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Metric problems for quadrics in multidimensional space

Alexei Yu. Uteshev, Marina V. Yashina

St. Petersburg State University, Universitetskij pr. 35, Petrodvoretc, 198504, St. Petersburg, Russia

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ABSTRACT

Given the equations of the first and the second order manifolds in $\mathbb{R}^n$, we construct the *distance equation*, i.e. a univariate algebraic equation one of the zeros of which (generically minimal positive) coincides with the square of the distance between these manifolds. To achieve this goal we employ Elimination Theory methods. In the frame of this approach we also deduce the necessary and sufficient algebraic conditions under which the manifolds intersect and propose an algorithm for finding the coordinates of their nearest points. The case of parameter dependent manifolds is also considered.

1. Introduction

We solve the problem of finding the distance d from ellipsoid given by equation

$$X^T\mathbf{A}_1X + 2B_1^TX - 1 = 0 \tag{1}$$

either to linear manifold given by the system of equations

$$C_1^TX = h_1, \dots, C_k^TX = h_k \tag{2}$$

or to quadric given by equation

$$X^T\mathbf{A}_2X + 2B_2^TX - 1 = 0. \tag{3}$$

E-mail addresses: alexeiuteshev@gmail.com (A.Yu. Uteshev), marina.yashina@gmail.com (M.V. Yashina).

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Here $X = [x_1, \dots, x_n]^T$ is the column of variables, $\{B_1, B_2, C_1, \dots, C_k\} \subset \mathbb{R}^n$ are the given columns, $\{h_1, \dots, h_k\} \subset \mathbb{R}$ are the given numbers, $\mathbf{A}_1$ and $\mathbf{A}_2$ are the given symmetric matrices and $\mathbf{A}_1$ is sign-definite. The distance is evaluated in Euclidean metrics $\|\cdot\|_2$, i.e.

$$d = \min \sqrt{(X-Y)^T(X-Y)}$$

subject to $\{X, Y\} \subset \mathbb{R}^n$, $X^T\mathbf{A}_1 X + 2B_1^T X - 1 = 0$ and

$$C_1^T Y = h_1, \dots, C_k^T Y = h_k \quad \text{or} \quad Y^T\mathbf{A}_2 Y + 2B_2^T Y - 1 = 0.$$

The traditional starting point for solving this nonlinear constrained optimization problem is the system of equations obtained on equating to zero the derivatives of the Lagrange function given in the form

$$(X-Y)^T(X-Y) - \lambda\left(X^T\mathbf{A}_1 X + 2B_1^T X - 1\right) - \lambda_1\left(C_1^T Y - h_1\right) - \dots - \lambda_k\left(C_k^T Y - h_k\right) \tag{4}$$

or

$$(X-Y)^T(X-Y) - \lambda_1\left(X^T\mathbf{A}_1 X + 2B_1^T X - 1\right) - \lambda_2\left(Y^T\mathbf{A}_2 Y + 2B_2^T Y - 1\right) \tag{5}$$

w.r.t. all the variables involved. For instance, in the second case, this system is as follows:

$$\begin{cases} X - Y - \lambda_1(\mathbf{A}_1 X + B_1) = \mathbb{O}, \ -X + Y - \lambda_2(\mathbf{A}_2 Y + B_2) = \mathbb{O}, \\ X^T\mathbf{A}_1 X + 2B_1^T X = 1, \ Y^T\mathbf{A}_2 Y + 2B_2^T Y = 1. \end{cases} \tag{6}$$

It can be resolved w.r.t. X and Y with the aid of suitable iterative procedures; we refer to Choi et al. (2003), Lin and Han (2002), Schneider and Eberly (2003), Tamasyan and Chumakov (2014). The main computational difficulty of this approach is well-known: it is the problem of distinguishing whether the obtained solution provides the global minimum of the distance function or just a local one.

An alternative approach comprises the symbolic transformation of the algebraic system (6) to reduce the number of involved variables. In Lennerz and Schömer (2002) an algorithm has been suggested; it entails reducing (6) to the nonlinear algebraic system w.r.t. the Lagrange multipliers λ_1 and λ_2. By the use of the *resultant* technique, one of these parameters can be eliminated, and this results in a univariate algebraic equation w.r.t. the remaining parameter. Once all the real roots of this equation are evaluated, one can find the corresponding values for the eliminated parameter and restore the corresponding pair for X and Y from the linear system given by the first and the second equations from (6).

The problem of distance evaluation between the given manifolds relates directly to the problem of finding conditions for manifold intersection. For the case of some classes of quadrics in $\mathbb{R}^2$ and $\mathbb{R}^3$ this problem can be reduced to the investigation of the real zeros of the *characteristic equation*

$$f(\lambda) \stackrel{def}{=} \det\left(\begin{bmatrix} \mathbf{A}_2 & B_2 \\ B_2^T & -1 \end{bmatrix} - \lambda \begin{bmatrix} \mathbf{A}_1 & B_1 \\ B_1^T & -1 \end{bmatrix}\right) = 0. \tag{7}$$

Let Eqs. (1) and (3) define ellipses in $\mathbb{R}^2$ (or ellipsoids in $\mathbb{R}^3$) with the interior of these curves (surfaces) given by inequalities

$$(X, 1)^T \begin{bmatrix} \mathbf{A}_j & B_j \\ B_j^T & -1 \end{bmatrix} \begin{pmatrix} X \\ 1 \end{pmatrix} < 0 \quad \text{for } j \in \{1, 2\}, \ X \in \mathbb{R}^2 \ \left(\text{or } \mathbb{R}^3\right).$$

These curves (surfaces) are said to be *separate* if their sets of interior points or boundary points do not intersect. It turns out that the necessary and sufficient conditions for this property can be expressed in the form of algebraic inequalities w.r.t. the coefficients of (1) and (3) (Etayo et al., 2006; Wang and Krasauskas, 2003; Wang et al., 2001).

In relation to the problem of distance evaluation between quadrics, the problem of finding the so called *distance of closest approach* of two ellipses in $\mathbb{R}^2$ or ellipsoids in $\mathbb{R}^3$ should be mentioned, i.e. the distance between the centers of these quadrics when they collide *externally* (in the sense correlating with the notion *separate* mentioned in the previous paragraph). The problem, which is useful in modeling and simulating systems of liquid crystals, has been solved analytically in Etayo et al. (2010), Zheng and Palffy-Muhoray (2007), Zheng et al. (2009).

The present paper is focused on the evaluation of the distance directly, i.e. without the prior determination of the coordinates of nearest points in the manifolds. This is achieved via introduction of a new variable z by the equation

$$z-(X-Y)^T(X-Y)=0.$$

Being attached to the system (6), this equation provides the critical values of the distance function. For the obtained algebraic system one can apply a symbolic procedure of elimination of all the variables except for z. This means it is possible to construct an algebraic univariate equation $\mathcal{F}(z)=0$ one of the zeros of which (generically minimal positive) coincides with the square of the distance we are looking for. We will call this equation *the distance equation*. Its construction can be performed via computation of either the Gröbner basis or of the resultant. We have chosen the second alternative and have succeeded in finding explicit expressions for the distance equation for each of the stated problems in n-dimensional space. Any of the real zeros of this equation correspond to a (generically unique) pair of points in the treated manifolds, and we also suggest an algorithm for the evaluation of their coordinates as rational functions of the value of z. We also deduce the intersection condition for the manifolds.

The paper is structured as follows. Section 2 contains some preliminary results from the classical Elimination Theory: definition and some properties of the *discriminant* of the univariate and bivariate polynomials. The necessity of this particular notion will be justified in the foregoing sections: for any treated combination of the manifolds, the distance equation is constructed with the aid of the computation of an appropriate discriminant.

In Section 3 we consider the distance evaluation problem between the ellipsoid (1) and the linear manifold (2). The main result of this section (Theorem 6) was first formulated in Uteshev and Yashina (2007) in a less general statement and its proof in the cited paper contained an essential gap connected with the *problem of multiple zero existence* for the distance equation. We fill this gap and provide an estimation for the degree of the distance equation.

Section 4 is devoted to a particular case from the previous section, namely the distance evaluation problem between the given point and the ellipsoid (1). The latter is known to be a bottleneck in the problem of finding the *fitting ellipse* (or *ellipsoid*) for a set of data points that appears in pattern recognition, particle physics, computer graphics and statistical error analysis (Kanatani and Rangarajan, 2011; Szpak et al., 2012).

In Section 5 we address the problem of finding the distance between the quadrics. This section contains two results, namely Theorems 10 and 11, which were first formulated in Uteshev and Yashina (2007) (although Theorem 10 lacked the proof in the cited paper). We disrupt here the natural *general-to-specific* order of presentation, beginning in Theorem 10 with the discussion of the particular case of quadrics given by the equations $X^T\mathbf{A}_1X=1$ and $X^T\mathbf{A}_2X=1$ (i.e. possessing the common center $X=\mathbb{O}$). The reason for this disruption is that we have failed to deduce a particular result via passing to a limit $B_1\to\mathbb{O}$, $B_2\to\mathbb{O}$ in the general one. Although both results involve the discriminant in the expressions for the distance equation, in the first case this is the discriminant of the univariate polynomial, while in the second case this is the discriminant of the bivariate one. In this section we also compare the conditions for intersection of the quadrics in $\mathbb{R}^3$ with those presented in Wang and Krasauskas (2003).

In Section 6 we discuss the problem of finding the distance from a point to a family of parameter-dependent manifolds. We show that if the parameterization of the family is of polynomial type, the distance equation can be constructed via the iterated discriminant computation.

Notation.

1. $\mathcal{D}(\cdot)$ (or $\mathcal{D}_x(\cdot)$) denotes the discriminant of the polynomial (subscript denotes the variable w.r.t. which the polynomial is treated);
2. $\mathbf{I}_k$ stands for the identity matrix of the order k;

3. **adj**$(\cdot)$ stands for the adjoint matrix;
4. **G** is the Gram matrix;
5. $\equiv$ means the identity.

Accuracy. In the numerical examples we give the results of approximate computations rounded to 10^{-5}.

2. Algebraic preliminaries

Univariate discriminant. For the univariate polynomial $g(x) = b_0x^N + b_1x^{N-1} + \ldots + b_N \in \mathbb{C}[x]$, $b_0 \neq 0$, $N \geq 2$ the discriminant is formally defined as

$$\begin{aligned} \mathcal{D}_x(g) &\stackrel{def}{=} (-1)^{N(N-1)/2} b_0^{N-2} \prod_{j=1}^{N} g'(\mu_j) \\ &= b_0^{2(N-1)} \prod_{1 \leq j < k \leq N} (\mu_k - \mu_j)^2 \end{aligned} \tag{8}$$

where $\{\mu_1, \ldots, \mu_N\}$ is a set of zeros of $g(x)$ counted with their multiplicities. We will also use an alternative definition of discriminant

$$\mathcal{D}_x(g) \stackrel{def}{=} (-1)^{N(N-1)/2} N^N b_0^{N-1} \prod_{j=1}^{N-1} g(\lambda_j),$$

where $\{\lambda_1, \ldots, \lambda_{N-1}\}$ is a set of zeros of $g'(x)$ counted with their multiplicities. The constructive computation of discriminant – in the form of polynomial function of the coefficients of $g(x)$ – can be performed with the aid of several determinantal representations. We will utilize the approach attributed to Bézout.[1] It is based on the coefficients of the remainders $g_\ell(x)$ of the division of $x^\ell g(x)$ by $g'(x)$:

$$x^\ell g(x) \equiv b_{\ell 0} + b_{\ell 1} x + \ldots + b_{\ell, N-2} x^{N-2} + q_\ell(x) g'(x), \quad q_\ell(x) \in \mathbb{C}[x]$$

for $\ell \in \{0, \ldots, N-2\}$. Compose the matrix from these coefficients

$$\mathfrak{B} \stackrel{def}{=} [b_{\ell j}]_{\ell, j=0}^{N-2}. \tag{9}$$

Denote by $\mathfrak{B}_{N-1,j}$ the cofactor to the corresponding entry of the last row of $\mathfrak{B}$.

Theorem 1. *One has*

$$\mathcal{D}_x(g) = (-1)^{N(N-1)/2} N^N b_0^{N-1} \det \mathfrak{B}.$$

The polynomial $g(x)$ possesses a multiple zero iff $\det \mathfrak{B} = 0$. Under the last condition, the multiple zero is unique and has multiplicity 2 iff $\mathfrak{B}_{N-1,1} \neq 0$; in this case it can be expressed rationally via the coefficients of $g(x)$:

$$\lambda = \frac{\mathfrak{B}_{N-1,2}}{\mathfrak{B}_{N-1,1}}. \tag{10}$$

For two different proofs of this theorem we refer to Bikker and Uteshev (1999), Gonzalez-Vega (1996).

[1] Although the true paternity of this approach has not been authentically traced by the authors of the present paper.

Example 1. Find the real values of the parameter α under which the polynomial

$$g(x) = x^5 + 6x^4 + 2x^3 + \alpha x^2 - x + 3$$

possesses a multiple zero, and evaluate this zero.

