Z}_+$, and all points of the boundary Ω_j are interior to Ω_{j+1}. For $j = 0$, the embedding is obvious. Further, applying the operator M^{*j} to $\Omega_0 \subset \Omega_1$, use that since the map M^{*j} is non-degenerate, it preserves a feature of embedding. Taking all integer points from Ω_j except the point a_j, as $D(M^{*j+1})$, for $k \in D(M^{*j+1})$, we have: $\mu_k^{j+1} = 2$, whenever $k \in \Omega_{j-1}\backslash\{a_{j-1}, b_{j-1}\}$; $\mu_k^{j+1} = \sqrt{2}$ whenever $k = a_{j-1}$ or $k = b_{j-1}$; $\mu_k^{j+1} = 0$, whenever $k \notin \Omega_{j-1}$. It is not difficult to see that the M^{*j+1}-periodic extension of μ_k^{j+1} with respect to the subindex is a sequence satisfying Φ4. We can take $\gamma_k^j = 1/2$ as the factors from Φ5 of Theorem 7, It is clear that $\gamma_k^j\widehat{\varphi}_j(n) = \widehat{\varphi}_{j+1}(M^*n)$ for all $j \in \mathbb{Z}_+$, $k \in \mathbb{Z}^2$, and $n \equiv k \pmod{M^{*j}}$. Thus the sequence of functions $\{\varphi_j\}_{j=0}^{\infty}$ satisfies to all items of Theorem 7. Hence this sequence is scaling, and the function φ_j are trigonometric polynomials with is a minimum possible symmetric spectra. Further, it is not difficult to check that

$$\|\omega_r^j\varphi_j\|^2 = \sum_{l\in\mathbb{Z}^d} |\widehat{\varphi}_j(M^{*j}l + r)|^2 = 4^{-j}. \tag{33}$$

Hence, by Proposition 16, the systems of functions $\{S_n^j\varphi\}_{n\in D(M^j)}$ is orthonormal.

Now we begin to construct the wavelet sequences. Fix $j \in \mathbb{Z}_+$ and a set $D(M^{*j})$ that coincides with $\Omega_{j-1}\backslash\{a_{j-1}\}$. As above, the set $D(M^*)$ consists of s_0, s_1, s_2 and s_3. Let $n \in D(M^{*j})$. If $n \neq b_{j-1}$, then $\mu_{n+M^{*j}s_0}^{j+1} = 2$ and $\mu_{n+M^{*j}s_k}^{j+1} = 0$ for $k = 1, 2, 3$, because the vectors $n + M^{*j}s_k$ are not contained in Ω_{j-1} and, moreover, are not congruent to elements from Ω_{j-1} modulo M^{*j+1}. The corresponding unitary 4×4-matrix is diagonal with twos on the diagonal. If $n = b_{j-1}$, then $\mu_{n+M^{*j}s_k}^{j+1} = \sqrt{2}$ as $k = 0$ or $k = 3$ (the vector $b_{j-1} + M^{*j}s_3$ is congruent to a_{j-1} modulo M^{*j+1}, what can be checked immediately), and $\mu_{n+M^{*j}s_k}^{j+1} = 0$ as $k = 1$ or $k = 2$, since the vectors $n + M^{*j}s_k$, $k = 1, 2$ are not contained in Ω_{j-1} and are not congruent to elements from Ω_{j-1} modulo M^{*j+1}, the corresponding unitary matrix is

$$\begin{pmatrix} \sqrt{2}/2 & 0 & 0 & \sqrt{2}/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sqrt{2}/2 & 0 & 0 & -\sqrt{2}/2 \end{pmatrix}.$$

