Analytical Solution for the Generalized Fermat–Torricelli Problem

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Abstract. We present an explicit analytical solution for the problem of minimization of the function

$$F(x, y) = \sum_{j=1}^{3} m_j \sqrt{(x - x_j)^2 + (y - y_j)^2},$$

i.e., we find the coordinates of the stationary point and the corresponding critical value as functions of $\{m_j, x_j, y_j\}_{j=1}^{3}$. In addition, we also discuss the inverse problem of finding such values for m_1, m_2, and m_3 for which the corresponding function F possesses a prescribed position of stationary point.

1. INTRODUCTION. Consider the following problem. Given the coordinates of three noncollinear points $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$, and $P_3 = (x_3, y_3)$ in the plane, find the coordinates of the point $P_* = (x_*, y_*)$ that gives a solution to the optimization problem

$$\min_{(x,y)\in\mathbb{R}^2} F(x, y) \quad \text{for} \quad F(x, y) = \sum_{j=1}^{3} m_j \sqrt{(x - x_j)^2 + (y - y_j)^2}. \tag{1}$$

Here m_1, m_2, and m_3 are assumed to be real positive numbers and will be subsequently referred to as *weights*.

The stated problem, in its particular case of equal weights $m_1 = m_2 = m_3 = 1$, has been known since 1643 as the (classical) Fermat–Torricelli problem. It has a unique solution that coincides either with one of the points P_1, P_2, P_3 or with the so-called *Fermat* or *Fermat–Torricelli point* [**2, 4**] of the triangle $P_1P_2P_3$; this point makes an angle of $2\pi/3$ with any two vertices of the triangle.

Generalization of the problem to the case of unequal weights has been investigated since the 19th century. This generalization is known under different names: *the Steiner problem, the Weber problem, the problem of railway junction* ((Germ.) *Problem des Knotenpunktes*) [**3, 8**], *the three factory problem* [**6**]. The last two names were inspired by a facility location problem such as the following. Let the cities P_1, P_2, and P_3 be the sources of iron ore, coal, and water, respectively. To produce one ton of steel, the steel works needs m_1 tons of iron, m_2 tons of coal, and m_3 tons of water. Assuming that the freight charge for a ton-kilometer is independent of the nature of the cargo, find the optimal position for the steel works connected with P_1, P_2, and P_3 via straight roads so as to minimize the transportation costs.

In the rest of the paper, this problem will be referred to as *the generalized Fermat–Torricelli problem*. Existence and uniqueness of its solution is guaranteed by the following result [**4**].

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Theorem 1. *Denote by α_1, α_2, and α_3 the corner angles of the triangle $P_1P_2P_3$. If the conditions*

$$\begin{cases} m_1^2 < m_2^2 + m_3^2 + 2m_2m_3\cos\alpha_1, \\ m_2^2 < m_1^2 + m_3^2 + 2m_1m_3\cos\alpha_2, \\ m_3^2 < m_1^2 + m_2^2 + 2m_1m_2\cos\alpha_3 \end{cases} \tag{2}$$

are fulfilled, then there exists a unique solution $P_* = (x_*, y_*) \in \mathbb{R}^2$ for the generalized Fermat–Torricelli problem lying inside the triangle $P_1P_2P_3$. This point is a stationary point for the function $F(x, y)$, i.e., a real solution of the system

$$\sum_{j=1}^{3} \frac{m_j(x - x_j)}{\sqrt{(x - x_j)^2 + (y - y_j)^2}} = 0,$$

$$\sum_{j=1}^{3} \frac{m_j(y - y_j)}{\sqrt{(x - x_j)^2 + (y - y_j)^2}} = 0. \tag{3}$$

If any of the conditions (2) *are violated, then $F(x, y)$ attains its minimum value at the corresponding vertex of the triangle.*

Let us overview some approaches for finding the point P_*. Historically, the first approach is geometrical: The point is found as the intersection point of a special construction of lines or circles. For the equal weighted case, Torricelli proved that the circles circumscribing the equilateral triangles constructed on the sides of and outside the triangle $P_1P_2P_3$ intersect at the point P_*; for an alternative Simpson construction of P_*, see [**5**]. For the general, i.e., unequal weighted case, see [**3, 8**].

The second approach is based on the mechanical model (sometimes incorrectly called *Pólya's mechanical model*): A horizontal board is drilled with holes at the points P_1, P_2, and P_3 (or at the vertices of a triangle similar to $P_1P_2P_3$). Three strings are tied together in a knot at one end, the loose ends are passed through the holes, and are attached to physical weights proportional to m_1, m_2, and m_3, respectively, below the board. The equilibrium position of the knot yields the solution [**3**].

The third approach, based on the gradient descent method, originated in the paper [**11**]; further developments and comments can be found in [**7, 9**].