Solution. We compute first the remainders $g_\ell(x)$ of the division of $x^\ell g(x)$ by $g'(x)$:

$$\begin{aligned}
g_0(x) &\equiv \frac{81}{25} + \left(-\frac{12}{25}\alpha - \frac{4}{5}\right)x + \left(\frac{3}{5}\alpha - \frac{36}{25}\right)x^2 - \frac{124}{25}x^3, \\
g_1(x) &\equiv \ldots, \\
g_2(x) &\equiv \ldots, \\
g_3(x) &\equiv -\frac{84}{125}\alpha - \frac{63\,884}{3125} + \left(\frac{168}{125}\alpha^2 + \frac{128\,143}{3125}\alpha + \frac{2796}{625}\right)x \\
&\quad + \left(-\frac{6}{25}\alpha^2 - \frac{3072}{625}\alpha + \frac{380\,204}{3125}\right)x^2 + \left(\frac{2174}{125}\alpha + \frac{1\,459\,461}{3125}\right)x^3.
\end{aligned}$$

Then we compose the matrix $\mathfrak{B}$ from the coefficients of powers of x and compute its determinant

$$\det \mathfrak{B} = 5^{-5}(\alpha + 7)\big(324\alpha^4 + 5481\alpha^3 - 87\,771\alpha^2 - 409\,817\alpha + 5\,759\,315\big).$$

The discriminant $\mathcal{D}_x(g(x))$ coincides with the numerator of the last fraction and it vanishes for the following values of the parameter:

$$\alpha_1 \approx -24.63939, \qquad \alpha_2 \approx -9.29644, \qquad \alpha_3 = -7.$$

To evaluate the corresponding multiple zero of $g(x)$, we utilize formula (10):

$$\lambda = -\frac{\frac{27}{625}\alpha^3 + \frac{18}{5}\alpha^2 + \frac{32\,537}{625}\alpha + \frac{2724}{625}}{-\frac{54}{625}\alpha^4 - \frac{1296}{625}\alpha^3 + \frac{4508}{625}\alpha^2 + \frac{17\,208}{125}\alpha - \frac{57\,532}{625}},$$

where the numerator and denominator are the minors to the entries of the last row of $\mathfrak{B}$. Substitution of the obtained values for α into this formula yields the corresponding values of multiple zeros:

$$\lambda_1 \approx -3.80947, \qquad \lambda_2 \approx 0.74648, \qquad \lambda_3 = -1.$$ △

Theorem 2. *One can find polynomials providing the so-called linear representation of the discriminant, i.e., the pair $\{u(x),\ v(x)\} \subset \mathbb{C}[x]$ satisfying the identity*

$$v(x)g(x) + u(x)g'(x) \equiv \det \mathfrak{B}. \tag{11}$$

Here $v(x)$ can be represented as the determinant of the matrix obtained by replacing the first column of $\mathfrak{B}$ by $[1, x, \ldots, x^{N-2}]^T$, while

$$u(x) = -\frac{1}{N}\left(x + \frac{b_1}{Nb_0}\right)v(x) - \frac{1}{Nb_0}\det \widehat{\mathfrak{B}},$$

where $\widehat{\mathfrak{B}}$ denotes the matrix obtained from $\mathfrak{B}$ by replacing its first column by

$$\big[0, b_{0,N-2}, b_{0,N-2}x + b_{1,N-2}, b_{0,N-2}x^2 + b_{1,N-2}x + b_{2,N-2}, \ldots,$$
$$b_{0,N-2}x^{N-3} + b_{1,N-2}x^{N-4} + \ldots + b_{N-3,N-2}\big]^T.$$

Polynomials $u(x)$ and $v(x)$ satisfy the restrictions $\deg u < N - 1$, $\deg v < N - 2$.

The proof can be found in Bikker and Uteshev (1999).

We list some technical and easily deduced statements that will be utilized in the proofs of results in foregoing sections.

Theorem 3. *Let $\phi(x) = p(x)/q(x)$ be a rational function with relatively prime $p(x)$ and $q(x)$. Functions $\phi(x)$ and $\phi'(x)$ posses a common zero iff $\mathcal{D}_x(p(x)) = 0$.*

Theorem 4. *For polynomial $g(x)$ of degree $N \ge 2$ and a constant $A \in \mathbb{C}$ one has:*

$$\mathcal{D}_x\big(A \cdot g(x)\big) = A^{2N-2}\mathcal{D}_x\big(g(x)\big), \tag{12}$$

$$\mathcal{D}_x\big(g(Ax)\big) = A^{N(N-1)}\mathcal{D}_x\big(g(x)\big), \tag{13}$$

$$\mathcal{D}_x\big(x \cdot g(x)\big) = \big[g(0)\big]^2\mathcal{D}_x\big(g(x)\big). \tag{14}$$

Proof. The proof can be done with the aid of the definition of the discriminant (8) and the following table:

	Zero set	Leading coefficient
$A \cdot g(x)$	$\{\mu_1, \dots, \mu_N\}$	Ab_0
$g(Ax)$	$\{\mu_1/A, \dots, \mu_N/A\}$	$A^N b_0$
$x \cdot g(x)$	$\{\mu_1, \dots, \mu_N, 0\}$	b_0

□

Bivariate discriminant. For the given polynomial $g(X) \in \mathbb{C}[X]$, $X = (x_1, x_2)$, $\deg g = N \ge 2$ we define its discriminant as

$$\mathcal{D}_X(g) \overset{def}{=} \prod_{j=1}^{\mathfrak{N}} g(\Lambda_j).$$

Here $\Lambda_j = (\lambda_{j_1}, \lambda_{j_2}) \in \mathbb{C}^2$ stands for the stationary point of $g(X)$, i.e. a zero of the system $\partial g/\partial x_1 = 0$, $\partial g/\partial x_2 = 0$. In a generic case, the latter possesses precisely $\mathfrak{N} = (N-1)^2$ (Bézout's number) zeros in $\mathbb{C}^2$. Constructive computation of $\mathcal{D}_X(g)$ is possible with the aid of an analogue to the division process utilized in the univariate case. Choose the set of $\mathfrak{N}$ power products in X:

$$\big\{\mathcal{M}_\ell(X)\big\}_{\ell=0}^{\mathfrak{N}-1} = \big\{x_1^{j_1}x_2^{j_2} \bigm| 0 \le j_1 < N-1,\ 0 \le j_2 \le 2(N-j_1-2)\big\}. \tag{15}$$

For instance, one has for $N = 4$:

$$\big\{\mathcal{M}_\ell(X)\big\}_{\ell=0}^{8} = \left\{\begin{array}{lllll} 1, & x_2, & x_2^2, & x_2^3, & x_2^4, \\ x_1, & x_1x_2, & x_1x_2^2, & & \\ x_1^2 & & & & \end{array}\right\}. \tag{16}$$

We will call the reduction of the polynomial $\mathcal{M}_\ell(X)g(X)$ modulo $\partial g/\partial x_1$ and $\partial g/\partial x_2$ its representation in the form

$$\mathcal{M}_\ell(X)g(X) \equiv b_{\ell 0}\mathcal{M}_0(X) + \ldots + b_{\ell,\mathfrak{N}-1}\mathcal{M}_{\mathfrak{N}-1}(X) + q_{\ell 1}(X)\partial g/\partial x_1 + q_{\ell 2}(X)\partial g/\partial x_2, \tag{17}$$

with $\{q_{\ell 1}(X), q_{\ell 2}(X)\} \subset \mathbb{C}[X]$. The theoretical possibility of such a representation as well as constructive algorithms for its implementation are discussed in Bikker and Uteshev (1999). We note only that in the case of reducibility, the coefficients $b_{\ell j}$ can be expressed as rational functions of the coefficients of $g(X)$. Reorder the set (15) in such a manner that $\mathcal{M}_0 = 1, \mathcal{M}_1 = x_1, \mathcal{M}_2 = x_2$ and make the matrix from the coefficients of the reductions (17) for $\ell \in \{0, \dots, \mathfrak{N}-1\}$, i.e. for all the power products from (15):

$$\mathfrak{B} = [b_{\ell j}]_{\ell,j=0}^{\mathfrak{N}-1}. \tag{18}$$

Denote by $\mathfrak{B}_{\mathfrak{N},j}$ the cofactor to the corresponding entry of the last row of $\mathfrak{B}$.

Theorem 5. *One has*

$$\mathcal{D}_X(g) = \det \mathfrak{B}.$$

The polynomial $g(X)$ possesses a multiple zero $\Lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2$ (i.e. the zero for which $g = 0$, $\partial g/\partial x_1 = 0$, $\partial g/\partial x_2 = 0$) iff $\det \mathfrak{B} = 0$. Under this condition, the multiple zero is unique if $\mathfrak{B}_{\mathfrak{N},1} \neq 0$; in this case, it can be expressed as

$$\lambda_1 = \mathfrak{B}_{\mathfrak{N},2}/\mathfrak{B}_{\mathfrak{N},1}, \qquad \lambda_2 = \mathfrak{B}_{\mathfrak{N},3}/\mathfrak{B}_{\mathfrak{N},1}. \tag{19}$$

Schur and Frobenius formulas. Subsequently we will use the following formulas for the determinant and the inversion of a block matrix (Horn and Johnson, 1986). These are Schur complement formula:

$$\det \begin{pmatrix} \mathbf{U} & \mathbf{V} \\ \mathbf{S} & \mathbf{T} \end{pmatrix} = \det \mathbf{U} \det \mathbf{K}, \tag{20}$$

and Frobenius formula:

$$\begin{pmatrix} \mathbf{U} & \mathbf{V} \\ \mathbf{S} & \mathbf{T} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{U}^{-1} + \mathbf{U}^{-1}\mathbf{V}\mathbf{K}^{-1}\mathbf{S}\mathbf{U}^{-1} & -\mathbf{U}^{-1}\mathbf{V}\mathbf{K}^{-1} \\ -\mathbf{K}^{-1}\mathbf{S}\mathbf{U}^{-1} & \mathbf{K}^{-1} \end{pmatrix}. \tag{21}$$

Here $\mathbf{U}$ and $\mathbf{T}$ are square matrices,

$$\mathbf{K} \stackrel{def}{=} \mathbf{T} - \mathbf{S}\mathbf{U}^{-1}\mathbf{V},$$

and in the equality (21), it is assumed that $\mathbf{U}$ and $\mathbf{K}$ are nonsingular matrices.

3. Distance between a quadric and a linear manifold

We treat the equations of the manifolds in the form (1) and (2) and assume the columns $C_1, \dots, C_k$ to be linearly independent (the latter results in the restriction $k \le n$). Compose the matrices

$$\mathbf{C} \stackrel{def}{=} [C_1, \dots, C_k], \qquad H \stackrel{def}{=} (h_1, \dots, h_k)^T$$

and

$$\mathbf{G} \stackrel{def}{=} \mathbf{C}^T \mathbf{C}, \tag{22}$$

i.e. $\mathbf{G}$ is the Gram matrix for the columns $C_1, \dots, C_k$. Due to the imposed restriction on $C_1, \dots, C_k$, the matrix $\mathbf{G}$ is nonsingular.

Theorem 6. *The condition*

$$0 \le \begin{vmatrix} \mathbf{A}_1 & B_1 & \mathbf{C} \\ B_1^T & -1 & -H^T \\ \mathbf{C}^T & -H & \mathbb{O} \end{vmatrix} \times \begin{cases} (-1)^{k-1}, & \text{if } \mathbf{A}_1 \text{ is positive definite}, \\ (-1)^n, & \text{if } \mathbf{A}_1 \text{ is negative definite} \end{cases} \tag{23}$$

is necessary and sufficient for the ellipsoid (1) to intersect the linear manifold (2). If this condition is not fulfilled, then the square of the distance between the ellipsoid and the linear manifold equals the minimal positive zero of the equation

$$\mathcal{F}(z) \stackrel{def}{=} \mathcal{D}_\mu \left(\mu^k \begin{vmatrix} \mathbf{A}_1 & B_1 & \mathbf{C} \\ B_1^T & -1 + \mu z & -H^T \\ \mathbf{C}^T & -H & \frac{1}{\mu}\mathbf{G} \end{vmatrix} \right) = 0 \tag{24}$$

provided that this zero is not a multiple one.

Proof. I. Finding the intersection condition. Let us first find the critical value of[2] $V(X)=X^T\mathbf{A}X+2B^TX-1$ in the manifold $\mathbf{C}^TX=H$. The critical point of the Lagrange function

$$X^T\mathbf{A}X+2B^TX-1-\nu_1\big(C_1^TX-h_1\big)-\ldots-\nu_k\big(C_k^TX-h_k\big)$$

satisfies the system of equations

$$2\mathbf{A}X+2B-\mathbf{C}[\nu_1,\ldots,\nu_k]^T=\mathbb{O},\qquad \mathbf{C}^TX=H.$$

Express X from the first matrix equation

$$X=-\mathbf{A}^{-1}B+\frac{1}{2}\mathbf{A}^{-1}\mathbf{C}[\nu_1,\ldots,\nu_k]^T \tag{25}$$

and substitute it into the second one:

$$[\nu_1,\ldots,\nu_k]^T=2\big(\mathbf{C}^T\mathbf{A}^{-1}\mathbf{C}\big)^{-1}\big[\mathbf{C}^T\mathbf{A}^{-1}B+H\big]. \tag{26}$$

Reverse substitution of (26) into (25) yields

$$X_e=-\mathbf{A}^{-1}B+\mathbf{A}^{-1}\mathbf{C}\big(\mathbf{C}^T\mathbf{A}^{-1}\mathbf{C}\big)^{-1}\big[\mathbf{C}^T\mathbf{A}^{-1}B+H\big]$$

and the corresponding critical value of $V(X)$ subject to $\mathbf{C}^TX=H$ equals

$$V(X_e)=-\big(B^T\mathbf{A}^{-1}B+1-\big[B^T\mathbf{A}^{-1}\mathbf{C}+H^T\big]\big(\mathbf{C}^T\mathbf{A}^{-1}\mathbf{C}\big)^{-1}\big[\mathbf{C}^T\mathbf{A}^{-1}B+H\big]\big).$$

With the aid of the Schur formula (20) one can transform the last expression into

$$V(X_e)=\frac{-\left|\begin{array}{cc}\mathbf{C}^T\mathbf{A}^{-1}\mathbf{C} & \mathbf{C}^T\mathbf{A}^{-1}B+H\\ B^T\mathbf{A}^{-1}\mathbf{C}+H^T & B^T\mathbf{A}^{-1}B+1\end{array}\right|}{\det(\mathbf{C}^T\mathbf{A}^{-1}\mathbf{C})}=\frac{(-1)^k\left|\begin{array}{ccc}\mathbf{A} & B & \mathbf{C}\\ B^T & -1 & -H^T\\ \mathbf{C}^T & -H & \mathbb{O}\end{array}\right|}{\det(\mathbf{A})\det(\mathbf{C}^T\mathbf{A}^{-1}\mathbf{C})}. \tag{27}$$

If $V(X_e)=0$ then the linear manifold (2) is tangent to the ellipsoid (1) at $X=X_e$. Otherwise let us compare the sign of $V(X_e)$ with the sign of $V(X)$ at infinity. These signs will be distinct iff the considered manifolds intersect. If $\mathbf{A}$ is a positive definite then $V(\infty)>0$, $\det(\mathbf{A})>0$ and $\det(\mathbf{C}^T\mathbf{A}^{-1}\mathbf{C})>0$. Therefore, $V(X_e)<0$ iff the numerator in (27) is negative. This confirms (23). In case of a negative definite matrix $\mathbf{A}$, one has $V(\infty)<0$ and the sign of $(-1)^k\det(\mathbf{C}^T\mathbf{A}^{-1}\mathbf{C})=\det(\mathbf{C}^T(-\mathbf{A})^{-1}\mathbf{C})$ is positive since $(-\mathbf{A})$ is positive definite. The sign of $\det\mathbf{A}$ equals $(-1)^n$. The condition $V(X_e)>0$ is fulfilled iff the second alternative of (23) is valid.

II. Construction of the distance equation. To simplify the deduction for the second statement of the theorem, we first prove it for the case of the linear manifold passing through the origin, i.e. we assume $H=\mathbb{O}$.

