Taylor Series Method for Second Order Polynomial ODEs

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Abstract—An algorithm for numerical integration of non-linear Lagrange equations is presented. Formulas for approximate solutions are derived using the Taylor Series Method. Radius of convergence estimates and error bounds are given.

I. INTRODUCTION

Lagrange equation of motion for a physical system is widely used in mechanics [1]. The initial value problem for the second-order Lagrange equations has the form

\[
D(q, \dot{q}) \ddot{q} + C(q, \dot{q}) \dot{q} + K(q, u) = 0
\]

(1)

with \(D\) and \(C\) being matrices with polynomial dependencies on \(q\) and non-linear dependencies on the generalized coordinates \(q\). \(K\) is the vector of generalized forces. All matrices and vectors are in bold.

Most of Lagrange equations have similar form [2], [3]. Based on Taylor series method [4] we develop a numerical algorithm to integrate the second-order equations

\[
A(q, \dot{q}) \dot{q} + b(q, \dot{q}) = 0
\]

(2)

with mild assumptions on \(A(q, \dot{q})\) matrix and \(b(q, \dot{q})\) vector which are satisfied for typical applications.

The general observation of [5] shows that ODEs with polynomial right-hand sides are quite common. Euler equations of rotational motion and differential Riccati equations [6] from control theory already have polynomial right-hand sides of degree 2. If the ODE has some non-polynomial functions, it usually can be reduced to the polynomial ODE with the introduction of auxiliary variables.

II. REDUCTION TO POLYNOMIAL SYSTEM: EXAMPLE

Let us show how to reduce the following non-linear ODE system to the polynomial form. The equations for double pendulum system are defined by the matrices

\[
D = \begin{pmatrix}
(m_1 + m_2) L_1^2 & m_2 L_1 L_2 c_{1-2} \\
 m_2 L_1 L_2 c_{1-2} & m_2 L_2^2
\end{pmatrix},
\]

\[
C = \begin{pmatrix}
m_2 L_1 L_2 s_{1-2} \dot{\theta}_2 & 0 \\
m_2 L_1 L_2 s_{1-2} \dot{\theta}_1 & 0
\end{pmatrix},
\]

\[
K = g \begin{pmatrix}
L_1 s_1 (m_1 + m_2) \\
m_2 L_2 s_2
\end{pmatrix}.
\]

Here \(c_i\) and \(s_i\) denote \(\cos \theta_i\) and \(\sin \theta_i\) respectively, \(c_{1-2}\) and \(s_{1-2}\) denote \(\cos(\theta_1 - \theta_2)\) and \(\sin(\theta_1 - \theta_2)\). The former trigonometric functions can be expressed with \(c_i\) and \(s_i\).

The independent variables are \(q_1 = \theta_1\) and \(q_2 = \theta_2\). Let us introduce additional variables \(q_3 = s_1, q_4 = c_1, q_5 = s_2\) and \(q_6 = c_2\) and reduce the original equations to the system with polynomial dependencies on the variables.

After taking the derivative with respect to \(t\) we get the following first-order equations:

\[
\begin{align*}
q_3 &= q_4 q_1 \\
q_4 &= -q_3 q_1 \\
q_5 &= q_6 q_2 \\
q_6 &= -q_5 q_2
\end{align*}
\]

(3)

Our goal is the introduction of the augmented second-order system

\[
\tilde{D} \ddot{q} + \tilde{C} \dot{q} + \tilde{K} = 0.
\]

Let us take the derivative of (3) with respect to \(t\). This gives a set of second-order equations

\[
\begin{align*}
\ddot{q}_3 &= q_4 \dot{q}_1 + q_1 \dot{q}_4 \\
\ddot{q}_4 &= -q_3 \dot{q}_1 - q_3 \dot{q}_1 \\
\ddot{q}_5 &= q_6 \dot{q}_2 + q_6 \dot{q}_2 \\
\ddot{q}_6 &= -q_5 \dot{q}_2 - q_5 \dot{q}_2
\end{align*}
\]