By Lemma 4, the vectors $n+M^{*j}s_k$ with $s_k \in D(M^*)$ and $n \in D(M^{*j})$, run the set $D(M^{*j+1})$. Therefore it suffices to find the Fourier coefficients of the wavelet functions for all integer vectors l which congruent to the vectors $n + M^{*j}s_k$ modulo M^{*j+1}:

$$\widehat{\psi}_j^{(1)}(l) := \begin{cases} 2^{-j}, & l \equiv n + M^{*j}s_1 \pmod{M^{*j+1}},\ n \in D(M^{*j}), l \in \Omega_j\backslash\{a_j, b_j\} \\ 2^{-j-1/2}, & l \equiv n + M^{*j}s_1 \pmod{M^{*j+1}},\ n \in D(M^{*j}), l = a_j,\ l = b_j \\ 0, & \text{otherwise}; \end{cases} \tag{34}$$

$$\widehat{\psi}_j^{(2)}(l) := \begin{cases} 2^{-j}, & l \equiv n + M^{*j}s_2 \pmod{M^{*j+1}},\ n \in D(M^{*j}), l \in \Omega_j\backslash\{a_j, b_j\} \\ 2^{-j-1/2}, & l \equiv n + M^{*j}s_2 \pmod{M^{*j+1}},\ n \in D(M^{*j}), l = a_j,\ l = b_j \\ 0, & \text{otherwise}; \end{cases} \tag{35}$$

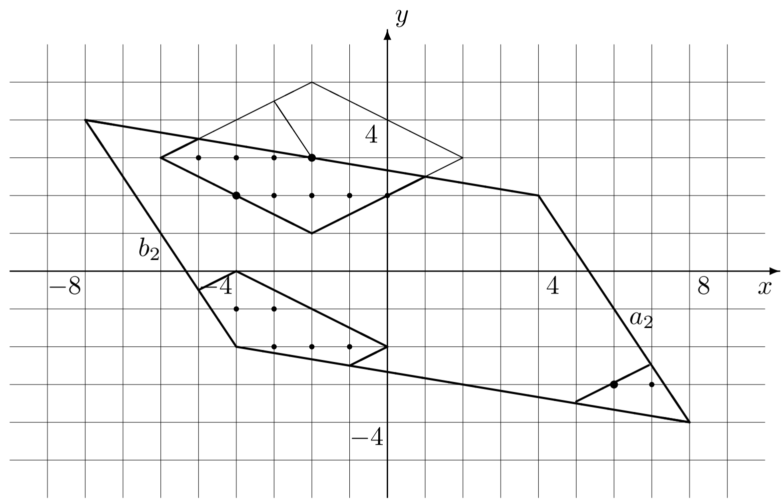

Figure 2: The spectrum of the function $\psi_2^{(1)}$ is market by points, in all emphasized points, the value of $\hat{\psi}_2^{(1)}$ is equal to 2^{-2}

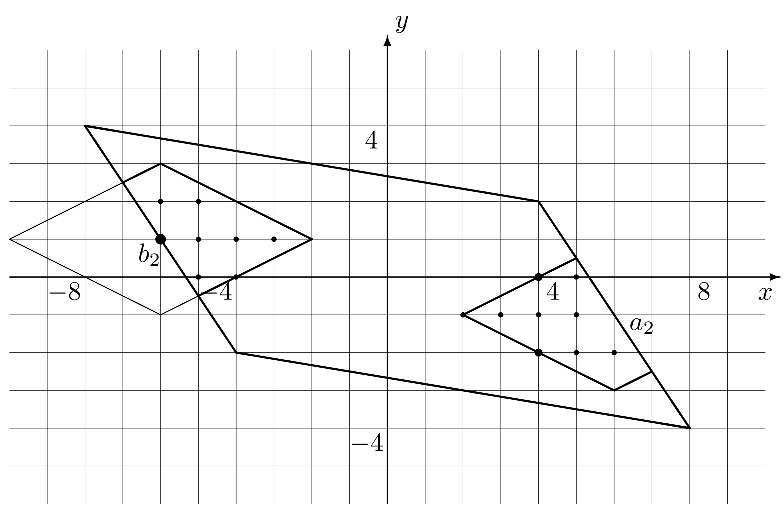

Figure 3: The spectrum of the function $\psi_2^{(2)}$ is marked by points, in all emphasized points except the point $b_2 = (-6, 1)$, the value of $\hat{\psi}_2^{(2)}$ is equal to 2^{-2} and $\hat{\psi}_2^{(2)}(b_2) = 2^{-5/2}$