Computing the derivatives of the Lagrange function

$$(X-Y)^T(X-Y)-\lambda\big(X^T\mathbf{A}_2X+2B_2^TX-1\big)-\lambda_1C_1^TY-\ldots-\lambda_kC_k^TY$$

we reduce the constrained optimization problem to the following system of algebraic equations

$$X-Y-\lambda\mathbf{A}X-\lambda B=\mathbb{O}, \tag{28}$$

$$X-Y+\frac{1}{2}\mathbf{C}[\lambda_1,\ldots,\lambda_k]^T=\mathbb{O}, \tag{29}$$

$$X^T\mathbf{A}X+2B^TX-1=0, \tag{30}$$

$$\mathbf{C}^TY=\mathbb{O}. \tag{31}$$

We also introduce a new variable responsible for the critical values of the distance function:

[2] To simplify the notation we will type matrices $\mathbf{A}$ and B without their indices.

$$z-(X-Y)^T(X-Y)=0. \tag{32}$$

Our aim is to eliminate all the variables except for z from the system (28)–(32). We first express X and Y from (28) and (29):

$$X=-\mathbf{A}^{-1}B-\frac{1}{2\lambda}\mathbf{A}^{-1}\mathbf{C}[\lambda_1,\dots,\lambda_k]^T, \tag{33}$$

$$Y=-\mathbf{A}^{-1}B-\frac{1}{2\lambda}\big(\mathbf{A}^{-1}-\lambda\mathbf{I}\big)\mathbf{C}[\lambda_1,\dots,\lambda_k]^T. \tag{34}$$

Then we substitute (34) into (31) to express $\lambda_1,\dots,\lambda_k$ via λ. This can be performed with the aid of the following symmetric matrix of the order k:

$$\mathbf{M}\stackrel{def}{=}\frac{1}{\lambda}\mathbf{C}^T\mathbf{A}^{-1}\mathbf{C}-\mathbf{C}^T\mathbf{C}=\mu\mathbf{C}^T\mathbf{A}^{-1}\mathbf{C}-\mathbf{G}, \tag{35}$$

with $\mathbf{G}$ defined by (22) and $\mu\stackrel{def}{=}1/\lambda$. Indeed, one has

$$\mathbf{M}[\lambda_1,\dots,\lambda_k]^T=-2\mathbf{C}^T\mathbf{A}^{-1}B \tag{36}$$

and, provided that $\mathbf{M}$ is nonsingular,

$$[\lambda_1,\dots,\lambda_k]^T=-2\mathbf{M}^{-1}\mathbf{C}^T\mathbf{A}^{-1}B. \tag{37}$$

Next, we substitute (37) into (29) and then the obtained result into (32):

$$z-B^T\mathbf{A}^{-1}\mathbf{C}\mathbf{M}^{-1}\mathbf{G}\mathbf{M}^{-1}\mathbf{C}^T\mathbf{A}^{-1}B=0. \tag{38}$$

Eq. (38) is a rational one with respect to the variables μ and z.

To find an extra equation for these variables, let us substitute (37) into (33):

$$X=-\mathbf{A}^{-1}B+\mu\mathbf{A}^{-1}\mathbf{C}\mathbf{M}^{-1}\mathbf{C}^T\mathbf{A}^{-1}B \tag{39}$$

and then this expression into (30):

$$\begin{aligned}0&=X^T\mathbf{A}X+2B^TX-1\\&=-B^T\mathbf{A}^{-1}B-1+\mu B^T\mathbf{A}^{-1}\mathbf{C}\mathbf{M}^{-1}\big(\mu\mathbf{C}^T\mathbf{A}^{-1}\mathbf{C}-\mathbf{G}+\mathbf{G}\big)\mathbf{M}^{-1}\mathbf{C}^T\mathbf{A}^{-1}B.\end{aligned}$$

Using (35) and (38), the last equation takes the form

$$\Psi(\mu,z)\stackrel{def}{=}-1+\mu z-B^T\mathbf{A}^{-1}B+\mu B^T\mathbf{A}^{-1}\mathbf{C}\mathbf{M}^{-1}\mathbf{C}^T\mathbf{A}^{-1}B=0. \tag{40}$$

Therefore, system (28)–(32) is reduced to Eqs. (38) and (40). It can be verified that the left-hand side of (38) is just the derivative with respect to μ of that of (40). Thus, it remains to eliminate μ from the system

$$\Psi(\mu,z)=0,\qquad \Psi'_\mu(\mu,z)=0.$$

Taking into account Theorem 3, one can perform this with the aid of the discriminant — and that is the reason for its appearance in the statement of the theorem.

Schur formula (20) aids once again in representing $\Psi(\mu,z)$ in the determinantal form:

$$\Psi(\mu,z)\equiv\frac{\begin{vmatrix}\mathbf{A}&B&\mathbf{C}\\B^T&-1+\mu z&\mathbb{O}\\\mathbf{C}^T&\mathbb{O}&\frac{1}{\mu}\mathbf{G}\end{vmatrix}}{\begin{vmatrix}\mathbf{A}&\mathbf{C}\\\mathbf{C}^T&\frac{1}{\mu}\mathbf{G}\end{vmatrix}}=\frac{\mu^k\begin{vmatrix}\mathbf{A}&B&\mathbf{C}\\B^T&-1+\mu z&\mathbb{O}\\\mathbf{C}^T&\mathbb{O}&\frac{1}{\mu}\mathbf{G}\end{vmatrix}}{\det(\mathbf{A})\det(\mathbf{M})}. \tag{41}$$

One may conclude that the set of zeros of the equation

$$\mathcal{F}(z) = \mathcal{D}_{\mu}\left(\mu^k \begin{vmatrix} \mathbf{A} & B & \mathbf{C} \\ B^T & -1+\mu z & \mathbb{O} \\ \mathbf{C}^T & \mathbb{O} & \frac{1}{\mu}\mathbf{G} \end{vmatrix}\right) = 0 \tag{42}$$

contain all the stationary values of the distance function (32). However, one extra condition has to be verified: the denominator of the fraction (41) should not have common zeros with $\mathcal{F}(z)$. This will be done in the part **IV**.

III. Finding the nearest points in the manifolds. At first glance, this aim appears to be irrelevant to the proof of the main result. To revise this attitude, imagine a situation when the *real* zero of the polynomial $\mathcal{F}(z)$ corresponds to an *imaginary* solution of the system (28)–(31). We postpone an illustrative example for this (as yet hypothetical) trouble for the next section. Let us extract from the part **II** formulas for the coordinates of the points X_* and Y_* in the manifolds corresponding to a zero $z = z_*$ of the distance equation (42).

For $z = z_*$, the polynomial in μ standing in the numerator of (41)

$$\Phi(\mu, z) \stackrel{def}{=} \mu^k \begin{vmatrix} \mathbf{A} & B & \mathbf{C} \\ B^T & -1+\mu z & \mathbb{O} \\ \mathbf{C}^T & \mathbb{O} & \frac{1}{\mu}\mathbf{G} \end{vmatrix} \tag{43}$$

has a multiple zero $\mu = \mu_*$. Provided that the multiple zero is unique and has a multiplicity 2, it can be expressed as rational function of the coefficients of this polynomial (and consequently in z_*) with the aid of (10). We substitute this value into (39), and this gives us the coordinate column for the point in the ellipsoid (1). To obtain the coordinates of the point in the linear manifold $\mathbf{C}^T X = \mathbb{O}$ we first substitute (37) into (34):

$$Y = -\mathbf{A}^{-1}B + \left(\mu\mathbf{A}^{-1} - \mathbf{I}\right)\mathbf{C}\mathbf{M}^{-1}\mathbf{C}^T\mathbf{A}^{-1}B \stackrel{(39)}{=} X - \mathbf{C}\mathbf{M}^{-1}\mathbf{C}^T\mathbf{A}^{-1}B \tag{44}$$

and then set here $\mu = \mu_*$. If the computations in the right-hand sides of the formulas (39) and (44) are executable, these formulas give one the coordinates of the points in the ellipsoid and in the linear manifold corresponding to the zero $z = z_*$ of the distance equation. If, in addition, z_* stands for the minimal positive zero for (42), then (39) and (44) provide the coordinates of the nearest points X_* and Y_* in the considered manifolds. By construction, the dependency X_* and Y_* on z_* is expressed via real rational functions.

IV. Nonsingularity of the matrix M. The algorithm for finding the nearest points in the considered manifolds fails when the matrix **M** becomes singular for $\mu = \mu_*$. In the present part we demonstrate that this problem is avoided by imposing the simplicity restriction for the minimal zero of the distance equation in the statement of the theorem.

In accordance with Theorem 2, the polynomial $\mathcal{F}(z)$, being the discriminant of $\Phi(\mu, z)$, permits the linear representation

$$\mathcal{F}(z) \equiv v(\mu, z)\Phi + u(\mu, z)\Phi'_{\mu}, \tag{45}$$

with the polynomials $\{v(\mu, z), u(\mu, z)\} \subset \mathbb{R}[\mu, z]$ satisfying the degree restrictions: $\deg_{\mu} u < \deg_{\mu} \Phi$, $\deg_{\mu} v < \deg_{\mu} \Phi'_{\mu}$.

If $z = z_*$ stands for a zero of $\mathcal{F}(z)$, then $\Phi(\mu, z_*)$ and $\Phi'_{\mu}(\mu, z_*)$ possesses a common zero $\mu = \mu_*$. Differentiate (45) with respect to z:

$$\mathcal{F}'(z) \equiv v'_z\Phi + v\Phi'_z + u'_z\Phi'_{\mu} + u\Phi''_{\mu z}$$

and substitute $\mu = \mu_*$, $z = z_*$:

$$\mathcal{F}'(z_*) = v\Phi'_z + u\Phi''_{\mu z}. \tag{46}$$

We intend to prove that $u(\mu_*, z_*) = 0$. To do this, we differentiate (45) with respect to μ:

$$0 \equiv v'_{\mu}\Phi + v\Phi'_{\mu} + u'_{\mu}\Phi'_{\mu} + u\Phi''_{\mu^2}$$

and substitute $\mu = \mu_*$, $z = z_*$

$$0 = u(\mu_*, z_*) \frac{\partial^2 \Phi}{\partial \mu^2}\bigg|_{(\mu_*, z_*)} \Leftrightarrow \tag{47}$$

$$u(\mu_*, z_*) = 0 \quad \text{or} \quad \frac{\partial^2 \Phi}{\partial \mu^2}\bigg|_{(\mu_*, z_*)} = 0. \tag{48}$$

The second alternative from (48) has the meaning that the zero $\mu = \mu_*$ is of multiplicity $\mathfrak{m} > 2$ for $\Phi(\mu, z_*)$. In this case, one has from (45):

$$0 \equiv v(\mu, z_*)\Phi(\mu, z_*) + u(\mu, z_*)\Phi'_\mu(\mu, z_*) \quad \Leftrightarrow$$
$$u(\mu, z_*)\Phi'_\mu(\mu, z_*) \equiv -v(\mu, z_*)\Phi(\mu, z_*). \tag{49}$$

Since the multiplicity of $\mu = \mu_*$ for $\Phi'_\mu(\mu, z_*)$ equals $\mathfrak{m} - 1$ it follows from (49) that its left-hand side is divisible by $(\mu - \mu_*)^{\mathfrak{m}}$ while one of its factors is divisible at most by $(\mu - \mu_*)^{\mathfrak{m}-1}$. Consequently, $u(\mu, z_*)$ is divisible by $\mu - \mu_*$ and hence $u(\mu_*, z_*) = 0$. Hence, in any case, the condition (47) implies that $u(\mu_*, z_*) = 0$. Formula (46) yields then that $\mathcal{F}'(z_*) = v(\mu_*, z_*)\partial\Phi/\partial z|_{(\mu_*, z_*)}$ and, provided that z_* is a simple zero for $\mathcal{F}(z)$, one has $\mathcal{F}'(z_*) \neq 0$, which results in $\partial\Phi/\partial z|_{(\mu_*, z_*)} \neq 0$. To obtain the expression for the last derivative, let us differentiate the determinantal representation (43)

$$\frac{\partial \Phi}{\partial z} = \mu^k \begin{vmatrix} \mathbf{A} & B & \mathbf{C} \\ \mathbb{O} & -\mu & \mathbb{O} \\ \mathbf{C}^T & \mathbb{O} & -\frac{1}{\mu}\mathbf{G} \end{vmatrix} = -\mu^{k+1} \begin{vmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C}^T & -\frac{1}{\mu}\mathbf{G} \end{vmatrix}$$
$$= -\mu^{k+1} \det(\mathbf{A}) \det\left(-\frac{1}{\mu}\mathbf{G} - \mathbf{C}^T\mathbf{A}^{-1}\mathbf{C}\right)$$
$$= (-1)^{k+1}\mu \det(\mathbf{A}) \det\big(\mathbf{G} + \mu\mathbf{C}^T\mathbf{A}^{-1}\mathbf{C}\big) = (-1)^{k+1}\mu \det(\mathbf{A}) \det(\mathbf{M}).$$

Since $\partial\Phi/\partial z \neq 0$ for $\mu = \mu_*$, $z = z_*$, the matrix $\mathbf{M}$ should be nonsingular for these values.

V. Construction of the distance equation for the case $\boldsymbol{H} \neq \mathbb{O}$. We now turn to the case of the general linear manifold $\mathbf{C}^T X = H$. We will reduce the treatment of this case to the one dealt with in part **II** with the aid of the substitution $X = \tilde{X} + Y_0$ where Y_0 is an arbitrary point in the linear manifold: $\mathbf{C}^T Y_0 = H$. This substitution yields the following representation for the equations of the manifolds:

$$\tilde{X}^T \tilde{\mathbf{A}} \tilde{X} + 2\tilde{B}\tilde{X} - 1 = 0 \quad \text{and} \quad \mathbf{C}^T \tilde{X} = \mathbb{O}.$$

Here

$$\tilde{\mathbf{A}} \stackrel{def}{=} \frac{1}{\varkappa}\mathbf{A}, \qquad \tilde{B} \stackrel{def}{=} \frac{1}{\varkappa}(\mathbf{A}Y_0 + B) \quad \text{with } \varkappa \stackrel{def}{=} -\big(Y_0^T\mathbf{A}Y_0 + 2B^T Y_0 - 1\big). \tag{50}$$

If the manifolds do not intersect then $\varkappa \neq 0$.