The present paper is devoted to the fourth approach, the analytical one. We look for explicit expressions for the coordinates of the stationary point P_* as functions of $\{m_j, x_j, y_j\}_{j=1}^3$. Although the existence of such a solution by radicals, i.e., in a finite number of operations like standard arithmetic ones and extraction of (positive integer) roots, is not questioned in any review article on the problem, we failed to find in the literature the constructive and universal version of an algorithm even for the classical, i.e., equal weighted, case.

2. ALGEBRA.

Theorem 2. *Under the conditions* (2)*, the coordinates of the stationary point (x_*, y_*) of the function $F(x, y)$ are as follows:*

$$x_* = \frac{K_1K_2K_3}{4|S|\sigma d}\left(\frac{x_1}{K_1} + \frac{x_2}{K_2} + \frac{x_3}{K_3}\right), \quad y_* = \frac{K_1K_2K_3}{4|S|\sigma d}\left(\frac{y_1}{K_1} + \frac{y_2}{K_2} + \frac{y_3}{K_3}\right) \tag{4}$$

with

$$F(x_*, y_*) = \min_{(x,y)\in\mathbb{R}^2} F(x, y) = \sqrt{d}.$$

Here

$$d = \frac{1}{2\sigma}(m_1^2K_1 + m_2^2K_2 + m_3^2K_3)\text{ , or alternatively,} \tag{5}$$

$$d = 2|S|\sigma + \frac{1}{2}\left[m_1^2(r_{12}^2+r_{13}^2 - r_{23}^2)+m_2^2(r_{23}^2+r_{12}^2-r_{13}^2)+m_3^2(r_{13}^2+r_{23}^2-r_{12}^2)\right], \tag{6}$$

$$r_{j\ell} = |P_jP_\ell| = \sqrt{(x_j - x_\ell)^2 + (y_j - y_\ell)^2} \quad \text{for } \{j, \ell\} \subset \{1, 2, 3\},$$

$$S = x_1y_2 + x_2y_3 + x_3y_1 - x_1y_3 - x_3y_2 - x_2y_1, \tag{7}$$

$$\sigma = \frac{1}{2}\sqrt{-m_1^4 - m_2^4 - m_3^4 + 2m_1^2m_2^2 + 2m_1^2m_3^2 + 2m_2^2m_3^2}, \tag{8}$$

and

$$\begin{cases} K_1 = (r_{12}^2 + r_{13}^2 - r_{23}^2)\sigma + (m_2^2 + m_3^2 - m_1^2)|S|, \\ K_2 = (r_{23}^2 + r_{12}^2 - r_{13}^2)\sigma + (m_1^2 + m_3^2 - m_2^2)|S|, \\ K_3 = (r_{13}^2 + r_{23}^2 - r_{12}^2)\sigma + (m_1^2 + m_2^2 - m_3^2)|S|. \end{cases} \tag{9}$$

Proof. First, we establish the validity of the equality

$$K_1K_2 + K_1K_3 + K_2K_3 = 4\sigma|S|d, \tag{10}$$

and the dual equality

$$r_{23}^2K_1 + r_{13}^2K_2 + r_{12}^2K_3 = 2|S|d \tag{11}$$

for (5). Second, let us deduce the following relationships

$$\sqrt{(x_* - x_j)^2 + (y_* - y_j)^2} = \frac{m_jK_j}{2\sigma\sqrt{d}} \quad \text{for } j \in \{1, 2, 3\}. \tag{12}$$

Here is the proof for the case $j = 1$:

$$\begin{aligned}
&(x_* - x_1)^2 + (y_* - y_1)^2 \\
&\overset{(10)}{=} \left(\frac{K_1K_2K_3}{4\sigma|S|d}\right)^2 \left[\left(\frac{x_2}{K_2} + \frac{x_3}{K_3} - \frac{x_1}{K_2} - \frac{x_1}{K_3}\right)^2 + \left(\frac{y_2}{K_2} + \frac{y_3}{K_3} - \frac{y_1}{K_2} - \frac{y_1}{K_3}\right)^2\right] \\
&= \left(\frac{K_1K_2K_3}{4\sigma|S|d}\right)^2 \left[\frac{(x_2 - x_1)^2 + (y_2 - y_1)^2}{K_2^2} + \frac{(x_3 - x_1)^2 + (y_3 - y_1)^2}{K_3^2}\right. \\
&\qquad \left. + 2\frac{(x_2 - x_1)(x_3 - x_1) + (y_2 - y_1)(y_3 - y_1)}{K_2K_3}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{K_1K_2K_3}{4\sigma|S|d}\right)^2 \left[\frac{r_{12}^2}{K_2^2} + \frac{r_{13}^2}{K_3^2} + 2\frac{1/2(r_{12}^2+r_{13}^2-r_{23}^2)}{K_2K_3}\right] \\
&= \frac{K_1^2}{(4\sigma|S|d)^2}\left[r_{12}^2K_3^2 + r_{13}^2K_2^2 + (r_{12}^2+r_{13}^2-r_{23}^2)K_2K_3\right] \\
&= \frac{K_1^2}{(4\sigma|S|d)^2}\left[(r_{12}^2K_3 + r_{13}^2K_2)(K_2+K_3) - r_{23}^2K_2K_3\right] \\
&\overset{(11)}{=} \frac{K_1^2}{(4\sigma|S|d)^2}\left[(2|S|d - r_{23}^2K_1)(K_2+K_3) - r_{23}^2K_2K_3\right] \\
&= \frac{K_1^2}{(4\sigma|S|d)^2}\left[2|S|d(K_2+K_3) - r_{23}^2(K_1K_2+K_1K_3+K_2K_3)\right] \\
&\overset{(10)}{=} \frac{K_1^2}{(4\sigma|S|d)^2}\left[2|S|d(K_2+K_3) - 4r_{23}^2\sigma|S|d\right] \\
&= \frac{2|S|dK_1^2}{(4\sigma|S|d)^2}\left[K_2+K_3-2r_{23}^2\sigma\right] \\
&\overset{(9)}{=} \frac{K_1^2}{8|S|d\sigma^2}\left[2m_1^2|S|\right] \\
&= \frac{m_1^2K_1^2}{4\sigma^2 d}.
\end{aligned}
$$