In this case we can take

\[
\tilde{D} = \begin{pmatrix}
d_{11} & d_{12} & 0 & 0 & 0 & 0 \\
d_{21} & d_{22} & 0 & 0 & 0 & 0 \\
-\dot{q}_3 & 0 & 1 & 0 & 0 & 0 \\
0 & q_6 & 0 & 0 & 1 & 0 \\
0 & -\dot{q}_5 & 0 & 0 & 0 & 1 \\
c_{11} & c_{12} & 0 & 0 & 0 & 0 \\
c_{21} & c_{22} & 0 & 0 & 0 & 0 \\
\dot{q}_4 & 0 & 0 & 0 & 0 & 0 \\
-\dot{q}_3 & 0 & 0 & 0 & 0 & 0 \\
0 & \dot{q}_6 & 0 & 0 & 0 & 0 \\
0 & -\dot{q}_5 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
\tilde{C} = \begin{pmatrix}
\end{pmatrix}.
\]

The \(\tilde{C}\) matrix has some new coefficients which we define later. The crucial thing here is the following observation: if \(D\) matrix is invertible then the \(\tilde{D}\) is also invertible because \(\det \tilde{D} = \det D\) since \(\tilde{D}\) only adds the identity matrix at the diagonal. \(\tilde{K}\) is extended with zeros.
III. REDUCTION TO POLYNOMIAL SYSTEM: GENERAL CASE

Let us suppose that all the non-linear terms and factors in Lagrange equations satisfy the polynomial ODEs.

E.g., if the source equation is $\dot{q} = f(q)$ and $f(q)$ includes sines and cosines, we introduce a $q_1 = \sin q$ and $q_2 = \cos q$ pair and get a first-order system

$$\begin{align*}
\dot{q}_1 &= q_2 \frac{\partial f}{\partial q} \\
\dot{q}_2 &= -q_1 \frac{\partial f}{\partial q}
\end{align*}$$

In many mechanical applications trigonometric functions and inverse values like $1/q^n$ are enough. The equation for $y = 1/q$ is

$$\dot{y} = -1/q^2 = -y^2$$

and has a quadratic polynomial as right-hand side.

From this point we assume that $q_1, \ldots, q_n$ are the source variables and we add the auxiliary variables $q_{n+1}, \ldots, q_{n+m}$ which satisfy the equations

$$\begin{align*}
\dot{q}_{n+1} &= P_1(q_1, \ldots, q_{n+m}, \dot{q}_1, \ldots, \dot{q}_n) \\
\ldots
\dot{q}_{n+m} &= P_m(q_1, \ldots, q_{n+m}, \dot{q}_1, \ldots, \dot{q}_n)
\end{align*}$$

where $P_i$ are polynomial functions.

Taking the derivative of (4) with respect to $t$ we get for each $i$ the relation

$$\ddot{q}_{n+i} = \sum_{j=1}^{n+m} \frac{\partial P_j}{\partial q_j} \dot{q}_j + \sum_{k=1}^{m} \frac{\partial P_j}{\partial q_k} \ddot{q}_k.$$

So we define

$$\ddot{D}_{n+i,j} = \frac{\partial P_j}{\partial q_j}, \quad \ddot{C}_{n+i,j} = \frac{\partial P_j}{\partial q_k}.$$

The rest of the entries are taken from $D$ and $C$ respectively and

$$\ddot{D}_{n+i,n+j} = 1, \quad \ddot{D}_{n+i,n+j} = 0, \quad \text{if} \quad i \neq j.$$

IV. TAYLOR SERIES METHOD FOR SECOND-ORDER ODES

Throughout this section we use $i$ and $j$ as indices for matrix and vector elements, $m$ to denote the degrees of power series coefficients for the $(t-t_0)^m$ terms, and all the other letters as summation indices.