Construct Eq. (42) for these manifolds. Our aim now is to express the determinant

$$\begin{vmatrix} \tilde{\mathbf{A}} & \tilde{B} & \mathbf{C} \\ \tilde{B}^T & -1 + \mu z & \mathbb{O} \\ \mathbf{C}^T & \mathbb{O} & \frac{1}{\mu}\mathbf{G} \end{vmatrix} \tag{51}$$

in terms of the matrices $\mathbf{A}$ and B. This will be done with the aid of tiresome manipulations with the blocks of the determinant. First, we reorder its columns and rows

$$= \begin{vmatrix} \tilde{\mathbf{A}} & \mathbf{C} & \tilde{B} \\ \mathbf{C}^T & \frac{1}{\mu}\mathbf{G} & \mathbb{O} \\ \tilde{B}^T & \mathbb{O} & -1 + \mu z \end{vmatrix}.$$

Then we utilize the Schur formula:

$$= \begin{vmatrix} \tilde{\mathbf{A}} & \mathbf{C} \\ \mathbf{C}^T & \frac{1}{\mu}\mathbf{G} \end{vmatrix} \cdot \left\{ -1 + \mu z - \left[\tilde{B}^T, \mathbb{O}_{1\times k}\right] \begin{pmatrix} \tilde{\mathbf{A}} & \mathbf{C} \\ \mathbf{C}^T & \frac{1}{\mu}\mathbf{G} \end{pmatrix}^{-1} \begin{bmatrix} \tilde{B} \\ \mathbb{O}_{k\times 1} \end{bmatrix} \right\}. \tag{52}$$

Next we use the Frobenius formula (21) to inverse the block matrix standing inside the braces:

$$\begin{pmatrix} \tilde{\mathbf{A}} & \mathbf{C} \\ \mathbf{C}^T & \frac{1}{\mu}\mathbf{G} \end{pmatrix}^{-1} = \begin{pmatrix} \tilde{\mathbf{A}}^{-1} + \tilde{\mathbf{A}}^{-1}\mathbf{C}\mathbf{K}^{-1}\mathbf{C}^T\tilde{\mathbf{A}}^{-1} & -\tilde{\mathbf{A}}^{-1}\mathbf{C}\mathbf{K}^{-1} \\ -\mathbf{K}^{-1}\mathbf{C}^T\tilde{\mathbf{A}}^{-1} & \mathbf{K}^{-1} \end{pmatrix}$$

where

$$\mathbf{K} \stackrel{def}{=} \varkappa\left(\frac{1}{\mu\varkappa}\mathbf{G} - \mathbf{C}^T\mathbf{A}^{-1}\mathbf{C}\right).$$

Thus the expression in the braces on the right-hand side of (52) equals

$$\begin{aligned}
&-1 + \mu z - \tilde{B}^T\left(\tilde{\mathbf{A}}^{-1} + \tilde{\mathbf{A}}^{-1}\mathbf{C}\mathbf{K}^{-1}\mathbf{C}^T\tilde{\mathbf{A}}^{-1}\right)\tilde{B} \\
&\stackrel{(50)}{=} -1 + \mu z - \frac{1}{\varkappa^2}(\mathbf{A}Y_0 + B)^T\left(\varkappa\mathbf{A}^{-1} + \varkappa^2\mathbf{A}^{-1}\mathbf{C}\mathbf{K}^{-1}\mathbf{C}^T\mathbf{A}^{-1}\right)(\mathbf{A}Y_0 + B) \\
&= -1 + \mu z - \frac{1}{\varkappa}(\mathbf{A}Y_0 + B)^T\mathbf{A}^{-1}(\mathbf{A}Y_0 + B) \\
&\quad - (\mathbf{A}Y_0 + B)^T\mathbf{A}^{-1}\mathbf{C}\mathbf{K}^{-1}\mathbf{C}^T\mathbf{A}^{-1}(\mathbf{A}Y_0 + B) \\
&= -1 + \mu z - \frac{1}{\varkappa}\left(Y_0^T\mathbf{A}Y_0 + 2\,B^TY_0 + B^T\mathbf{A}^{-1}B\right) \\
&\quad - \left(Y_0^T\mathbf{C}\mathbf{K}^{-1}\mathbf{C}^TY_0 + 2\,B^T\mathbf{A}^{-1}\mathbf{C}\mathbf{K}^{-1}\mathbf{C}^TY_0 + B^T\mathbf{A}^{-1}\mathbf{C}\mathbf{K}^{-1}\mathbf{C}^T\mathbf{A}^{-1}B\right) \\
&\stackrel{(50)}{=} -1 + \mu z - \frac{1}{\varkappa}\left(1 - \varkappa + B^T\mathbf{A}^{-1}B\right) \\
&\quad - \left(H^T\mathbf{K}^{-1}H + 2\,B^T\mathbf{A}^{-1}\mathbf{C}\mathbf{K}^{-1}H + B^T\mathbf{A}^{-1}\mathbf{C}\mathbf{K}^{-1}\mathbf{C}^T\mathbf{A}^{-1}B\right) \\
&= \frac{1}{\varkappa}\Big(-1 + \mu\varkappa z - B^T\mathbf{A}^{-1}B - H^T\left(\varkappa\mathbf{K}^{-1}\right)H - 2\,B^T\mathbf{A}^{-1}\mathbf{C}\left(\varkappa\mathbf{K}^{-1}\right)H \\
&\quad - B^T\mathbf{A}^{-1}\mathbf{C}\left(\varkappa\mathbf{K}^{-1}\right)\mathbf{C}^T\mathbf{A}^{-1}B\Big).
\end{aligned}$$

Now we utilize the Frobenius formula (21) in the reverse direction:

$$= \frac{1}{\varkappa}\left(-1 + \mu\varkappa z - \left[B^T, -H^T\right]\begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C}^T & \frac{1}{\mu\varkappa}\mathbf{G} \end{pmatrix}^{-1}\begin{bmatrix} B \\ -H \end{bmatrix}\right)$$

and finally the Schur formula (20) in the reverse direction:

$$= \frac{1}{\varkappa\begin{vmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C}^T & \frac{1}{\mu\varkappa}\mathbf{G} \end{vmatrix}}\begin{vmatrix} \mathbf{A} & \mathbf{C} & B \\ \mathbf{C}^T & \frac{1}{\mu\varkappa}\mathbf{G} & -H \\ B^T & -H^T & -1 + \mu\varkappa z \end{vmatrix}.$$

It remains to substitute the last expression instead of the braces into the right-hand side of (52). Since

$$\begin{vmatrix} \tilde{\mathbf{A}} & \mathbf{C} \\ \mathbf{C}^T & \frac{1}{\mu}\mathbf{G} \end{vmatrix} = \begin{vmatrix} \frac{1}{\varkappa}\mathbf{A} & \mathbf{C} \\ \mathbf{C}^T & \frac{1}{\mu}\mathbf{G} \end{vmatrix} = \frac{1}{\varkappa^n}\begin{vmatrix} \mathbf{A} & \varkappa\mathbf{C} \\ \mathbf{C}^T & \frac{1}{\mu}\mathbf{G} \end{vmatrix} = \frac{\varkappa^k}{\varkappa^n}\begin{vmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C}^T & \frac{1}{\mu\varkappa}\mathbf{G} \end{vmatrix},$$

one finds that the determinant (51) equals

$$\frac{\varkappa^k}{\varkappa^{n+1}}\begin{vmatrix} \mathbf{A} & \mathbf{C} & B \\ \mathbf{C}^T & \frac{1}{\mu\varkappa}\mathbf{G} & -H \\ B^T & -H^T & -1 + \mu\varkappa z \end{vmatrix}.$$

If we multiply this determinant by μ^k and take the discriminant of the obtained polynomial w.r.t. μ, then, due to the properties (12) and (13) of the discriminant, the resulting expression will differ from (24) only by the power of $\varkappa$. $\square$

Corollary 1. *If the system of columns $C_1, \dots, C_k$ is orthonormal, then the expression under the discriminant sign in (24) can be reduced to*

$$\begin{vmatrix} \mathbf{A}_1 - \mu \mathbf{C}\mathbf{C}^T & B_1 + \mu \mathbf{C} H \\ B_1^T + \mu H^T \mathbf{C}^T & -1 + \mu z - \mu H^T H \end{vmatrix}. \tag{53}$$

Proof. For this case one has: $\mathbf{G} = \mathbf{I}_k$. Further reasoning is based on the Schur formula:

$$\begin{vmatrix} \mathbf{A}_1 & B_1 & \mathbf{C} \\ B_1^T & -1+\mu z & -H^T \\ \mathbf{C}^T & -H & \frac{1}{\mu}\mathbf{I} \end{vmatrix} = \begin{vmatrix} \frac{1}{\mu}\mathbf{I} & \mathbf{C}^T & -H \\ \mathbf{C} & \mathbf{A}_1 & B_1 \\ -H^T & B_1^T & -1+\mu z \end{vmatrix}$$

$$\stackrel{(20)}{=} \det\left(\frac{1}{\mu}\mathbf{I}\right) \det\left(\begin{bmatrix} \mathbf{A}_1 & B_1 \\ B_1^T & -1+\mu z \end{bmatrix} - \mu \begin{bmatrix} \mathbf{C} \\ -H^T \end{bmatrix} [\mathbf{C}^T, -H]\right). \quad \square$$

Example 2. Find the distance between the x_1-axis and the ellipsoid

$$7x_1^2 + 6x_2^2 + 5x_3^2 - 4x_1x_2 - 4x_2x_3 - 37x_1 - 12x_2 + 3x_3 + 54 = 0.$$

Solution. Here

$$\mathbf{A}_1 = \begin{pmatrix} -7/54 & 1/27 & 0 \\ 1/27 & -1/9 & 1/27 \\ 0 & 1/27 & -5/54 \end{pmatrix}, \qquad B_1 = \begin{pmatrix} 37/108 \\ 1/9 \\ -1/36 \end{pmatrix}, \qquad \mathbf{C} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$H = (0,0,0)^T$, and the matrix $\mathbf{A}_1$ is negatively definite. The x_1-axis does not intersect the ellipsoid since the condition (23) is not fulfilled:

$$\begin{vmatrix} \mathbf{A}_1 & B_1 & \mathbf{C} \\ B_1^\top & -1 & \mathbb{O} \\ \mathbf{C}^\top & \mathbb{O} & \mathbb{O} \end{vmatrix} \times (-1)^3 = -\frac{143}{11664} < 0.$$

The determinant (53) takes the form

$$\begin{vmatrix} -7/54 & 1/27 & 0 & 37/108 \\ 1/27 & -1/9-\mu & 1/27 & 1/9 \\ 0 & 1/27 & -5/54-\mu & -1/36 \\ 37/108 & 1/9 & -1/36 & -1+\mu z \end{vmatrix}$$
$$= -\frac{7}{54}\mu^3 z - \frac{73}{2916}\mu^2 z + \frac{143}{11\,664}\mu^2 - \frac{1}{972}\mu z - \frac{1069}{314\,928}\mu - \frac{1621}{4\,251\,528}.$$

Distance equation (42)

$$\mathcal{F}(z) = 2^{-16}3^{-30}\big(1\,331\,935\,488 z^4 - 38\,807\,307\,008 z^3 + 245\,988\,221\,152 z^2 - 1\,086\,769\,525\,104 z + 61\,289\,436\,065\big) = 0$$

has two real zeros: $z_1 \approx 0.05712$ and $z_2 \approx 22.54561$. Hence, the distance equals $\sqrt{z_1} \approx 0.23901$. $\triangle$

To complete the present section, we provide an estimation for the degree of the distance equation.

Theorem 7. *The degree of the polynomial from (24) generically equals $2k$.*

Proof. Let $\Phi(\mu, z)$ be defined by (43). As a matter of fact, we want to find the leading term of the resultant of polynomials $\Phi(\mu, z)$ and $\Phi'_\mu(\mu, z)$ by eliminating the variable μ. To do this, we exploit the idea of the proof of Bézout's theorem on the number of zeros of a system of two bivariate algebraic equations, which can be found in Brill (1925). This idea considers the terms of the highest degree in the expansion of both equations in decreasing powers of the variables.

For simplicity we will assume the system of columns $C_1, \ldots, C_k$ to be orthonormal. We expand $\Phi(\mu, z)$ in powers of z:

$$\Phi(\mu, z) = z\mu^{k+1} \begin{vmatrix} \mathbf{A}_1 & \mathbf{C} \\ \mathbf{C}^T & \frac{1}{\mu}\mathbf{I}_k \end{vmatrix} + \mu^k \begin{vmatrix} \mathbf{A}_1 & B_1 & \mathbf{C} \\ B_1^T & -1 & -H^T \\ \mathbf{C}^T & -H & \frac{1}{\mu}\mathbf{I}_k \end{vmatrix}$$

and utilize the results of Theorems 1 and 4. The leading term of $\mathcal{F}(z) = \mathcal{D}_\mu(\Phi(\mu, z))$ coincides with

$$\mathcal{D}_\mu \left(z\mu^{k+1} \begin{vmatrix} \mathbf{A}_1 & \mathbf{C} \\ \mathbf{C}^T & \frac{1}{\mu}\mathbf{I} \end{vmatrix} \right). \tag{54}$$

In order to evaluate the degree of the last expression w.r.t. variable z, we may exploit formula (12). For this aim, it is necessary to find the degree w.r.t. μ of the polynomial under the discriminant sign. Application of the Schur formula (20) in the way corresponding to (53) yields

$$z\mu^{k+1} \begin{vmatrix} \mathbf{A}_1 & \mathbf{C} \\ \mathbf{C}^T & \frac{1}{\mu}\mathbf{I} \end{vmatrix} \equiv z\mu \det\bigl(\mathbf{A}_1 - \mu\mathbf{C}\mathbf{C}^T\bigr),$$

which is not useful for our purpose since the matrix $\mathbf{C}\mathbf{C}^T$ is singular for $k < n$. Let us use the Schur formula in an alternative way:

$$\equiv z\mu \det \mathbf{A}_1 \det\bigl(\mathbf{I} - \mu\mathbf{C}^T\mathbf{A}_1^{-1}\mathbf{C}\bigr).$$

The last determinant is of the order k with all of its entries depending linearly on μ. We expand it in decreasing powers of μ:

$$\equiv (-1)^{k+1} z\mu^{k+1} \det \mathbf{A}_1 \det\bigl(\mathbf{C}^T\mathbf{A}_1^{-1}\mathbf{C}\bigr) + \ldots$$

Since, by assumption, matrix $\mathbf{A}_1$ from Eq. (1) is sign-definite, so are the matrices $\mathbf{A}_1^{-1}$ and $\mathbf{C}^T\mathbf{A}_1^{-1}\mathbf{C}$. Therefore, $\det(\mathbf{C}^T\mathbf{A}_1^{-1}\mathbf{C}) \neq 0$ and the degree of the polynomial under the discriminant sign in (54) equals $k+1$. Thus, the leading term of $\mathcal{F}(z)$ equals generically

$$z^{2k}(\det \mathbf{A}_1)^{2k} \mathcal{D}_\mu\bigl(\det\bigl(\mathbf{I} - \mu\mathbf{C}^T\mathbf{A}_1^{-1}\mathbf{C}\bigr)\bigr). \qquad \square \tag{55}$$

4. Distance from a point to a quadric

We address here in more detail an important particular case of the general result from the previous section. Let the linear manifold (2) degenerate into a point.