Similar arguments hold for $j \in \{2, 3\}$ in (12). To complete the proof of these equalities, it should be additionally verified that the values K_1, K_2, and K_3 are nonnegative. This will be done in the next section.

To prove the first statement of the theorem, we will utilize the following alternative representation for x_* and y_*:

$$
\begin{aligned}
x_* &\overset{(10)}{=} \frac{1}{\frac{1}{K_1}+\frac{1}{K_2}+\frac{1}{K_3}}\left(\frac{x_1}{K_1}+\frac{x_2}{K_2}+\frac{x_3}{K_3}\right), \quad \text{and} \\
y_* &\overset{(10)}{=} \frac{1}{\frac{1}{K_1}+\frac{1}{K_2}+\frac{1}{K_3}}\left(\frac{y_1}{K_1}+\frac{y_2}{K_2}+\frac{y_3}{K_3}\right).
\end{aligned} \tag{13}
$$

We substitute (4) into the left-hand side of the first equation of (3). The resulting expression can be reduced with the aid of (12) to

$$
\frac{x_*-x_1}{K_1}+\frac{x_*-x_2}{K_2}+\frac{x_*-x_3}{K_3} = x_*\left(\frac{1}{K_1}+\frac{1}{K_2}+\frac{1}{K_3}\right)-\left(\frac{x_1}{K_1}+\frac{x_2}{K_2}+\frac{x_3}{K_3}\right) \overset{(13)}{=} 0.
$$

Similar arguments are valid for the second equation from (3). Finally, we compute $F(x_*, y_*)$:

$$
F(x_*, y_*) = \sum_{j=1}^{3} m_j\sqrt{(x_*-x_j)^2+(y_*-y_j)^2} \overset{(12)}{=} \sum_{j=1}^{3}\frac{m_j^2K_j}{2\sigma\sqrt{d}} \overset{(5)}{=} \frac{2\sigma d}{2\sigma\sqrt{d}} = \sqrt{d}. \qquad \blacksquare
$$

Some test values are provided in Table 1.

	P_1 / m_1	P_2 / m_2	P_3 / m_3	P_* / $\sqrt{d}$
1.	$(2, 6)$ 2	$(1, 1)$ 3	$(5, 1)$ 4	$\left(\frac{4103+1833\sqrt{15}}{2866}, \frac{29523-4481\sqrt{15}}{8598}\right)$ $\approx (3.9086, 1.4152)$ $\sqrt{d} = 2\sqrt{79+15\sqrt{15}} \approx 23.4174$
2.	$(2, 6)$ 3	$(1, 1)$ 5	$(5, 1)$ 4	$\left(\frac{751}{485}, \frac{647}{485}\right) \approx (1.5484, 1.3340)$ $\sqrt{d} = \sqrt{970} \approx 31.1448$
3.	$(0, 0)$ $3/2$	$(2, 0)$ 2	$(-\sqrt{2}, \sqrt{2})$ 2	$\left(1 - \frac{1}{\sqrt{2}} - \frac{3}{\sqrt{110}}, \frac{1}{\sqrt{2}} - \frac{3}{\sqrt{55}} - \frac{3}{\sqrt{110}}\right)$ $\approx (0.0068, 0.0165)$ $\sqrt{d} = \sqrt{32 + \frac{23}{\sqrt{2}} + 3\sqrt{\frac{55}{2}}} \approx 7.9997$

Table 1.

3. GEOMETRY. Let us give an interpretation for some constants that appeared in Theorem 2. First, on rewriting (7) in determinantal form

$$S = \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix},$$

we recognize that $|S| = 2S_{\Delta P_1 P_2 P_3}$, where $S_{\Delta P_1 P_2 P_3}$ stands for the area of triangle $P_1 P_2 P_3$. As for the constant (8), factorization of the radicand on the right-hand side leads to the form

$$\sigma = 2\left[\frac{m_1+m_2+m_3}{2}\left(\frac{m_1+m_2+m_3}{2} - m_1\right)\left(\frac{m_1+m_2+m_3}{2} - m_2\right)\right.$$
$$\left.\times\left(\frac{m_1+m_2+m_3}{2} - m_3\right)\right]^{1/2},$$

which can be treated as the Heron formula for twice the area of a triangle formed by the triple of weights m_1, m_2, and m_3. Under the restrictions (2), such a triangle exists. Construct this triangle and denote its angles, as shown in Figure 1.