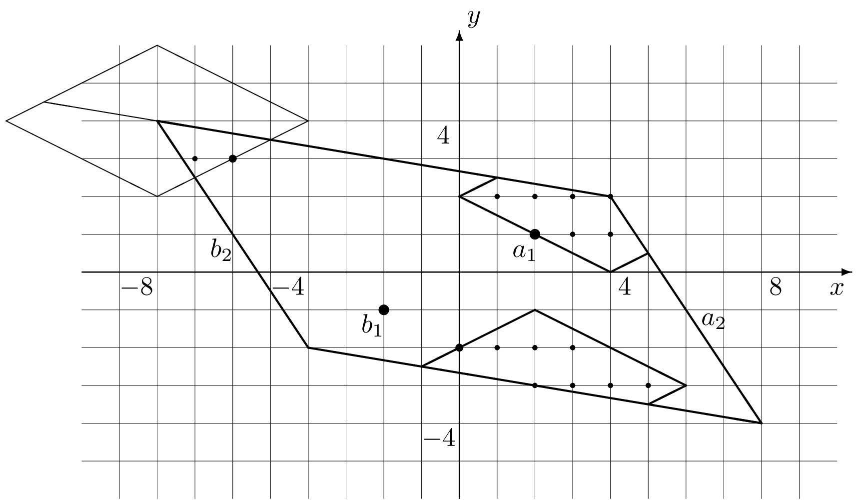

Figure 4: The spectrum of the function $\psi_2^{(3)}$ is marked by points, in all emphasized points except the points $a_1=(2,1)$ and $b_1=(-2,-1)$, the value of $\hat{\psi}_2^{(3)}$ is equal to 2^{-2} and $\hat{\psi}_2^{(3)}(b_1)=2^{-5/2}$

$$\widehat{\psi}_j^{(3)}(l) := \begin{cases} 2^{-j}, & l \equiv n + M^{*j}s_3 \pmod{M^{*j+1}},\ n\in D(M^{*j}),\ n\neq b_{j-1}, l\in\Omega_j\backslash\{a_j,b_j\} \\ 2^{-j-1/2}, & l \equiv n + M^{*j}s_3 \pmod{M^{*j+1}},\ n\in D(M^{*j}),\ n\neq b_{j-1}, l=a_j,\ l=b_j \\ 2^{-j-1/2}, & l\equiv b_{j-1} \pmod{M^{*j+1}},\ l\in\Omega_j\backslash\{a_j,b_j\} \\ -2^{-j-1/2}, & l\equiv a_{j-1} \pmod{M^{*j+1}},\ l\in\Omega_j\backslash\{a_j,b_j\} \\ 0, & \text{otherwise.} \end{cases} \tag{36}$$

The following areas are represented on Figure 2: Ω_2 (large parallelogram); Ω_1 shifted to M^*s_1 (small parallelogram). All spectrum of the function $\psi^{(1)}$ is located in three separated areas inside of the large parallelogram, which form a set congruent to the small parallelogram modulo M^{*2}. Similarly, the spectra of the functions $\psi^{(2)}$ and $\psi^{(3)}$ are respectively represented on Figures 3 and 4.