Theorem 8. *Let the point $X_0 \in \mathbb{R}^n$ not lie in the ellipsoid (1), i.e. $X_0^T\mathbf{A}_1X_0 + 2B_1^TX_0 - 1 \neq 0$. The square of the distance from X_0 to the ellipsoid coincides with the minimal positive zero of the distance equation*

$$\mathcal{F}(z) \stackrel{def}{=} \mathcal{D}_\mu \left(\det\left(\begin{bmatrix} \mathbf{A}_1 & B_1 \\ B_1^T & -1 \end{bmatrix} + \mu \begin{bmatrix} -\mathbf{I} & X_0 \\ X_0^T & z - X_0^TX_0 \end{bmatrix} \right) \right) = 0 \tag{56}$$

provided that this zero is not a multiple one.

Proof. Set $\mathbf{C} = \mathbf{I}_n$, $H = X_0$ in (53). $\square$

Under the conditions of Theorem 8, the nearest point to X_0 in the ellipsoid (1) can be found as

$$X_* = -\mathbf{A}_1^{-1}B_1 - \mu_*(\mathbf{A}_1 - \mu_*\mathbf{I})^{-1}\left(\mathbf{A}_1^{-1}B_1 + X_0\right). \tag{57}$$

Here μ_* denotes the multiple zero of the polynomial in μ standing under the discriminant sign in (56) when the variable z is set to be equal to the minimal positive zero of the distance equation.

Corollary 2. *The square of the distance from the origin $X = \mathbb{O}$ to the ellipsoid (1) coincides with the minimal positive zero of the equation*

$$\mathcal{F}(z) \stackrel{def}{=} \mathcal{D}_\mu\left((\mu z - 1)\det(\mathbf{A}_1 - \mu\mathbf{I}) - B_1^T\mathbf{adj}(\mathbf{A}_1 - \mu\mathbf{I})B_1\right) = 0 \tag{58}$$

provided that this zero is not a multiple one. Here **adj** *stands for the adjoint matrix.*

Proof. Proof can be performed with the aid of the Schur formula (20). □

Remark. For large n, one can simultaneously compute $\det(\mathbf{A}_1 - \mu\mathbf{I})$ and $\mathbf{adj}(\mathbf{A}_1 - \mu\mathbf{I})$ with the aid of the Leverrier–Faddeev method (Faddeev and Faddeeva, 1963).

Remark. For the case $B_1 = \mathbb{O}$ and positive definite $\mathbf{A}_1$, the polynomial (58) equals $\mathcal{F}(z) \equiv \mathcal{D}(f)[z^n f(1/z)]^2$ with $f(\mu) \stackrel{def}{=} \det(\mathbf{A}_1 - \mu\mathbf{I})$. This correlates with the well-known fact that the distance to the ellipsoid $X^T\mathbf{A}_1X = 1$ from its center coincides with the square root of the reciprocal of the largest eigenvalue of the matrix $\mathbf{A}_1$.

As a consequence of Theorem 7, one can evaluate the degree of the polynomial $\mathcal{F}(z)$:

Theorem 9. *For the polynomial $\mathcal{F}(z)$ from (56), the leading term equals generically*

$$z^{2n}(\det\mathbf{A}_1)^2\mathcal{D}_\mu\left(\det(\mathbf{A}_1 - \mu\mathbf{I}_n)\right).$$

Proof. Substitute $\mathbf{C} = \mathbf{I}$, $k = n$ into (55). Then the leading term of (56) equals

$$\begin{aligned} z^{2n}(\det\mathbf{A}_1)^{2n}\mathcal{D}_\mu\left(\det\left(\mathbf{I} - \mu\mathbf{A}_1^{-1}\right)\right) &= z^{2n}(\det\mathbf{A}_1)^{2n}\mathcal{D}_\mu\left(\det(\mathbf{A}_1 - \mu\mathbf{I})/\det\mathbf{A}_1\right) \\ &\stackrel{(12)}{=} z^{2n}(\det\mathbf{A}_1)^2\mathcal{D}_\mu\left(\det(\mathbf{A}_1 - \mu\mathbf{I})\right). \end{aligned}$$

It vanishes iff the matrix $\mathbf{A}_1$ possesses a multiple eigenvalue. □

We exploit the result of Theorem 8 to elucidate the importance of the simplicity restriction imposed on the minimal positive zero for $\mathcal{F}(z)$; this assumption will also appear in the foregoing results.

Example 3. Find the distance equation for the ellipse $x^2/4 + y^2 = 1$ and the point (x_0, y_0).

Solution. For the ellipse $x^2/a^2 + y^2/b^2 = 1$, the polynomial $\mathcal{F}(z)$ from (56) is computed as

$$\begin{aligned} \mathcal{F}(z) = \mathcal{D}_\mu\Bigg(z\mu^3 &+ \left\{-z\left(\frac{1}{a^2} + \frac{1}{b^2}\right) + \left(\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} - 1\right)\right\}\mu^2 \\ &+ \frac{z + a^2 + b^2 - x_0^2 - y_0^2}{a^2b^2}\mu - \frac{1}{a^2b^2}\Bigg), \end{aligned}$$

which, for our particular case $a = 2$, $b = 1$, yields (up to a factor 2^{-8})

$$\begin{aligned}\mathcal{F}(z,x_0,y_0) = 9z^4 &- 6\left(2x_0^2+7y_0^2+15\right)z^3 \\ &+ \left(-2x_0^4+62x_0^2y_0^2+73y_0^4-90x_0^2+270y_0^2+297\right)z^2 \\ &+ \left(4x_0^6-30x_0^4y_0^2-90x_0^2y_0^4-56y_0^6\right. \\ &\left.-62x_0^4+140x_0^2y_0^2-248y_0^4+270x_0^2-360y_0^2-360\right)z \\ &+ 16\left(x_0^4+2x_0^2y_0^2+y_0^4-6x_0^2+6y_0^2+9\right)\left(x_0^2/4+y_0^2-1\right)^2. \end{aligned} \tag{59}$$

Let us evaluate its zeros for $y_0 = 0$, i.e. for the points in the x-axis:

$$\mathcal{F}(z,x_0,0) \equiv \left(z-(x_0-2)^2\right)\left(z-(x_0+2)^2\right)\left(3z-\left(3-x_0^2\right)\right)^2.$$

Multiple zero $z_2 = 1 - x_0^2/3$ is positive for $0 \le x_0 < \sqrt{3}$. Moreover, for these values of x_0, zero z_2 is the minimal one for the distance equation. Nevertheless, for $x_0 > 3/2$, the square of the distance from $(x_0, 0)$ to the ellipse equals $z_1 = (x_0 - 2)^2$. Explanation of this phenomenon is as follows: the multiple zero z_2 corresponds to the pair of points $(4x_0/3, \pm\sqrt{1-4x_0^2/9})$ in the ellipse. These points are real for $0 \le x_0 < 3/2$ and imaginary (complex-conjugate) for $x_0 > 3/2$. △

Let us use the computations presented in the solution of the previous example to illuminate a potential application of the theory. Let a set of experimentally determined points $\{(x_j, y_j)\}_{j=1}^M \subset \mathbb{R}^2$ be given, which is assumed to be an approximation of some ellipse in the plane. We do need to choose the best fitting ellipse among some given samples (see Fig. 1).

Example 4. Find the number of points of the set

$$\left\{\begin{array}{cccccc} (-1.85,-0.04), & (-1.80,-0.97), & (-1.32,-0.28), & (-1.31,0.15), & (-1.19,1.20), & (-1.10,-0.54), \\ (-0.61,0.29), & (-0.59,-0.50), & (-0.51,0.90), & (-0.46,-0.50), & (-0.32,0.79), & (-0.20,0.28), \\ (-0.20,0.54), & (2.11,-1.06), & (0.44,-0.91), & (0.51,0.98), & (0.77,-0.18), & (0.82,-0.70), \\ (1.13,-0.91), & (1.28,0.79), & (1.39,-0.66), & (1.39,0.68), & (1.47,0.19), & (1.66,-0.81), \\ (0.92,0.94) & & & & & \end{array}\right\}$$

lying in the 0.25-vicinity of the ellipse $x^2/4 + y^2 = 1$.

Solution. A straightforward approach consists in evaluating the distance from every given point to the ellipse. This is not a very constructive solution in the case of a large number of points. Let us try an alternative algorithm consisting in representing the 0.25-vicinity of the ellipse analytically. We are looking for the *equidistant curve* for the ellipse, i.e. the curve containing all the points (x_0, y_0) with the prescribed distance d from them to the ellipse. Analytical expression for these curves is already obtained: they are given implicitly by an algebraic equation $\mathcal{F}(d^2, x_0, y_0) = 0$ with $\mathcal{F}$ defined by (59).

For our example, in order to find whether a particular point (x_j, y_j) lie inside the 0.25-vicinity of the ellipse, one should check the sign of $\mathcal{F}(1/16, x_j, y_j)$ – it should be negative. The number of points we are looking for equals 12. △

To conclude the discussion of the proposed approach for the metric problem of the present section, let us illuminate its relationship to an ancient one concerning conic sections.

Example 5. Estimate the number of real zeros of the polynomial (59) in its dependency of x_0 and y_0 treated as parameters.

Solution. The sign of the discriminant of the polynomial is significant. One has

$$\Xi(x_0,y_0) \stackrel{def}{=} \mathcal{D}_z\left(\mathcal{F}(z,x_0,y_0)\right) \equiv -2^{10}3^2x_0^2y_0^2\left[\left(4x_0^2+y_0^2-9\right)^3+972x_0^2y_0^2\right]^3.$$

Drawn in the (x_0, y_0)-plane, the curve $\Xi(x_0, y_0) = 0$ consists of three branches: the coordinate axes and the curve known as an *astroid* (marked in red in Fig. 2). The latter was first treated by

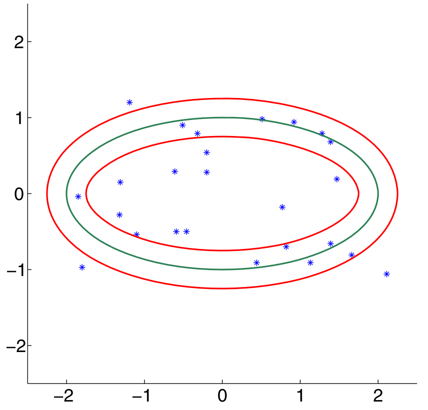

Fig. 1. Equidistant curve for ellipse.

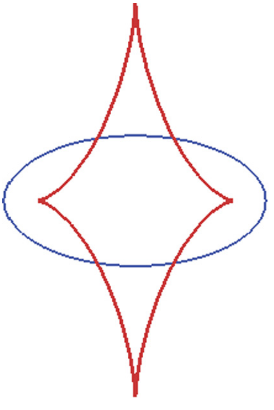

Fig. 2. Astroid. (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

Apollonius in the 3rd century BC (Hartmann and Jantzen, 2003), in connection with the problem of finding the number of normals drawn from the given point (x_0, y_0) to the ellipse.

In terms of the zeros of polynomial (59), the answer is as follows: for the points (x_0, y_0) inside the astroid the polynomial $\mathcal{F}(z)$ possesses four real zeros, for those outside – just two. The exceptional points lie in the axes: one gets four real zeros for corresponding polynomial $\mathcal{F}(z)$ (with two of them becoming negative outside the astroid). △

5. Distance between quadrics

Consider first the case of quadrics (1) and (3) centered at the origin: $B_1 = \mathbb{O}$, $B_2 = \mathbb{O}$.

Theorem 10. *Let the matrix* $\mathbf{A}_1$ *be positive definite. The quadrics* $X^T\mathbf{A}_1X = 1$ *and* $X^T\mathbf{A}_2X = 1$ *intersect iff the matrix* $\mathbf{A}_1 - \mathbf{A}_2$ *is not sign-definite. If the quadrics do not intersect then the square of the distance between them coincides with the minimal positive zero of the equation*

$$\mathcal{F}(z) \stackrel{def}{=} \mathcal{D}_\lambda\big(\det\big(\lambda\mathbf{A}_1 + (z-\lambda)\mathbf{A}_2 - \lambda(z-\lambda)\mathbf{A}_2\mathbf{A}_1\big)\big) = 0 \qquad (60)$$

provided that this zero is not a multiple one.

Proof. I. Intersection condition. This condition can be found as an exercise in the problem book (Proskuryakov, 1978).

II. Construction of the distance equation. If the intersection condition is not valid, then the distance problem becomes nontrivial and we apply the Lagrange multipliers method for the objective function in the form

$$(X-Y)^T(X-Y)-\lambda_1\left(X^T\mathbf{A}_1X-1\right)-\lambda_2\left(Y^T\mathbf{A}_2Y-1\right).$$

The corresponding system of algebraic equations is as follows

$$X-Y-\lambda_1\mathbf{A}_1X=\mathbb{O},\qquad -X+Y-\lambda_2\mathbf{A}_2Y=\mathbb{O}, \tag{61}$$

$$X^T\mathbf{A}_1X=1,\qquad Y^T\mathbf{A}_2Y=1. \tag{62}$$

This system yields

$$\lambda_1\mathbf{A}_1X+\lambda_2\mathbf{A}_2Y=\mathbb{O}, \tag{63}$$

$$X-Y=\lambda_1\mathbf{A}_1X, \tag{64}$$

and

$$(\lambda_1\lambda_2\mathbf{A}_2\mathbf{A}_1-\lambda_1\mathbf{A}_1-\lambda_2\mathbf{A}_2)X=\mathbb{O}, \tag{65}$$

$$(\lambda_1\lambda_2\mathbf{A}_1\mathbf{A}_2-\lambda_1\mathbf{A}_1-\lambda_2\mathbf{A}_2)Y=\mathbb{O} \tag{66}$$

(one of the last two equations can be considered redundant since it follows from another one and (63)–(64)).

Since the matrices of the systems (65) and (66) differ only by transposition, their determinants are equal. Their common value should be zero

$$\det(\lambda_1\lambda_2\mathbf{A}_1\mathbf{A}_2-\lambda_1\mathbf{A}_1-\lambda_2\mathbf{A}_2)=0 \tag{67}$$

due to the fact that we are looking for nontrivial solutions of homogeneous systems.