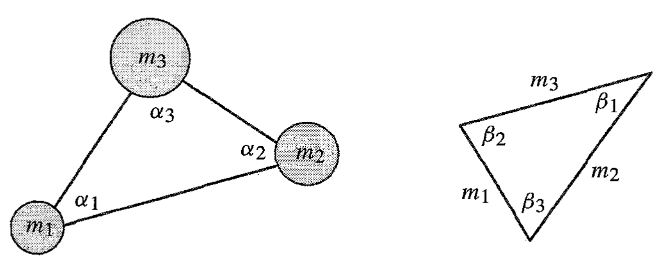

Figure 1. Two triangles generated by the problem

The first formula from (9) can thus be represented with the aid of the law of cosines as

$$K_1 = \sigma|S|\left(\frac{r_{12}^2 + r_{13}^2 - r_{23}^2}{|S|} + \frac{m_2^2 + m_3^2 - m_1^2}{\sigma}\right)$$
$$= \sigma|S|\left(\frac{2r_{12}r_{13}\cos\alpha_1}{|S|} + \frac{2m_2m_3\cos\beta_1}{\sigma}\right)$$
$$= 2\sigma|S|(\cot\alpha_1 + \cot\beta_1).$$

Rewriting the first condition from (2) in the form $\cos\alpha_1 + \cos\beta_1 > 0$, we can conclude that $\cot\alpha_1 + \cot\beta_1 > 0$ and, thus, $K_1 > 0$. In a similar way, the expressions for K_2 and K_3 can be deduced, and we can establish that, under the restrictions (2), they are both positive. This completes the proof of Theorem 2.

Remark 1. We set the *dual* generalized Fermat–Torricelli problem. Let the triangle be composed of the sides with the lengths equal to m_1, m_2, and m_3; let the weights r_{12}, r_{23}, and r_{13} be placed in its vertices, as shown in Figure 2.

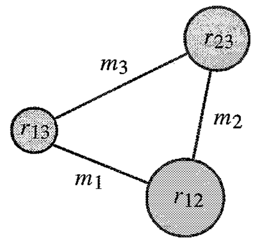

Figure 2. Dual problem

The minimum value for the objective function will be the same as in the direct problem, since (6) is equivalent to

$$2|S|\sigma + \frac{1}{2}\left[r_{12}^2(m_1^2 + m_2^2 - m_3^2) + r_{13}^2(m_1^2 + m_3^2 - m_2^2) + r_{23}^2(m_2^2 + m_3^2 - m_1^2)\right].$$

4. CLASSICAL FERMAT–TORRICELLI PROBLEM. Consider now the equal weighted case $m_1 = m_2 = m_3 = 1$.

Theorem 3. *Let all the angles of the triangle $P_1P_2P_3$ be less than $2\pi/3$, or, equivalently,*

$$r_{12}^2 + r_{13}^2 + r_{12}r_{13} - r_{23}^2 > 0,$$
$$r_{23}^2 + r_{12}^2 + r_{12}r_{23} - r_{13}^2 > 0,$$
$$r_{13}^2 + r_{23}^2 + r_{13}r_{23} - r_{12}^2 > 0.$$

The coordinates of the Fermat–Torricelli point for this triangle are as follows:

$$x_* = \frac{k_1k_2k_3}{2\sqrt{3}|S|d}\left(\frac{x_1}{k_1} + \frac{x_2}{k_2} + \frac{x_3}{k_3}\right), \quad y_* = \frac{k_1k_2k_3}{2\sqrt{3}|S|d}\left(\frac{y_1}{k_1} + \frac{y_2}{k_2} + \frac{y_3}{k_3}\right), \tag{14}$$

with the corresponding minimum value of the objective function

$$F(x_*, y_*) = \min_{(x,y)\in\mathbb{R}^2} \sum_{j=1}^{3} \sqrt{(x - x_j)^2 + (y - y_j)^2} = \sqrt{d}.$$

Here,

$$d = \frac{1}{\sqrt{3}}(k_1 + k_2 + k_3) = \frac{r_{12}^2 + r_{13}^2 + r_{23}^2}{2} + \sqrt{3}\,|S| \tag{15}$$

and

$$k_1 = \frac{\sqrt{3}}{2}(r_{12}^2 + r_{13}^2 - r_{23}^2) + |S|,$$

$$k_2 = \frac{\sqrt{3}}{2}(r_{23}^2 + r_{12}^2 - r_{13}^2) + |S|,$$

$$k_3 = \frac{\sqrt{3}}{2}(r_{13}^2 + r_{23}^2 - r_{12}^2) + |S|,$$

with the rest of the parameters coinciding with those from Theorem 2.