6. Wavelet Expansion of Functions

Fix a (p,q)-pair satisfying the conditions of Theorems 19. By this theorem, the systems of wavelet functions

$$\{S_r^j\psi_j^{(\nu)},\ j\in\mathbb{Z}_+,\ r\in D(M^j), \nu=1,\dots,m-1\},\quad \{S_r^j\widetilde{\psi}_j^{(\nu)},\ j\in\mathbb{Z}_+,\ r\in D(M^j), \nu=1,\dots,m-1\}$$

are biorthonormal. For functions $f\in L_p(\mathbb{T}^d)$, we can consider expansions with respect to the corresponding Fourier series

$$\langle f, \widetilde{\varphi}_0\rangle\varphi_0 + \sum_{j=0}^{\infty}\sum_{\nu=1}^{m-1}\sum_{r\in D(M^j)}\langle f, S_r^j\widetilde{\psi}_j^{(\nu)}\rangle S_r^j\psi_j^{(\nu)}. \tag{37}$$

Enumerated the sets of digits $D(M^j)=\{r_l\}_{l=0}^{m^j-1}$ in arbitrary way, denote the partial sums of this series by $s_n(f)$, and define convergence of the series as convergence of the sequences $s_n(f)$.

Theorem 21 *Let $\{V_j\}_{j=0}^{\infty}$, $\{\widetilde{V}_j\}_{j=0}^{\infty}$ form a $(\infty,1)$-pair with scaling sequences $\{\varphi_j\}_{j=0}^{\infty}$, $\{\widetilde{\varphi}_j\}_{j=0}^{\infty}$ such that $\{S_n^j\varphi_j\}_{n\in D(M^j)}$ and $\{S_k^j\widetilde{\varphi}_j\}_{n\in D(M^j)}$ are biorthonormal systems, and let $\{\psi_j^{(\nu)}\}_{j=0}^{\infty}$, $\{\widetilde{\psi}_j^{(\nu)}\}_{j=0}^{\infty}$, $\nu=1,\dots,m-1$ be the corresponding sequences of wavelet functions. If*

$$\sup_j \|\widetilde{\varphi}_j\|_1, \sup_{j,\nu} \|\widetilde{\psi}_j^{(\nu)}\|_1 < \infty \tag{38}$$

and there exists a monotone decreasing function K defined on $[0,\infty)$ such that

$$\int\limits_{\mathbb{R}^d} K(|x|)\,dx < \infty \tag{39}$$

and

$$|\varphi_j(x)|\,, \left|\psi_j^{(\nu)}(x)\right| \le K(|M^j x|) \tag{40}$$

for all $x\in[-1/2,1/2]^d$, then for each $f\in C(\mathbb{T}^d)$, the series (37) uniformly converges to f, and for each $g\in L(\mathbb{T}^d)$, the series

$$\langle\, f,\varphi_0\,\rangle\,\widetilde{\varphi}_0 + \sum_{j=0}^{\infty}\sum_{\nu=1}^{m-1}\sum_{r\in D(M^j)} \langle\, f, S_r^j\psi_j^{(\nu)}\,\rangle\, S_r^j\widetilde{\psi}_j^{(\nu)}. \tag{41}$$

converges to g in the norm of $L(\mathbb{T}^d)$.

Proof. Let $N=\kappa m^j+n$, $j\in\mathbb{Z}_+$, $\kappa=1,\dots,m-2$, $n=0,\dots,m^j-1$, then the partial sum $s_N(f,x)$ of the series (37) is represented as

$$s_N(f) = \langle\, f,\widetilde{\varphi}_0\,\rangle\,\varphi_0 + \sum_{i=0}^{j-1}\sum_{\nu=1}^{m-1}\sum_{r\in D(M^i)} \langle\, f, S_r^i\widetilde{\psi}_i^{(\nu)}\,\rangle\, S_r^i\psi_i^{(\nu)} \quad +$$

$$\sum_{\nu=1}^{\kappa}\sum_{r\in D(M^j)} \langle\, f, S_r^j\widetilde{\psi}_j^{(\nu)}\,\rangle\, S_r^j\psi_j^{(\nu)} + \sum_{l=0}^{n}\langle\, f, S_{r_l}^j\widetilde{\psi}_j^{\kappa+1}\,\rangle\, S_{r_l}^j\psi_j^{\kappa+1} \quad =$$

$$s_N^{(0)}(f) + \sum_{\nu=1}^{\kappa} s_N^{(\nu)}(f) + s_N^{(\kappa+1)}(f). \tag{42}$$