Let us introduce the vector

$$Z\stackrel{def}{=}X-Y \tag{68}$$

and the matrix

$$\mathbf{M}\stackrel{def}{=}\mathbf{I}-\frac{1}{\lambda_1}\mathbf{A}_1^{-1}-\frac{1}{\lambda_2}\mathbf{A}_2^{-1}. \tag{69}$$

For the values λ_1,λ_2 satisfying the system (63)–(64) the matrix should be singular: $\det\mathbf{M}=0$. In the notation just introduced, and with the aid of (64), Eqs. (65) and (66) can be rewritten into the equivalent form

$$\mathbf{M}Z=\mathbb{O}\quad\Leftrightarrow\quad Z=\left(\frac{1}{\lambda_1}\mathbf{A}_1^{-1}+\frac{1}{\lambda_2}\mathbf{A}_2^{-1}\right)Z, \tag{70}$$

while the conditions (62) in the form

$$\frac{1}{\lambda_j^2}Z^T\mathbf{A}_j^{-1}Z=1\quad\text{for } j\in\{1,2\}. \tag{71}$$

Let us introduce a new variable z responsible for the critical values of the distance function

$$z=(X-Y)^T(X-Y)\stackrel{(68)}{=}Z^TZ\stackrel{(70)}{=}\frac{1}{\lambda_1}Z^T\mathbf{A}_1^{-1}Z+\frac{1}{\lambda_2}Z^T\mathbf{A}_2^{-1}Z\stackrel{(71)}{=}\lambda_1+\lambda_2. \tag{72}$$

Thus, we have eliminated the variables X and Y from the system (61)–(62) with the resulting equations assuming the forms (67) and (72). To deduce an extra relationship between λ_1 and λ_2, one should start with the identity

$$\mathbf{M}\cdot\mathbf{adj}(\mathbf{M})=\mathbf{I}\cdot\det\mathbf{M}.$$

By differentiation of this as to λ_j, one obtains

$$\frac{\partial\mathbf{M}}{\partial\lambda_j}\mathbf{adj}(\mathbf{M})+\mathbf{M}\frac{\partial\mathbf{adj}(\mathbf{M})}{\partial\lambda_j}\equiv\frac{\partial\det\mathbf{M}}{\partial\lambda_j}\mathbf{I}.$$

Multiply this by Z^T from the left-hand side and by Z from the right-hand side, with Z standing for any nontrivial solution to the system (70):

$$Z^T\frac{\partial\mathbf{M}}{\partial\lambda_j}\mathbf{adj}(\mathbf{M})Z+Z^T\mathbf{M}\frac{\partial\mathbf{adj}(\mathbf{M})}{\partial\lambda_j}Z\equiv\frac{\partial\det\mathbf{M}}{\partial\lambda_j}Z^TZ. \quad (73)$$

Taking into account (70) and the symmetry of the matrix $\mathbf{M}$, one arrives at

$$Z^T\mathbf{M}=(\mathbf{M}Z)^T=\mathbb{O},$$

and, therefore, identity (73) turns to

$$Z^T\frac{\partial\mathbf{M}}{\partial\lambda_j}\mathbf{adj}(\mathbf{M})Z=\frac{\partial\det\mathbf{M}}{\partial\lambda_j}Z^TZ, \quad (74)$$

or, in view of (69):

$$\frac{1}{\lambda_j^2}Z^T\mathbf{A}_j^{-1}\mathbf{adj}(\mathbf{M})Z=\frac{\partial\det\mathbf{M}}{\partial\lambda_j}Z^TZ. \quad (75)$$

Now, our aim is to prove that

$$\mathbf{adj}(\mathbf{M})Z=\gamma Z \quad (76)$$

for a certain scalar γ. Indeed,

$$\mathbf{adj}(\mathbf{M})\mathbf{M}Z=\mathbb{O} \quad\Leftrightarrow\quad \mathbf{M}\big(\mathbf{adj}(\mathbf{M})Z\big)=\mathbb{O}.$$

If $\mathbf{rank}(\mathbf{M})=n-1$, then any solution U to the system of homogeneous equations $\mathbf{M}U=\mathbb{O}$ should equal just a multiple of Z; therefore

$$\mathbf{adj}(\mathbf{M})Z=\gamma Z.$$

The case $\mathbf{rank}(\mathbf{M})<n-1$ is trivial, since $\mathbf{adj}(\mathbf{M})=\mathbb{O}_{n\times n}$. (It can be proved that in any case $\gamma=\mathbf{M}_{11}+\mathbf{M}_{22}+\ldots+\mathbf{M}_{nn}$ with $\mathbf{M}_{jj}$ standing for the cofactor to the corresponding entry of $\mathbf{M}$.)

Hence, formula (75) is transformed into

$$\frac{\gamma}{\lambda_j^2}Z^T\mathbf{A}_j^{-1}Z=\frac{\partial\det\mathbf{M}}{\partial\lambda_j}Z^TZ,$$

where from it can be deduced (with the aid of (71)) that

$$\frac{\partial\det\mathbf{M}}{\partial\lambda_1}=\frac{\partial\det\mathbf{M}}{\partial\lambda_2}. \quad (77)$$

Recalling now that λ_1 and λ_2 are connected via condition (72), we substitute $\lambda_1=z-\lambda_2$ into (77) and obtain

$$\frac{\partial\det\mathbf{M}}{\partial\lambda_2}\frac{d\lambda_2}{d\lambda_1}=\frac{\partial\det\mathbf{M}}{\partial\lambda_2} \quad\Rightarrow\quad \frac{\partial\det\mathbf{M}}{\partial\lambda_2}=0.$$

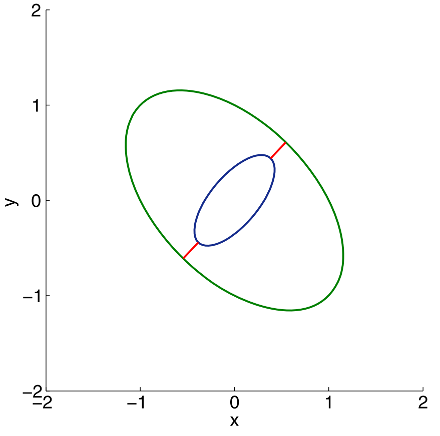

Fig. 3. Distance between central ellipses.

Thus, the process of elimination of variables from system (62)–(66) and (72) terminates when we get two equations: the first one is

$$\det\left(\mathbf{I}-\frac{1}{z-\lambda_2}\mathbf{A}_1^{-1}-\frac{1}{\lambda_2}\mathbf{A}_2^{-1}\right)=0, \tag{78}$$

while the second is obtained by differentiating its left-hand side as to λ_2. Elimination of λ_2 from these equations can be performed in the traditional manner, i.e. via discriminant. Using the result of Theorem 3, we turn from rational functions to polynomial ones. Multiplication of (78) by $\det(\mathbf{A}_1\mathbf{A}_2)$ and substitution $\lambda = z - \lambda_2$ completes the proof.

III. Finding the nearest points. To find the nearest points in the quadrics we suggest the following approach. Compose the matrix

$$\widetilde{\mathbf{M}}(\lambda, z) \stackrel{def}{=} \lambda\mathbf{A}_1 + (z-\lambda)\mathbf{A}_2 - \lambda(z-\lambda)\mathbf{A}_2\mathbf{A}_1 \tag{79}$$

from the distance equation (60) and denote

$$\Phi(\lambda, z) \stackrel{def}{=} \det\widetilde{\mathbf{M}}(\lambda, z).$$

Once the real zero $z = z_*$ of (60) is evaluated, one can find the corresponding value $\lambda = \lambda_*$, which is a multiple zero for $\Phi(\lambda, z_*)$. Under the assumption of the theorem, this zero is unique and can be expressed rationally in terms of the coefficients of the polynomial $\Phi(\lambda, z_*)$ with the aid of (10). Furthermore, the coordinate column X_* of the point in the quadric $X^T\mathbf{A}_1X = 1$ is a solution for the system of homogeneous equations

$$\widetilde{\mathbf{M}}(\lambda_*, z_*)X = \mathbb{O}, \tag{80}$$

which possesses an infinite number of solutions since its determinant vanishes. From the solution set one should choose a representative satisfying the condition $X^T\mathbf{A}_1X = 1$. Due to symmetry of the problem (v. Fig. 3), there exists a pair of such solutions.

Similarly, the coordinate column for the point in the second quadric satisfies the system

$$\widetilde{\mathbf{M}}(\lambda_*, z_*)^T Y = \mathbb{O}. \tag{81}$$

In order to solve both systems (80) and (81) it suffices to treat the columns and the rows of the matrix $\mathbf{adj}(\widetilde{\mathbf{M}}(\lambda_*, z_*))$. Indeed, X_* equals just a multiple of any nonzero column of this matrix while Y_*^T coincides with a multiple of any its nonzero row. By a suitable selection of the mentioned multipliers, one can provide the fulfillment of the conditions $X^T\mathbf{A}_1X = 1$ and $Y^T\mathbf{A}_2Y = 1$. The obtained pairs of points should be adjusted according to the condition

$$(X_* - Y_*)^T (X_* - Y_*) = z_*.$$

IV. Non-vanishing of the matrix $\mathbf{adj}(\widetilde{\mathbf{M}}(\lambda_*, z_*))$. The algorithm for finding the nearest points fails when $\mathbf{adj}(\widetilde{\mathbf{M}}(\lambda_*, z_*)) = \mathbb{O}_{n\times n}$. In the present part of the proof we demonstrate that this problem is avoided by imposing the simplicity restriction for the minimal zero of the distance equation in the statement of the theorem. We intend to exploit essentially the arguments from the corresponding part of the proof of Theorem 6. Following the arguments beginning at (45) (and taking λ instead of μ) we arrive at the conclusion that $\partial\Phi/\partial z|_{(\lambda_*,z_*)} \ne 0$ if $\mathcal{F}'(z_*) \ne 0$. Differentiate now the matrix identity

$$\mathbf{adj}\big(\widetilde{\mathbf{M}}(\lambda, z)\big) \cdot \widetilde{\mathbf{M}}(\lambda, z) \equiv \mathbf{I}_n \det \widetilde{\mathbf{M}}(\lambda, z) \equiv \mathbf{I}_n \Phi(\lambda, z)$$

w.r.t. z and substitute $\lambda = \lambda_*, z = z_*$:

$$\frac{\partial \mathbf{adj}(\widetilde{\mathbf{M}})}{\partial z}\bigg|_{(\lambda_*,z_*)} \widetilde{\mathbf{M}}(\lambda_*, z_*) + \mathbf{adj}\big(\widetilde{\mathbf{M}}(\lambda_*, z_*)\big) \frac{\partial \widetilde{\mathbf{M}}}{\partial z}\bigg|_{(\lambda_*,z_*)} = \mathbf{I} \frac{\partial \Phi}{\partial z}\bigg|_{(\lambda_*,z_*)}.$$

Assume now that $\mathbf{adj}(\widetilde{\mathbf{M}}(\lambda_*, z_*)) = \mathbb{O}$ and, under this assumption, compute the determinant of both parts of the last matrix equality:

$$\det\left(\frac{\partial \mathbf{adj}(\widetilde{\mathbf{M}})}{\partial z}\right)\bigg|_{(\lambda_*,z_*)} \Phi(\lambda_*, z_*) = \left[\frac{\partial \Phi}{\partial z}\bigg|_{(\lambda_*,z_*)}\right]^n.$$

Here $\Phi(\lambda_*, z_*) = 0$ while $\partial\Phi/\partial z|_{(\lambda_*,z_*)} \ne 0$. The contradiction proves that $\mathbf{adj}(\widetilde{\mathbf{M}}(\lambda_*, z_*))$ is a nonzero matrix. □

Example 6. Find the distance between the ellipses

$$10x_1^2 - 12x_1x_2 + 8x_2^2 = 1 \quad \text{and} \quad x_1^2 + x_1x_2 + x_2^2 = 1$$

and the coordinates of their nearest points.

Solution. Here

$$\mathbf{A}_1 = \begin{pmatrix} 10 & -6 \\ -6 & 8 \end{pmatrix}, \qquad \mathbf{A}_2 = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$$

and the matrix $\mathbf{A}_1 - \mathbf{A}_2$ is positive definite. Thus, the ellipses do not intersect.

Compose the matrix (79):

$$\widetilde{\mathbf{M}}(\lambda, z) = \begin{pmatrix} 7\lambda^2 + (-7z + 9)\lambda + z & -2\lambda^2 + (2z - \frac{13}{2})\lambda + \frac{1}{2}z \\ -\lambda^2 + (z - \frac{13}{2})\lambda + \frac{1}{2}z & 5\lambda^2 + (-5z + 7)\lambda + z \end{pmatrix}. \tag{82}$$

The discriminant of its determinant

$$\Phi(\lambda, z) = 33\lambda^4 + \left(-66z + \frac{149}{2}\right)\lambda^3 + \left(33z^2 - 61z + \frac{83}{4}\right)\lambda^2 + \left(-\frac{27}{2}z^2 + \frac{45}{2}z\right)\lambda + \frac{3}{4}z^2$$

w.r.t. λ equals

$$\mathcal{F}(z) = \frac{3}{16}z^2\big(936\,086\,976z^6 - 10\,969\,697\,376z^5 + 50\,706\,209\,664z^4 - 115\,515\,184\,664z^3 + 130\,176\,444\,432z^2 - 59\,826\,725\,574z + 2\,866\,271\,785\big).$$

Its positive zeros are as follows:

$$z_1 \approx 0.05394, \qquad z_2 \approx 1.33405, \qquad z_3 \approx 1.95921, \qquad z_4 \approx 2.87858.$$

Hence, $z_* = z_1$ and $d = \sqrt{z_*} \approx 0.23226$.