It turns out that the right-hand sides of the expressions (14), being represented as rational fractions with respect to $\{x_j, y_j\}_{j=1}^3$, can be reduced further to the form where denominators become "area free."

Corollary. *Under conditions of Theorem* 3, *the coordinates of the Fermat–Torricelli point are as follows:*

$$x_* = \frac{1}{2\sqrt{3}d}\Bigg[(x_1 + x_2 + x_3)|S| + \sqrt{3}\left(x_1 r_{23}^2 + x_2 r_{13}^2 + x_3 r_{12}^2\right) \tag{16}$$

$$+ 3\,\mathrm{sgn}(S)\left|\begin{array}{ccc} 1 & 1 & 1 \\ y_1 & y_2 & y_3 \\ x_2x_3 + y_2y_3 & x_1x_3 + y_1y_3 & x_1x_2 + y_1y_2 \end{array}\right|\Bigg],$$

$$y_* = \frac{1}{2\sqrt{3}d}\Bigg[(y_1 + y_2 + y_3)|S| + \sqrt{3}\left(y_1 r_{23}^2 + y_2 r_{13}^2 + y_3 r_{12}^2\right) \tag{17}$$

$$- 3\,\mathrm{sgn}(S)\left|\begin{array}{ccc} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_2x_3 + y_2y_3 & x_1x_3 + y_1y_3 & x_1x_2 + y_1y_2 \end{array}\right|\Bigg].$$

Remark 2. The result of the last corollary can be extended to the generalized Fermat–Torricelli problem. Numerators and denominators in the right-hand sides of the formulas (4) can be reduced by the common factor $|S|$. We do not present the resulting expressions here, since they are inelegantly cumbersome.

Remark 3. One of the referees of the present paper suggested that the author "provide some motivation or insight of how he found the explicit expressions in Theorem 2."

Frankly speaking, the historical development of the investigation went in the direction opposite to what has been presented up to this point. First, the formulas (16)–(17) were obtained as the solution of a linear system of equations arising from the feature of the Fermat–Torricelli point to make an angle of $2\pi/3$ with any two vertices of the triangle. Next, in a similar way, the formulas mentioned in Remark 2 were obtained for the generalized Fermat–Torricelli problem, i.e., for the coordinates x_*, y_*. Although these formulas looked awful, they permitted us to deduce the explicit expression (6) for the value of minimal distance. Moreover, we noticed the appearance of this value in the expressions for denominators of the formulas for x_* and y_*. Next, we intended to perform an additional verification of the obtained results via direct substitution into the equations (3). At this moment, the following lucky guess came to mind: the radicand of

$$\sqrt{(x_* - x_j)^2 + (y_* - y_j)^2}$$

should be a perfect square! The only remaining trick was to discover the values (9).

5. INVERSE PROBLEM. Given the coordinates of the point $P_* = (x_*, y_*)$, we wish to find the values for the weights m_1, m_2, and m_3 with the aim for the corresponding objective function (1) to posses a minimum point precisely at P_*.

Theorem 4. *Let the vertices of the triangle $P_1P_2P_3$ be counted counterclockwise. Then for the choice*

$$m_1^* = |P_*P_1| \cdot \begin{vmatrix} 1 & 1 & 1 \\ x_* & x_2 & x_3 \\ y_* & y_2 & y_3 \end{vmatrix},$$

$$m_2^* = |P_*P_2| \cdot \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_* & x_3 \\ y_1 & y_* & y_3 \end{vmatrix}, \quad \textit{and}$$

$$m_3^* = |P_*P_3| \cdot \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_* \\ y_1 & y_2 & y_* \end{vmatrix} \tag{18}$$

the function

$$F(x, y) = \sum_{j=1}^{3} m_j^* \sqrt{(x - x_j)^2 + (y - y_j)^2}$$

has its stationary point at P_*. Provided that the latter is chosen inside the triangle $P_1P_2P_3$, the values (18) *are all positive, and*

$$F(x_*, y_*) = \min_{(x,y)\in\mathbb{R}^2} F(x, y) = \begin{vmatrix} 1 & 1 & 1 & 1 \\ x_* & x_1 & x_2 & x_3 \\ y_* & y_1 & y_2 & y_3 \\ x_*^2 + y_*^2 & x_1^2 + y_1^2 & x_2^2 + y_2^2 & x_3^2 + y_3^2 \end{vmatrix}. \tag{19}$$

Proof. Substitute $x = x_*$, $y = y_*$ and the values (18) into the left-hand side of the first equation from (3) as follows:

$$(x_*-x_1)\begin{vmatrix}1&1&1\\x_*&x_2&x_3\\y_*&y_2&y_3\end{vmatrix}+(x_*-x_2)\begin{vmatrix}1&1&1\\x_1&x_*&x_3\\y_1&y_*&y_3\end{vmatrix}+(x_*-x_3)\begin{vmatrix}1&1&1\\x_1&x_2&x_*\\y_1&y_2&y_*\end{vmatrix}. \tag{20}$$