Since s_N^1 is a projective operator onto the space V_j, the sum $s_N^{(0)}(f)$ in a right hand side can be reexpended with respect to the shifts of the function φ_j:

$$s_N^{(0)}(f) = \sum_{r\in D(M^j)} \langle\, f, S_r^j\widetilde{\varphi}_j\,\rangle\, S_r^j\varphi_j. \tag{43}$$

Using (38), we have

$$|s_N^{(0)}(f,x)| = \left|\int\limits_{\mathbb{T}^d} f(t)\sum_{l=0}^{m^j-1}\widetilde{\varphi}_j(t+M^{-j}r_l)\varphi_j(x+M^{-j}r_l)\,dt\right| \le \|f\|_\infty\|\widetilde{\varphi}_j\|_1\sum_{l=0}^{m^j-1}|\varphi_j(x+M^{-j}r_l)|.$$

Set

$$g_j(t) = \begin{cases} \varphi_j(M^{-j}t), & t\in M^j\mathbb{T}^d,\\ 0, & t\notin M^j\mathbb{T}^d. \end{cases}$$

It is clear that

$$|g_j(x)| \le K(|x|) \tag{44}$$

and

$$\varphi_j(x) = \sum_{k\in\mathbb{Z}^d} g_j(M^j x + M^j k) \tag{45}$$

for all $x \in \mathbb{R}^d$. The relations (44), (45) give

$$\sum_{l=0}^{m^j-1} |\varphi_j(x + M^{-j} r_l)| \le \sum_{l=0}^{m^j-1} \sum_{k\in\mathbb{Z}^d} K(|M^j x + M^j k + r_l|) = \sum_{k\in\mathbb{Z}^d} K(|M^j x + k|).$$

The monotonicity of K and (39) yields uniform the boundedness of this sum. Thus is proved that, for $\nu = 0$, the operators $s_N^{(\nu)}$ taking $C(\mathbb{T}^d)$ to $C(\mathbb{T}^d)$, are uniformly bounded in norm. The uniform boundedness of the operators $s_N^{(\nu)}$ for $\nu = 1, \dots, m-1$ can be proved similarly. Therefore,

$$\|s_N(f)\| \le C, \tag{46}$$

where C is an absolute constant.

Given $\epsilon > 0$, by the property $MR2$ of Definition 5, there exists $F \in V_{j_0}$ such that

$$\|f - F\|_\infty < \epsilon.$$

Due to Theorem 19, $s_N(F) = F$ for $N \ge m^{j_0}$. Therefore, by (46), we have

$$|f - s_N(f)| = |f - F - s_N(f - F)| \le (C+1)\|f - F\| \le (C+1)\epsilon.$$

Thus the first statement of the theorem is proved. The second statement can be proved similarly if estimating the sums $s_N^{(\nu)}(f)$ we exchange the roles of φ_j, $\psi_j^{(\nu)}$ and x respectively with $\widetilde{\varphi}_j$, $\widetilde{\psi}_j$ and t. ■