To find the nearest points in the given ellipses, establish first the multiple zero of the polynomial $\Phi(\lambda, z_*)$ with the aid of Theorem 1:

$$\lambda_* = -\frac{-725\,274\,z_*^5 + 1\,455\,894\,z_*^4 + \frac{11\,286\,981}{2}z_*^3 - \frac{26\,486\,523}{2}z_*^2 + \frac{42\,000\,075}{8}z_*}{17\,591\,706\,z_*^4 - 109\,992\,894\,z_*^3 + \frac{450\,450\,691}{2}z_*^2 - \frac{315\,606\,253}{2}z_* + \frac{77\,466\,805}{8}}$$
$$\approx -0.13576.$$

Secondly, for the obtained pair of values λ_* and z_* the determinant of the matrix (82) vanishes and, therefore, both systems (80) and (81) posses nontrivial solutions. To find these solutions, take the first column and the first row of the matrix $\mathbf{adj}(\widetilde{\mathbf{M}}(\lambda_*, z_*))$:

$$X = \begin{pmatrix} 5\lambda_*^2 + (-5z+7)\lambda_* + z_* \\ \lambda_*^2 - (z_* - \frac{13}{2})\lambda_* - \frac{1}{2}z_* \end{pmatrix} \approx \begin{pmatrix} -0.76760 \\ -0.88366 \end{pmatrix},$$
$$Y = \begin{pmatrix} 5\lambda_*^2 + (-5z+7)\lambda_* + z_* \\ 2\lambda_*^2 - (2z_* - \frac{13}{2})\lambda_* - \frac{1}{2}z_* \end{pmatrix} \approx \begin{pmatrix} -0.76760 \\ -0.85790 \end{pmatrix}.$$

Every such point defines a line passing through the origin. To find intersection points with the corresponding ellipses, one should make normalization

$$X_* = \frac{\pm X}{\sqrt{X^T\mathbf{A}_1X}} \approx \pm\begin{pmatrix} -0.38383 \\ -0.44186 \end{pmatrix},$$
$$Y_* = \frac{\pm Y}{\sqrt{Y^T\mathbf{A}_2Y}} \approx \pm\begin{pmatrix} -0.54499 \\ -0.60911 \end{pmatrix}.$$

These formulas provide two pairs of nearest points in the ellipses. △

Let us now treat the general form for the quadric equations.

Theorem 11. *The quadrics $X^T\mathbf{A}_1X + 2B_1^TX - 1 = 0$ and $X^T\mathbf{A}_2X + 2B_2^TX - 1 = 0$ intersect iff among the real zeros of the equation*

$$\Theta(z) \stackrel{def}{=} \mathcal{D}_\lambda\left(\det\left(\begin{bmatrix} \mathbf{A}_2 & B_2 \\ B_2^T & -1-z \end{bmatrix} - \lambda\begin{bmatrix} \mathbf{A}_1 & B_1 \\ B_1^T & -1 \end{bmatrix}\right)\right) = 0 \tag{83}$$

there are values of different signs or 0. *If the quadrics do not intersect then the square of the distance between them coincides with the minimal positive zero of the equation*

$$\mathcal{F}(z) \stackrel{def}{=} \mathcal{D}_{\mu_1,\mu_2}\left(\det\left(\mu_1\begin{bmatrix} \mathbf{A}_1 & B_1 \\ B_1^T & -1 \end{bmatrix} + \mu_2\begin{bmatrix} \mathbf{A}_2 & B_2 \\ B_2^T & -1 \end{bmatrix} - \begin{bmatrix} \mathbf{A}_2\mathbf{A}_1 & \mathbf{A}_2B_1 \\ B_2^T\mathbf{A}_1 & B_2^TB_1 - \mu_1\mu_2 z \end{bmatrix}\right)\right) = 0, \tag{84}$$

provided that this zero is not a multiple one.

Proof. I. Finding the intersection condition. Extrema of the function $X^T\mathbf{A}_2X + 2B_2^TX - 1$ in the ellipsoid (1) are all of the similar sign iff the quadrics (1) and (3) do not intersect. We state the problem of finding the extremal values of $V(X) = X^T\mathbf{A}_2X + 2B_2^TX - 1$ subject to (1). The critical point of the Lagrange function

$$X^T\mathbf{A}_2X + 2B_2^TX - 1 - \lambda\left(X^T\mathbf{A}_1X + 2B_1^TX - 1\right)$$

satisfies the system

$$\mathbf{A}_2X + B_2 - \lambda(\mathbf{A}_1X + B_1) = \mathbb{O} \quad \Longleftrightarrow \quad (\mathbf{A}_2 - \lambda\mathbf{A}_1)X = -(B_2 - \lambda B_1)$$

where from it follows that

$$X = -(\mathbf{A}_2 - \lambda\mathbf{A}_1)^{-1}(B_2 - \lambda B_1), \tag{85}$$

provided that the matrix $\mathbf{A}_2 - \lambda\mathbf{A}_1$ is nonsingular. Substitute this into (1):

$$(B_2 - \lambda B_1)^T(\mathbf{A}_2 - \lambda\mathbf{A}_1)^{-1}\mathbf{A}_1(\mathbf{A}_2 - \lambda\mathbf{A}_1)^{-1}(B_2 - \lambda B_1) \\ - 2B_1^T(\mathbf{A}_2 - \lambda\mathbf{A}_1)^{-1}(B_2 - \lambda B_1) - 1 = 0 \tag{86}$$

and into the function $V(X)$:

$$\begin{aligned}
&(B_2 - \lambda B_1)^T(\mathbf{A}_2 - \lambda\mathbf{A}_1)^{-1}\mathbf{A}_2(\mathbf{A}_2 - \lambda\mathbf{A}_1)^{-1}(B_2 - \lambda B_1) \\
&\quad - 2B_1^T(\mathbf{A}_2 - \lambda\mathbf{A}_1)^{-1}(B_2 - \lambda B_1) - 1 \\
&= (B_2 - \lambda B_1)^T(\mathbf{A}_2 - \lambda\mathbf{A}_1)^{-1}(\mathbf{A}_2 - \lambda\mathbf{A}_1 + \lambda\mathbf{A}_1)(\mathbf{A}_2 - \lambda\mathbf{A}_1)^{-1}(B_2 - \lambda B_1) \\
&\quad - 2B_1^T(\mathbf{A}_2 - \lambda\mathbf{A}_1)^{-1}(B_2 - \lambda B_1) - 1 \\
&= (B_2 - \lambda B_1)^T(\mathbf{A}_2 - \lambda\mathbf{A}_1)^{-1}(B_2 - \lambda B_1) \\
&\quad + \lambda(B_2 - \lambda B_1)^T(\mathbf{A}_2 - \lambda\mathbf{A}_1)^{-1}\mathbf{A}_1(\mathbf{A}_2 - \lambda\mathbf{A}_1)^{-1}(B_2 - \lambda B_1) \\
&\quad - 2B_1^T(\mathbf{A}_2 - \lambda\mathbf{A}_1)^{-1}(B_2 - \lambda B_1) - 1 \\
&\overset{(86)}{=} (B_2 - \lambda B_1)^T(\mathbf{A}_2 - \lambda\mathbf{A}_1)^{-1}(B_2 - \lambda B_1) \\
&\quad + \lambda\big(2B_2^T(\mathbf{A}_2 - \lambda\mathbf{A}_1)^{-1}(B_2 - \lambda B_1) + 1\big) \\
&\quad - 2B_1^T(\mathbf{A}_2 - \lambda\mathbf{A}_1)^{-1}(B_2 - \lambda B_1) - 1 \\
&= -(B_2 - \lambda B_1)^T(\mathbf{A}_2 - \lambda\mathbf{A}_1)^{-1}(B_2 - \lambda B_1) + \lambda - 1.
\end{aligned}$$

Introduce now a variable z responsible for the critical values of the function $V(X)$ on the ellipsoid (1). Then one has

$$z + (B_2 - \lambda B_1)^T(\mathbf{A}_2 - \lambda\mathbf{A}_1)^{-1}(B_2 - \lambda B_1) - \lambda + 1 = 0. \tag{87}$$

It turns out that the derivative of the left-hand side of (87) w.r.t. λ coincides with that one of (86). In order to eliminate λ from these equations, one can use Theorem 3. The last step is to show that the expression for the numerator of the rational function (87) coincides with the one standing under the discriminant sign in (83). This can be done with the (traditional for the present paper) application of the Schur formula (20).

II. Construction of the distance equation. To prove the second part of the theorem, we resolve Eq. (6) w.r.t. X and Y:

$$(-\lambda_1\mathbf{A}_1 - \lambda_2\mathbf{A}_2 + \lambda_1\lambda_2\mathbf{A}_2\mathbf{A}_1)X = \lambda_1 B_1 + \lambda_2 B_2 - \lambda_1\lambda_2\mathbf{A}_2 B_1, \tag{88}$$

$$(-\lambda_1\mathbf{A}_1 - \lambda_2\mathbf{A}_2 + \lambda_1\lambda_2\mathbf{A}_1\mathbf{A}_2)Y = \lambda_1 B_1 + \lambda_2 B_2 - \lambda_1\lambda_2\mathbf{A}_1 B_2. \tag{89}$$

Take the matrix $\mathbf{M}$ as defined by (69), set

$$Q \overset{def}{=} -\mathbf{A}_1^{-1}B_1 + \mathbf{A}_2^{-1}B_2, \tag{90}$$

and transform these equations into

$$X = -\mathbf{A}_1^{-1}B_1 + \frac{1}{\lambda_1}\mathbf{A}_1^{-1}\mathbf{M}^{-1}Q, \qquad Y = -\mathbf{A}_2^{-1}B_2 - \frac{1}{\lambda_2}\mathbf{A}_2^{-1}\mathbf{M}^{-1}Q, \tag{91}$$

$$-B_j^T\mathbf{A}_j^{-1}B_j + \frac{1}{\lambda_j^2}Q^T\mathbf{M}^{-1}\mathbf{A}_j^{-1}\mathbf{M}^{-1}Q - 1 = 0 \quad \text{for } j \in \{1,2\}. \tag{92}$$

Now introduce the variable z by the equation

$$z = (X - Y)^T(X - Y).$$

If the pair X,Y corresponds to the stationary point of the Lagrange function (5), then, from the equalities (91), it follows that $X-Y=\mathbf{M}^{-1}Q$ and the last equation transforms into

$$z-Q^T\mathbf{M}^{-2}Q=0. \tag{93}$$

Now multiply every equation from (92) by the corresponding λ_j and add them to (93); from (69) we deduce that

$$-\lambda_1 B_1^T\mathbf{A}_1^{-1}B_1-\lambda_2 B_2^T\mathbf{A}_2^{-1}B_2-Q^T\mathbf{M}^{-1}Q-\lambda_1-\lambda_2+z=0. \tag{94}$$

The derivative of the left-hand side of (94) with respect to λ_j coincides with that one of (92). The substitution

$$\mu_1=1/\lambda_2, \qquad \mu_2=1/\lambda_1$$

and the use of the Schur formula enable one to reduce (94) to the determinantal representation from (84). Let us show this starting from (84):

$$\begin{aligned}
&\left|\begin{matrix}\mu_1\mathbf{A}_1+\mu_2\mathbf{A}_2-\mathbf{A}_2\mathbf{A}_1 & \mu_1B_1+\mu_2B_2-\mathbf{A}_2B_1\\ (\mu_1B_1+\mu_2B_2-\mathbf{A}_1B_2)^T & -\mu_1-\mu_2+\mu_1\mu_2z-B_2^TB_1\end{matrix}\right|\\
&\overset{(20)}{=}\det(\mu_1\mathbf{A}_1+\mu_2\mathbf{A}_2-\mathbf{A}_2\mathbf{A}_1)\big\{-\mu_1-\mu_2+\mu_1\mu_2z-B_2^TB_1\\
&\quad-(\mu_1B_1+\mu_2B_2-\mathbf{A}_1B_2)^T(\mu_1\mathbf{A}_1+\mu_2\mathbf{A}_2-\mathbf{A}_2\mathbf{A}_1)^{-1}(\mu_1B_1+\mu_2B_2-\mathbf{A}_2B_1)\big\}.
\end{aligned} \tag{95}$$

We now transform the expression

$$\begin{aligned}
&(\mu_1B_1+\mu_2B_2-\mathbf{A}_1B_2)^T(\mu_1\mathbf{A}_1+\mu_2\mathbf{A}_2-\mathbf{A}_2\mathbf{A}_1)^{-1}(\mu_1B_1+\mu_2B_2-\mathbf{A}_2B_1)\\
&\overset{(69)}{=}(\mu_1B_1+\mu_2B_2-\mathbf{A}_1B_2)^T\mathbf{A}_1^{-1}(-\mathbf{M}^{-1})\mathbf{A}_2^{-1}(\mu_1B_1+\mu_2B_2-\mathbf{A}_2B_1)\\
&=-\left(\frac{1}{\lambda_2}\mathbf{A}_1^{-1}B_1+\frac{1}{\lambda_1}\mathbf{A}_1^{-1}B_2-B_2\right)^T\mathbf{M}^{-1}\left(\frac{1}{\lambda_2}\mathbf{A}_2^{-1}B_1+\frac{1}{\lambda_1}\mathbf{A}_2^{-1}B_2-B_1\right)\\
&\overset{(69)}{=}-\left(-\mathbf{M}B_2-\frac{1}{\lambda_2}\mathbf{A}_2^{-1}B_2+\frac{1}{\lambda_2}\mathbf{A}_1^{-1}B_2\right)^T\mathbf{M}^{-1}\left(-\mathbf{M}B_1-\frac{1}{\lambda_1}\mathbf{A}_1^{-1}B_1+\frac{1}{\lambda_1}\mathbf{A}_2^{-1}B_2\right)\\
&\overset{(90)}{=}-\left(-\mathbf{M}B_2-\frac{1}{\lambda_2}Q\right)^T\mathbf{M}^{-1}\left(-\mathbf{M}B_1+\frac{1}{\lambda_1}Q\right)\\
&=-B_2^T\mathbf{M}B_1+\frac{1}{\lambda_1}B_2^TQ-\frac{1}{\lambda_2}Q^TB_1+\frac{1}{\lambda_1\lambda_2}Q^T\mathbf{M}^{-1}Q\\
&=-B_2^T\mathbf{M}B_1+\left(\frac{1}{\lambda_1}B_2^T-\frac{1}{\lambda_2}B_1^T\right)Q+\frac{1}{\lambda_1\lambda_2}Q^T\mathbf{M}^{-1}Q\\
&\overset{(90)}{=}-B_2^T\mathbf{M}B_1-\frac{1}{\lambda_1}B_2^T\mathbf{A}_1^{-1}B_1+\frac{1}{\lambda_1}B_2^T\mathbf{A}_2^{-1}B_2+\frac{1}{\lambda_2}B_1^T\mathbf{A}_1^{-1}B_1-\frac{1}{\lambda_2}B_1^T\mathbf{A}_2^{-1}B_2\\
&\quad+\frac{1}{\lambda_1\lambda_2}Q^T\mathbf{M}^{-1}Q\\
&=B_2^T\left(-\mathbf{M}-\frac{1}{\lambda_1}\mathbf{A}_1^{-1}-\frac{1}{\lambda_2}\mathbf{A}_2^{-1}\right)B_1+\frac{1}{\lambda_1}B_2^T\mathbf{A}_2^{-1}B_2+\frac{1}{\lambda_2}B_1^T\mathbf{A}_1^{-1}B_1+\frac{1}{\lambda_1\lambda_2}Q^T\mathbf{M}^{-1}Q\\
&\overset{(69)}{=}-B_2^TB_1+\frac{1}{\lambda_1}B_2^T\mathbf{A}_2^{-1}B_2+\frac{1}{\lambda_2}B_1^T\mathbf{A}_1^{-1}B_1+\frac{1}{\lambda_1\lambda_2}Q^T\mathbf{M}^{-1}Q.
\end{aligned}$$

Substituting this expression into (95), we arrive at (94).