Represent this combination of the third-order determinants in the form of the fourth-order determinant, namely

$$\begin{vmatrix}1&1&1&1\\x_*&x_1&x_2&x_3\\y_*&y_1&y_2&y_3\\0&x_*-x_1&x_*-x_2&x_*-x_3\end{vmatrix}$$

(expansion by its last row coincides with (20)). Now add the second row to the last to obtain the following:

$$\begin{vmatrix}1&1&1&1\\x_*&x_1&x_2&x_3\\y_*&y_1&y_2&y_3\\x_*&x_*&x_*&x_*\end{vmatrix}.$$

In this determinant, the first row is proportional to the last one; therefore, the determinant equals just zero. The second equality from (3) can be verified in a similar manner.

Let us evaluate $F(x_*, y_*)$:

$$\begin{aligned}F(x_*,y_*)=\left[(x_*-x_1)^2+(y_*-y_1)^2\right]&\begin{vmatrix}1&1&1\\x_*&x_2&x_3\\y_*&y_2&y_3\end{vmatrix}\\+\left[(x_*-x_2)^2+(y_*-y_2)^2\right]&\begin{vmatrix}1&1&1\\x_1&x_*&x_3\\y_1&y_*&y_3\end{vmatrix}\\+\left[(x_*-x_3)^2+(y_*-y_3)^2\right]&\begin{vmatrix}1&1&1\\x_1&x_2&x_*\\y_1&y_2&y_*\end{vmatrix}.\end{aligned}$$

To prove the equality (19), let us split it into the x-part and the y-part. First, keep the x-terms in brackets of the previous formula:

$$(x_*-x_1)^2\begin{vmatrix}1&1&1\\x_*&x_2&x_3\\y_*&y_2&y_3\end{vmatrix}+(x_*-x_2)^2\begin{vmatrix}1&1&1\\x_1&x_*&x_3\\y_1&y_*&y_3\end{vmatrix}+(x_*-x_3)^2\begin{vmatrix}1&1&1\\x_1&x_2&x_*\\y_1&y_2&y_*\end{vmatrix}.$$

Similar to the proof of the first part of the theorem, represent this linear combination as the determinant of the fourth order:

$$\begin{vmatrix}1&1&1&1\\x_*&x_1&x_2&x_3\\y_*&y_1&y_2&y_3\\0&(x_*-x_1)^2&(x_*-x_2)^2&(x_*-x_3)^2\end{vmatrix}.$$

Multiply the first row by $(-x_*^2)$, the second one by $2x_*$ and add the obtained rows to the last one:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ x_* & x_1 & x_2 & x_3 \\ y_* & y_1 & y_2 & y_3 \\ x_*^2 & x_1^2 & x_2^2 & x_3^2 \end{vmatrix}. \tag{21}$$

The y-part of the equality (19) can be proven in exactly the same manner with the resulting determinant differing from (21) only in its last row. The linear property of determinant with respect to its rows completes the proof of (19). ∎

Remark 4. The solution of the inverse problem is determined up to a common positive multiplier, i.e., the solution triple (m_1, m_2, m_3) is defined by the value of the ratio $m_1 : m_2 : m_3$. (In the language of the facility location problem mentioned in the Introduction, this statement is equivalent to the fact that the optimal position of the steel works is independent of the currency of the state.) Up to this remark, the solution of the inverse problem is unique. We have proven this statement via direct computations starting from formulas (4).

Example 1. Let $P_1 = (2, 6)$, $P_2 = (1, 1)$, $P_3 = (5, 1)$, and

$$P_* = \left(\frac{1}{2866}\left(4103 + 1833\sqrt{15}\right), \frac{1}{8598}\left(29523 - 4481\sqrt{15}\right)\right).$$

Find the values for the weights m_1^*, m_2^*, and m_3^* from Theorem 4.

Solution. Formulas (18) give:

$$m_1^* = \frac{2(20925 - 4481\sqrt{15})}{18481401}\sqrt{316380606 + 35999826\sqrt{15}},$$

$$m_2^* = \frac{2(15105 - 2342\sqrt{15})}{6160467}\sqrt{75400161 - 9169767\sqrt{15}},$$

and

$$m_3^* = \frac{8(-1185 + 15988\sqrt{15})}{18481401}\sqrt{8335761 - 2050623\sqrt{15}},$$

with

$$F(x_*, y_*) = \frac{1}{4299}(-333980 + 193436\sqrt{15}).$$

Now, compare the obtained result with the one represented in test 1 from Section 2. According to Remark 4, we might expect that

$$m_1^* : m_2^* : m_3^* = 2 : 3 : 4.$$

We leave the verification of this fact as an exercise for the inquisitive reader.

The next example originated from the question posed by one of the referees of the present paper: What will happen to the result of Theorem 4 if we take $P_* = P_j$?

Example 2. Show how to choose the values for the weights m_1, m_2, and m_3 in order for the point P_* to coincide with the given point on a side of the triangle from Example 1.