We now consider the equation

$$A(q, \dot{q}) \ddot{q} + b(q, \dot{q}) = 0. \quad (5)$$

We have the matrix $A = (a_{ij})^n_{i,j=1}$ and vector $b = (b_1, \ldots, b_n)^T$. Here $a_{ij}$ and $b_i$ are polynomials of degree $\alpha_{ij}$ and $\beta_i$ respectively. We use compact multi-index notation where

$$q^p = q_1^{p_1} \ldots q_n^{p_n}, \quad |p| = p_1 + \ldots + p_n, \quad |r| = r_1 + \ldots + r_n, \quad a_{p_1p_2\ldots p_n,r_1,r_2\ldots r_n;i,j} = a_{p;i,j}, \quad b_{p_1p_2\ldots p_n,r_1,r_2\ldots r_n;i} = b_{p;i}. \quad (6)$$

The expansions for $q$ are

$$q_i(t) = \sum_{m=0}^{+\infty} q_{i|m}(t-t_0)^m. \quad (7)$$

If we have two series $f(t) = \sum_{m=0}^{+\infty} f_m(t-t_0)^m$ and $g(t) = \sum_{m=0}^{+\infty} g_m(t-t_0)^m$, the coefficients of their product $h = \sum_{m=0}^{+\infty} h_m(t-t_0)^m$ are given by Cauchy formula

$$h = f \cdot g = \sum_{m=0}^{+\infty} \sum_{k=0}^{m} (f_k g_{m-k}) (t-t_0)^m.$$

We denote the convolution of the series as $*$.

$$h_m = (f_m * g_m) = \sum_{k=0}^{m} f_k g_{m-k}. \quad (9)$$

The arbitrary power of the $q_i(t)$ is expressed recursively as

$$q_i^p(t) = \sum_{m=0}^{+\infty} q_{i|m}^p(t-t_0)^m, \quad (q_{i|m})^p = \sum_{k=0}^{m} \left( q_{i|k}^{p-1} q_{i|m-k} \right).$$

The derivatives of $q_i(t)$ are

$$\ddot{q}_i(t) = \sum_{m=0}^{+\infty} (m+1) q_{i|m+1}(t-t_0)^m, \quad \ddot{q}_i(t) = \sum_{m=0}^{+\infty} (m+1)(m+2) q_{i|m+2}(t-t_0)^m. \quad (10)$$

Finally the equation (2) is rewritten as

$$\sum_{j=1}^{n} (a_{ij} * ((m+1)(m+2)q_{j|m+2}) + b_{i|m} = 0. \quad (12)$$

The explicit equations for $q_k(t)$ coordinates are

$$\sum_{k=0}^{m} \sum_{j=1}^{n} (a_{ij}|m-k)((k+1)(k+2)q_{j|k+2}) + b_{i|m} = 0. \quad (13)$$

The $a_{ij|m}$ and $b_{i|m}$ values in the above formula depend on $q_0, \ldots, q_{m+1}$ and have the following expressions.

$$a_{ij|m} = \sum_{s=0}^{+\infty} \sum_{|p|+|r|=s} a_{p;i,j} q_m^p \cdot [(m+1) q_{m+1}]^r. \quad (8)$$

$$b_{i|m} = \sum_{s=0}^{+\infty} \sum_{|p|+|r|=s} b_{p;i} q_{m+1}^p \cdot [(m+1) q_{m+1}]^r. \quad (9)$$
Introducing new variables $z$ whose elements are polynomials. The compact notation for convolution power uses multi-indices from (6).

$$q_{m}^{p} = q_{1}^{p_{1}} \ast q_{2}^{p_{2}} \ast \ldots \ast q_{n}^{p_{n}}$$

Finally, the $m$-th coefficient for $q_{i}^{p}(t)$ series is given by

$$q_{i}^{p}_{m} = \sum_{k=0}^{m} q_{ij}^{p_{i}-1} q_{i|m-k} = q_{i|m} * \ldots * q_{i|m}.$$  

For $q_{m+2}$ we get the equation

$$(m+1)(m+2)A_{0}q_{m+2} + c_{m} = 0.$$  

Here we have

$$A_{0} = \{a_{ij}|0\}_{i,j=1}^{n}, \quad c_{i} = b_{i|0}, \quad \text{and for } m > 0$$

$$c_{i;m} = \sum_{k=0}^{m} \sum_{j=1}^{n} (a_{ij|m-k}((k+1)(k+2)q_{j|k+2}) + b_{i|m}.$$  

V. ALGORITHM SUMMARY

To calculate series expansions we start from the given initial conditions $q_{0}$ and $q_{1}$. Then for each $m$ from 0 to $M - 2$ we calculate the expansion coefficients.

$$q_{m+2} = \frac{-A_{0}^{-1}c_{m}}{(m+1)(m+2)}, \quad A_{0} = \{a_{ij}|0\}_{i,j=1}^{n}.$$  

The $a_{ij|m}$ and $c_{i;m}$ values in the formula above could be obtained from (8) and (10) with respect to (9).