Theorem 22 *Let $\{V_j\}_{j=0}^\infty$, $\{\widetilde{V}_j\}_{j=0}^\infty$ form a (p,q)-pair with scaling sequences $\{\varphi_j\}_{j=0}^\infty$, $\{\widetilde{\varphi}_j\}_{j=0}^\infty$ such that $\{S_n^j \varphi_j\}_{n\in D(M^j)}$ and $\{S_k^j \widetilde{\varphi}_j\}_{n\in D(M^j)}$ are biorthonormal systems, and let $\{\psi_j^{(\nu)}\}_{j=0}^\infty, \{\widetilde{\psi}_j^{(\nu)}\}_{j=0}^\infty$, $\nu = 1, \dots, m-1$ be the corresponding sequences of wavelet functions. If there exists a monotone decreasing function K defined on $[0,\infty)$ and satisfying (39) such that*

$$|\varphi_j(x)|, \ \left|\psi_j^{(\nu)}(x)\right|, \ \left|m^{-j}\widetilde{\varphi}_j(x)\right|, \ \left|m^{-j}\widetilde{\psi}_j^{(\nu)}(x)\right| \le K(|M^j x|) \tag{47}$$

for all $x \in [-1/2, 1/2]^d$, then, for each function $f \in L_p(\mathbb{T}^d)$, the series (37) converges to f at each its Lebesgue point.

Lemma 23 *[20, Lemma 2.7] Let K be a non-negative monotone decreasing function defined on $[0,\infty)$ and satisfying (39). Then there exists a constant C depending only on the dimension of the spaces d such that*

$$\sum_{k\in\mathbb{Z}^d} K(|x+k|)K(|y+k|) \le CK\left(\frac{|x-y|}{5}\right) \tag{48}$$

for all $x, y \in \mathbb{R}^d$.

Proof of Theorem 22. Let x be a Lebesgue point of f, $N = \kappa m^j + n$, $j \in \mathbb{Z}_+$, $\kappa = 1, \dots, m-2$, $n = 0, \dots, m^j - 1$. Since the space V_0 consists only of constants, $s_N(h, x) = h$ for all $h \equiv$ const. From this, using (42) (, 43), we have

$$f(x) - s_N(f,x) = \int_{\mathbb{T}^d} (f(x) - f(x+t)) \sum_{r\in D(M^j)} S_r^j \widetilde{\varphi}_j(x+t) \,\rangle\, S_r^j \varphi_j(x)\, dt +$$

$$\sum_{\nu=1}^{\kappa}\int_{\mathbb{T}^d}(f(x)-f(x+t))\sum_{r\in D(M^j)}S_r^j\widetilde{\psi}_j^{(\nu)}(x+t)S_r^j\psi_j^{(\nu)}(x)\,dt+$$

$$\int_{\mathbb{T}^d}(f(x)-f(x+t))\sum_{l=0}^{n}S_{r_l}^j\widetilde{\psi}_j^{\kappa+1}(x+t)S_{r_l}^j\psi_j^{\kappa+1}(x)\,dtI_0+\sum_{\nu=1}^{\kappa}I_\nu+I_{\kappa+1}. \quad (49)$$

Applying (45), (44) and similar relations for $\widetilde{\varphi}_j$, we get

$$I_0\le m^j\int_{\mathbb{T}^d}|f(x)-f(x+t)|\sum_{r\in D(M^j)}\sum_{\ell\in\mathbb{Z}^d}K\left(M^j(x+t)+M^j\ell+r\right)\sum_{k\in\mathbb{Z}^d}K\left(M^jx+M^jk+r\right)\,dt=$$

$$m^j\int_{\mathbb{R}^d}|f(x)-f(x+t)|\sum_{k\in\mathbb{Z}^d}K(M^j(x+t)+k)K\left(M^jx+k\right)\,dt.$$

It follows from Lemma 23 that

$$I_0\le Cm^j\int_{\mathbb{R}^d}|f(x)-f(x+t)|K\left(\frac{M^jt}{5}\right)\,dt.$$

This and a minor modification of Theorem 1.8 [25] yield $I_0\underset{j\to\infty}{\longrightarrow}0$. For $\nu=1,\dots,m-1$, the relation $I_2\underset{j\to\infty}{\longrightarrow}0$ can be proved similarly. Combining these relations with (49), we get

$$\lim_{N\to\infty}s_N(f,x)=f(x).\quad\blacksquare$$

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