Solution. If we take $P_* = P_2$, the formulas (18) give zero values for all the weights; however, the "weights" of these zeros are different. To explain this causistry, take $P_* = P_2 + (\mu, \mu)$ for the infinitely small $\mu > 0$. For this case, formulas (18) give:

$$m_1^*(\mu) = 4\mu\sqrt{26 - 12\mu + 2\mu^2} = 4\sqrt{26}\mu + o(\mu),$$
$$m_2^*(\mu) = 4\sqrt{2}\mu(5 - 2\mu),$$
$$m_3^*(\mu) = 4\mu\sqrt{16 - 8\mu + 2\mu^2} = 16\mu + o(\mu).$$

The weight $m_2^*(\mu)$ dominates over $m_1^*(\mu)$ and $m_3^*(\mu)$ when $\mu \to +0$. As a matter of fact, the true values of these weights do not influence the position of the point P_*; the latter depends only on the value of the ratio $m_1^*(\mu) : m_2^*(\mu) : m_3^*(\mu)$. Thus, the choice $m_1^* = 4\sqrt{26}$, $m_2^* = 20\sqrt{2}$, $m_3^* = 16$ provides us with $P_* = P_2$.

Let us now manipulate the weights with the aim of extruding the point P_* to an internal point of the side P_2P_3. This manipulation is not trivial, as in the previous case. First, we utilize formulas (18) and then simplify the obtained result with the aid of formulas (4). Finally, the variable weights

$$m_1^*(\mu) = t\mu,\ m_2^*(\mu) = 1 + \mu,\ m_3^*(\mu) = 1 - \mu$$

with a fixed $t > \sqrt{104}$, provide the following asymptotics as $\mu \to +0$:

$$P_* \longrightarrow \left(2 - \frac{10}{\sqrt{t^2 - 4}}, 1\right).$$

Thus, the two "essential" weights $m_2^*(\mu)$ and $m_3^*(\mu)$ guarantee delivery of P_* to the side P_2P_3, while the negligible weight $m_1^*(\mu)$ ensures the fine-tuning of this delivery to the particular point within the open line segment $P_2P_\perp$. Here $P_\perp = (2, 1)$ is the foot of the altitude of the triangle $P_1P_2P_3$ through the point P_1.

Let us discuss the geometrical meaning of the constants from Theorem 4. The value m_1^* equals twice the product of the distance $|P_1P_*|$ by the area of the triangle $P_*P_2P_3$. The first statement of the theorem is equivalent to the geometrical equality

$$\overrightarrow{P_*P_1} \cdot S_{\triangle P_*P_2P_3} + \overrightarrow{P_*P_2} \cdot S_{\triangle P_*P_3P_1} + \overrightarrow{P_*P_3} \cdot S_{\triangle P_*P_1P_2} = \vec{\mathbb{O}}.$$

Finally, the constant (19) is connected with

$$h = -\frac{1}{S}\begin{vmatrix} 1 & 1 & 1 & 1 \\ x_* & x_1 & x_2 & x_3 \\ y_* & y_1 & y_2 & y_3 \\ x_*^2 + y_*^2 & x_1^2 + y_1^2 & x_2^2 + y_2^2 & x_3^2 + y_3^2 \end{vmatrix},$$

which is known [**10**, pp. 251–252] as *the power of the point P_* with respect to the circle* through the points P_1, P_2, and P_3 (the circumscribed circle of the triangle). If we denote the circumcenter of the triangle $P_1P_2P_3$ by C, then

$$h = |CP_*|^2 - |CP_j|^2 \quad \text{for } j \in \{1, 2, 3\}, \tag{22}$$

and, provided that P_* lies inside this triangle, the value h is negative.

Results of the present section can evidently be extended to the case of three (and more) dimensions.

Theorem 5. *Let the points* $\{P_j = (x_j, y_j, z_j)\}_{j=1}^4$ *be noncoplanar, and be counted in such a manner that the value of the determinant*

$$V = \begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix} \tag{23}$$

is positive. Then for the choice

$$\left\{m_j^* = |P_*P_j| \cdot V_j\right\}_{j=1}^4, \tag{24}$$

where V_j *equals the determinant obtained on replacing the* j*th column of* (23) *by the column* $[1, x_*, y_*, z_*]^\top$ *(here* $^\top$ *denotes transposition), the function*

$$F(x, y, z) = \sum_{j=1}^{4} m_j^* \sqrt{(x - x_j)^2 + (y - y_j)^2 + (z - z_j)^2}$$

has its stationary point at $P_* = (x_*, y_*, z_*)$. *If* P_* *lies inside the tetrahedron* $P_1P_2P_3P_4$, *then the values* (24) *are all positive, and*

$$\begin{aligned} F(x_*, y_*, z_*) &= \min_{(x,y,z)\in\mathbb{R}^3} F(x, y, z) \\ &= -\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ x_* & x_1 & x_2 & x_3 & x_4 \\ y_* & y_1 & y_2 & y_3 & y_4 \\ z_* & z_1 & z_2 & z_3 & z_4 \\ x_*^2 + y_*^2 + z_*^2 & x_1^2 + y_1^2 + z_1^2 & x_2^2 + y_2^2 + z_2^2 & x_3^2 + y_3^2 + z_3^2 & x_4^2 + y_4^2 + z_4^2 \end{vmatrix}. \end{aligned} \tag{25}$$