It might seem strange why we don’t use the first-order system to obtain series coefficients directly which is indeed possible and useful. The most important thing is that this method allows us not to calculate the inverse matrix for $A$, whose elements are polynomials.

VI. ERROR ESTIMATES

To use the method for actual solution of Cauchy problems we need some error estimates which are given in [7].

The equation (5) can be rewritten as

$$\ddot{q}_{k} = -(det A)^{-1}B_{i}, \quad i = 1, \ldots, n,$$

where

$$B_{i} = \begin{bmatrix}
  a_{11} & \ldots & a_{1,i-1} & b_{1} & a_{1,i+1} & \ldots & a_{1n} \\
  a_{21} & \ldots & a_{2,i-1} & b_{2} & a_{2,i+1} & \ldots & a_{2n} \\
  \vdots & \ddots & \vdots & \ddots & \ddots & \ddots & \vdots \\
  a_{n,1} & \ldots & a_{n,i-1} & b_{n} & a_{n,i+1} & \ldots & a_{nn}
\end{bmatrix}.$$  

Introducing new variables $z_{j} = \dot{q}_{j}$, $z_{n+j} = q_{j}$, $j = 1, \ldots, n$, $z_{2n+1} = 1/\det A$ reduce the original $n$-dimensional system to $2n + 1$-dimensional first-order system

\[
\begin{align*}
\dot{z}_{i} &= z_{n+j}, \quad j = 1, \ldots, n \\
\dot{z}_{n+j} &= -z_{2n+1}B_{j}, \quad j = 1, \ldots, n, \\
\dot{z}_{2n+1} &= z_{2n+1} - \sum_{k=1}^{n} \frac{\partial}{\partial z_{k}} B_{k} z_{n+k} - \sum_{k=1}^{n} \frac{\partial}{\partial z_{k}} A \frac{z_{n+k}}{z_{2n+1}}
\end{align*}
\]

Since the functions in the right hand side of system (11) are polynomials $f_{i}(z) = \sum_{k=0}^{l_{i}+1} \sum_{|l|=k} f_{i} l_{i} z^{l}$ we can use the theory from [7] to obtain the radius of convergence and rigorous local error estimates.

After introducing the norm

$$\|z\| = ||(z_{1}, \ldots, z_{2n+1})|| = \max_{j=1,\ldots,2n+1} |z_{j}|$$

and the auxiliary function

$$s(\gamma) = \max_{i=1,\ldots,2n+1} \sum_{m=1}^{L+1} (\gamma^{m-1} \sum_{|l|=m} |f_{i}|)$$

we have the following estimate on the radius of convergence $\rho$ for the series (7):

$$\rho = \frac{1}{Ls(||z_{0}||)}.$$  

We denote the $M$-th order approximation to the solution of (11) as

$$T_{M}z(t) = \sum_{m=0}^{M} z_{m}(t - t_{0})^{m}.$$  

The error estimate for each $t$ in the region of convergence is

$$||\delta_{M}(t)|| = ||z(t) - T_{M}z(t)|| \leq \frac{||z(t_{0})||_{1}^{M+1}}{(1 - t_{1})L},$$

where

$$t_{1} = |t - t_{0}|/\rho.$$  

VIII. CONCLUSION

An algorithm for integration of Lagrange ODE system is presented. The formulas of this method do not require the inversion of $D$ matrix and use only the linear algebra operations even if the original equations contain some non-linear and non-polynomial functions like the trigonometric ones. Error estimates and radius of convergence estimates are calculated directly from the system without the need for any bounds on the functions in the equations.

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