III. Finding the nearest points. Once a real zero $z=z_*$ of (84) is evaluated, one can find the corresponding value $\mu_1=\mu_{1*},\mu_2=\mu_{2*}$, which is a multiple zero for the polynomial standing under the discriminant sign. Based on the assumption of the theorem, this zero is unique and can be expressed rationally in terms of the coefficients of this polynomial with the aid of (19). Define the matrix

$$\widetilde{\mathbf{M}}(\mu_1,\mu_2) \stackrel{def}{=} \mu_1\mathbf{A}_1 + \mu_2\mathbf{A}_2 - \mathbf{A}_2\mathbf{A}_1. \tag{96}$$

Then, from (88)–(89), one gets:

$$\begin{cases} X_* = \widetilde{\mathbf{M}}(\mu_{1*},\mu_{2*})^{-1}(-\mu_{1*}B_1 - \mu_{2*}B_2 + \mathbf{A}_2B_1), \\ Y_* = \left[\widetilde{\mathbf{M}}(\mu_{1*},\mu_{2*})^{-1}\right]^T(-\mu_{1*}B_1 - \mu_{2*}B_2 + \mathbf{A}_1B_2). \qquad \square \end{cases} \tag{97}$$

Example 7. Find the distance between the ellipses

$$-\frac{1}{2}x_1^2 + \frac{1}{2}x_1x_2 - \frac{3}{2}x_2^2 + \frac{5}{2}x_1 + 4x_2 = 1 \quad \text{and} \quad -\frac{1}{84}x_1^2 - \frac{4}{189}x_2^2 - \frac{1}{3}x_1 = 1$$

and the coordinates of their nearest points.

Solution. Intersection condition from Theorem 11 is not satisfied since the polynomial

$$\begin{aligned}\Theta(z) = 2^{-14}3^{-14}7^{-6}\big(&505\,118\,694\,465z^4 - 1\,023\,679\,248\,858z^3 - 7\,568\,287\,236\,783z^2 \\ &+ 33\,720\,131\,260\,536z + 34\,005\,894\,083\,152\big)\end{aligned}$$

has both its real zeros negative. To compute the discriminant in the distance equation (84) we use the result of Theorem 5 with the matrix $\mathfrak{B}$ of the order 9 constructed for the set (16):

$$\begin{aligned}\mathcal{F}(z) = \det(\mathfrak{B}) \equiv 157^3(20\,090z + 3\,526\,681)^2 \\ &\times \big(12\,866\,891\,832\,025z^{12} - 2\,445\,505\,463\,588\,880z^{11} \\ &- 10\,867\,111\,637\,549\,652\,716z^{10} - 3\,123\,865\,087\,697\,933\,253\,136z^9 \\ &+ 1\,561\,852\,119\,815\,441\,835\,822\,424z^8 \\ &+ 1\,041\,845\,279\,230\,362\,476\,059\,640\,640z^7 \\ &+ 302\,844\,249\,329\,911\,871\,856\,294\,474\,624z^6 \\ &+ 50\,781\,476\,668\,832\,773\,753\,935\,668\,661\,952z^5 \\ &+ 2\,215\,513\,880\,036\,430\,404\,751\,762\,329\,796\,624z^4 \\ &- 646\,131\,957\,386\,364\,232\,922\,218\,724\,008\,039\,168z^3 \\ &- 99\,189\,074\,464\,451\,279\,399\,168\,578\,577\,559\,865\,856z^2 \\ &- 5\,789\,019\,527\,920\,299\,026\,625\,801\,973\,715\,386\,789\,888z \\ &+ 60\,730\,952\,901\,233\,749\,068\,462\,660\,878\,127\,980\,941\,312\big).\end{aligned}$$

The positive zeros of $\mathcal{F}(z)$ are as follows:

$$z_1 \approx 9.01839, \qquad z_2 \approx 121.59673, \qquad z_3 \approx 582.35840, \qquad z_4 \approx 1031.42118.$$

Thus, the minimal positive zero is $z_* = z_1$ and the distance between the given ellipses equals $\sqrt{z_*} \approx 3.00306$.

For the obtained value of z_*, the polynomial in μ_1 and μ_2 from (84) possesses a multiple zero that can be expressed rationally in terms of z_* with the aid of the minors of $\mathfrak{B}$ via formulas (19). Substitution of the obtained values $\mu_{1*} \approx 0.04209$, $\mu_{2*} \approx 0.59321$ into (97) yields the coordinates of the nearest points in the given ellipses:

$$X_* \approx [-0.48247, 1.10651]^T \quad \text{and} \quad Y_* \approx [-3.46263, 0.73630]^T.$$

△

Let us compare now the conditions for the intersection of quadrics from Theorem 11 with the following results from Wang and Krasauskas (2003) (we restrict our discussion to the planar case and recall the notion of *separate* ellipses mentioned in Introduction):

Theorem 12. *Let (1) and (3) be the equations of two ellipses with the characteristic polynomial (7). Then*

(a) *$f(\lambda)$ has a positive zero;*
(b) *(1) and (3) are separate iff $f(\lambda)$ has two distinct negative zeros;*
(c) *(1) and (3) touch each externally iff $f(\lambda)$ has a negative double zero.*

The last condition can be interpreted in the discriminant form. The necessary condition for the touching of two quadrics is $\mathcal{D}_\lambda(f(\lambda)) = 0$. Exactly the same result is obtained by substituting $z = 0$ into the polynomial $\Theta(z)$ given by (83). In comparison with the polynomial $\Theta(z)$, the characteristic polynomial $f(\lambda)$ is simpler: the degree of the former equals 4 (cf. Example 7) while that of the latter equals 3. Therefore, the second condition of Theorem 12 is easier to verify than the intersection condition from Theorem 11. However, this advantage is lost when attempting to distinguish the case of existence of (nontrivial) intersections of the ellipses with the situation when these ellipses are *nested*. This problem cannot be solved in terms of the zero set of the characteristic polynomial $f(\lambda)$. To resolve it, in Wang and Krasauskas (2003), a nontrivial analysis in terms of the *index function* is attracted. As for the intersection condition from Theorem 11, it is insensitive to the relative position of the quadrics.

6. Parameter dependent manifolds

The problem of distance estimation between moving objects in 3D space is of importance to astronomy, robotics and computer graphics. To illuminate the perspectives of the approach developed in the previous sections for such problems dealing with quadrics, we will address the following problem.

Find the distance from the point $X_0 \in \mathbb{R}^n$ to the nearest point of the family of ellipsoids in $\mathbb{R}^n$

$$\left\{ X^T \mathbf{A}_1(t) X + 2B_1^T(t) X - 1 = 0 \,\middle|\, t \in [a,b] \right\} \tag{98}$$

with the coefficients of $\mathbf{A}_1(t)$ and $B_1(t)$ polynomially dependent on the parameter t.

Theorem 13. *The square of the distance from X_0 to (98) coincides with the minimal positive zero of one of the equations*

$$\widetilde{\mathcal{F}}(z) \stackrel{def}{=} \mathcal{D}_t\big(\mathcal{F}(z,t)\big) = 0, \qquad \mathcal{F}(z,a) = 0, \qquad \mathcal{F}(z,b) = 0 \tag{99}$$

where $\mathcal{F}(z,t)$ is polynomial (56), and the mentioned zero is not a multiple one.

In short: the stated problem can be solved with the aid of an *iterated* discriminant.

Proof. For any given value of t, the square of the distance from X_0 to the corresponding ellipsoid of the family (98) is evaluated as a zero of Eq. (56)

$$\mathcal{F}(z,t) = 0. \tag{100}$$

Due to imposed restrictions on the coefficients of the family, $\mathcal{F}$ is a polynomial function in t. Eq. (100) can be treated as defining an implicit function $z(t)$. It is known that zeros of a polynomial are continuously differentiable functions of the coefficients of this polynomial (except for coefficient specializations annihilating the discriminant) (Shilov, 1972). Consequently, for any zero $z = z_*(t)$ of (100) there exists the derivative $dz_*(t)/dt$. Differentiation of the equality $\mathcal{F}(z_*(t),t) \equiv 0$ w.r.t. t results in

$$\left.\frac{\partial \mathcal{F}}{\partial z}\right|_{z=z_*(t)} \cdot \frac{dz_*(t)}{dt} + \left.\frac{\partial \mathcal{F}}{\partial t}\right|_{z=z_*(t)} \equiv 0. \tag{101}$$

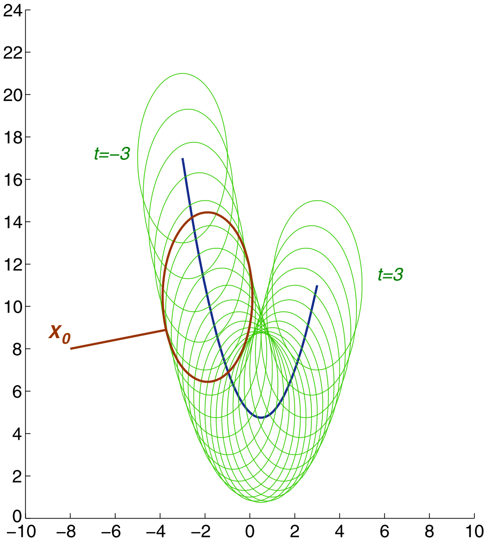

Fig. 4. Distance from a point to a family of ellipses.

For $t \in [a, b]$, the minimum of the function $z_*(t)$ is attained either at the end points of the interval or in the stationary point $t = \tilde{t}$ at which $dz_*/dt = 0$. In the latter case, it follows from (101) that

$$\partial \mathcal{F} / \partial t = 0 \tag{102}$$

at $t = \tilde{t}$. The two conditions (100) and (102) provide an algebraic system w.r.t. both variables z and t. One can eliminate the variable t with the aid of the discriminant. □

Example 8. Find the distance from the point $(-8, 8)$ to the family of ellipses

$$\left\{ \frac{(x-t)^2}{4} + \frac{(y-(t^2-t+5))^2}{16} = 1 \;\middle|\; t \in \mathbb{R} \right\}.$$

Solution. Representation of the family in the form (98) yields a rational dependency of the coefficients, not a polynomial one. Utilizing the result of Theorem 3, one may expect the appearance of some extraneous factors in the distance equation (99):

$$\begin{aligned}\widetilde{\mathcal{F}}(z) = {} & 2^{184}3^8 z^4 (z+1583)^2 (3z^2 - 700z + 35\,824)^2 \left[P(z)\right]^3 \\ & \times \big(589\,824z^8 - 419\,905\,536z^7 + 127\,025\,497\,600z^6 - 21\,258\,728\,389\,952z^5 \\ & + 2\,143\,340\,532\,031\,865z^4 - 132\,447\,231\,702\,193\,528z^3 \\ & + 4\,849\,349\,464\,298\,970\,936z^2 - 94\,448\,162\,098\,872\,830\,560z \\ & + 720\,285\,341\,211\,499\,815\,952\big).\end{aligned}$$

Here $P(z)$ stands for the polynomial of the degree 12. The *true* distance equation is provided by the last factor. Its minimal positive zero equals $z_* \approx 19.03255$. The distance from the given point to the family equals $\sqrt{z_*} \approx 4.36263$. One can find the ellipse of the family at which the distance is attained via the traditional application of the multiple zero evaluation formula (10): $t_* \approx -1.88563$ (see Fig. 4). The nearest, to $(-9, 9)$, point in this ellipse can be found via (57):

$$X_* \approx (-3.73014, 8.89491).$$ △

7. Conclusions

We have treated the problem of distance evaluation between algebraic manifolds in $\mathbb{R}^n$ via inversion of the traditional approach based on first finding the coordinates of the nearest points in the manifolds. This inversion has been performed via introduction of an extra variable responsible for the critical values of distance function and the application of Elimination Theory methods. Such an approach was first suggested in Uteshev and Cherkasov (1998) for the general polynomial optimization problem in $\mathbb{R}^n$. Its employment for the treated metric problems has led us to the following result: the construction for every distance equation is based on an appropriate discriminant as the cornerstone. Moreover, the necessary and sufficient condition for the intersection of the quadrics (Theorem 11) are also formulated with the aid of a discriminant.

We would like to point out two advantages of the proposed approach over the traditional scheme. The first one is that the problem is reduced to the evaluation of a single zero of a univariate algebraic equation instead of dealing with a multidimensional constrained optimization problem. In addition, introduction of an extra variable z into the problem provides one with a nice (i.e. rational) parameterization of the nearest points coordinates. The second advantage is connected with the problem of finding the functional dependency of the distance on the coefficients of the manifold equations. From the formal point of view, the problem is solved: the distance equation provides such a dependency. Unfortunately, this dependency is *implicit* and, in general, rather complicated for immediate real-life application (for instance, the ellipse fitting problem mentioned in Introduction).

Thus, the first (and the most vital) problem in the list of those remaining for further investigation is

(**a**) To resolve the distance equation, i.e. to obtain approximate (but explicit) formulas for its zero.

The next problem concerns the assumption appearing in all the theorems where the distance equations have been introduced:

(**b**) To estimate the stiffness of the simplicity condition for the minimal positive zero for the distance equation.

One of the referees of the present paper suggested some arguments for the conjecture that the square of the distance between an ellipsoid and a linear manifold is always attained at the least positive zero of *odd multiplicity*. These arguments (based on the convexity reasons) seem to be sound and can probably be extended to the case of *separated* ellipsoids (1) and (3). However, they fail when these ellipsoids are *nested* or when one of the quadrics is not an ellipsoid.

The next problem is

(**c**) To estimate the degree of distance equations constructed for the metric problems of Section 5.

We conjecture that $\deg \mathcal{F}(z) = n(n+1)$ for (60) and $\deg \mathcal{F}(z) = 2n(n+1)$ for (84), with these estimations valid on excluding some extraneous factor (e.g. in case of (60), this factor is just $z^{n(n-1)}$). For $n \in \{2, 3\}$ these conjectures correlate with estimations for the degrees of the irreducible univariate equations experimentally obtained in Lennerz and Schömer (2002). The structure of extraneous factors should also be clarified.

Discussion in the last paragraph of Section 5 generates one extra the problem

(**d**) To determine the relative position of the quadrics via the analysis of the set of real zeros of Eq. (83).

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