Geometrical meanings of the values appearing in the last theorem are similar to their counterparts from Theorem 4. For instance, the value (23) equals six times the volume of tetrahedron $P_1P_2P_3P_4$, while the value (25) divided by V is known [**10**, p. 255] as *the power of the point* P_* *with respect to a sphere* circumscribed to that tetrahedron; it is equivalent to (22), where C this time stands for the circumcenter of the tetrahedron while $j \in \{1, 2, 3, 4\}$.

6. CONCLUSIONS. An analytical solution for the generalized Fermat–Torricelli problem and its inversion is presented. The three-point case is completely solved using "extended radicals": In addition to elementary and extraction of roots operations, the sign function is utilized in the formulas. The treatment of the multidimensional $n > 3$ point case requires further investigation, although some theoretical results like [**1**] give little reason to hope for a nice, e.g., extended radicals, solution.

7. APPENDIX. We prove here the equalities (10) and (11). We have

$$
\begin{aligned}
&K_1K_2 + K_1K_3 + K_2K_3 \\
&\quad = \frac{1}{2}\left[K_1(K_2+K_3) + K_2(K_1+K_3) + K_3(K_1+K_2)\right] \\
&\quad \overset{(9)}{=} K_1(r_{23}^2\sigma + m_1^2|S|) + K_2(r_{13}^2\sigma + m_2^2|S|) + K_3(r_{12}^2\sigma + m_3^2|S|) \\
&\quad = \sigma^2\left[(r_{12}^2 + r_{13}^2 - r_{23}^2)r_{23}^2 + (r_{23}^2 + r_{12}^2 - r_{13}^2)r_{13}^2 + (r_{13}^2 + r_{23}^2 - r_{12}^2)r_{12}^2\right] \\
&\qquad + S^2\left[m_1^2(m_2^2 + m_3^2 - m_1^2) + m_2^2(m_1^2 + m_3^2 - m_2^2) + m_3^2(m_1^2 + m_2^2 - m_3^2)\right] \\
&\qquad + \sigma|S|\Big[m_1^2(r_{12}^2 + r_{13}^2 - r_{23}^2) + m_2^2(r_{12}^2 + r_{23}^2 - r_{13}^2) + m_3^2(r_{13}^2 + r_{23}^2 - r_{12}^2) \\
&\qquad + r_{23}^2(m_2^2 + m_3^2 - m_1^2) + r_{13}^2(m_1^2 + m_3^2 - m_2^2) + r_{12}^2(m_1^2 + m_2^2 - m_3^2)\Big] \\
&\quad = 4\sigma^2S^2 + 4\sigma^2S^2 \\
&\qquad + 2\sigma|S|\left[m_1^2(r_{12}^2 + r_{13}^2 - r_{23}^2) + m_2^2(r_{13}^2 + r_{23}^2 - r_{12}^2) + m_3^2(r_{13}^2 + r_{23}^2 - r_{12}^2)\right].
\end{aligned}
$$

Here we have utilized (8) and the equality

$$
4S^2 = (r_{12}^2 + r_{13}^2 - r_{23}^2)r_{23}^2 + (r_{13}^2 + r_{23}^2 - r_{12}^2)r_{12}^2 + (r_{23}^2 + r_{12}^2 - r_{13}^2)r_{13}^2, \qquad (26)
$$

which can be verified either directly or with the aid of the Heron formula for the area of a triangle (see Section 3). Reference to the definition (6) of the constant d completes the proof of (10).

We now deduce formula (11):

$$
\begin{aligned}
&r_{23}^2K_1 + r_{13}^2K_2 + r_{12}^2K_3 \\
&\quad = \sigma\left[(r_{12}^2 + r_{13}^2 - r_{23}^2)r_{23}^2 + (r_{13}^2 + r_{23}^2 - r_{12}^2)r_{12}^2 + (r_{23}^2 + r_{12}^2 - r_{13}^2)r_{13}^2\right] \\
&\qquad + |S|\left[r_{23}^2(m_2^2 + m_3^2 - m_1^2) + r_{13}^2(m_1^2 + m_3^2 - m_2^2) + r_{12}^2(m_1^2 + m_2^2 - m_3^2)\right] \\
&\quad \overset{(26)}{=} 4\sigma S^2 + |S|\left[m_1^2(r_{12}^2+r_{13}^2-r_{23}^2) + m_2^2(r_{23}^2+r_{12}^2-r_{13}^2) + m_3^2(r_{13}^2+r_{23}^2-r_{12}^2)\right] \\
&\quad \overset{(6)}{=} 2|S|d.
\end{aligned}
$$

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