On Construction of Multivariate Wavelet Frames [1]

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Abstract

Construction of wavelet frames with matrix dilation is studied. We found a necessary condition and a sufficient condition under which a given pair of refinable functions generates dual wavelet systems with a given number of vanishing moments.

For image compression and some other applications, it is very desirable to have wavelets with vanishing moment property. In particular, vanishing moments are closely related to the approximation order of wavelet frames (see, e.g., [1]). If wavelet system is a basis, the number of its vanishing moments depends only on the dual generating refinable function. Situation is essentially different for frames. Two pairs of dual wavelet frames may be generated by the same refinable functions and have different number of vanishing moments.

The goal of this paper is to describe refinable functions generating dual wavelet systems with vanishing moments and to present an explicit method for construction compactly supported wavelet frames with arbitrary number of vanishing moments. Very close problem were investigated by Ming-Jun Lai and A. Petukhov[2] for univariate wavelet frames. Their technique is not appropriate for multi-dimensional investigations because zero properties of multivariate masks can not be described by means of factorization in contrast to the one-dimensional case .

Throughout the paper we will use the following notations.

$\mathbb{N}$ is the set of positive integers, $\mathbb{R}^d$ denotes the d-dimensional Euclidean space, $x = (x_1, \dots, x_d)$, $y = (y_1, \dots, y_d)$ are its elements (vectors), $(x, y) = x_1y_1 + \ldots + x_dy_d$, $|x| = \sqrt{(x,x)}$, $\mathbf{e}_j = (0, \dots, 1, \dots, 0)$ is the j-th unit vector in $\mathbb{R}^d$, $\mathbf{0} = (0, \dots, 0) \in \mathbb{R}^d$, $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^d$; $\mathbb{Z}^d$ is the integer lattice in $\mathbb{R}^d$. For $x, y \in \mathbb{R}^d$, we write $x > y$ if $x_j > y_j$, $j = 1, \dots, d$; $\mathbb{Z}^d_+ = \{x \in Z^d : \ x \geq \mathbf{0}\}$.

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If $\alpha, \beta \in \mathbb{Z}_+^d$, $a, b \in \mathbb{R}^d$, we set $\alpha! = \prod_{j=1}^{d} \alpha_j!$, $\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}$, $a^b = \prod_{j=1}^{d} {a_j}^{b_j}$, $[\alpha] = \sum_{j=1}^{d} \alpha_j$, $D^\alpha f = \frac{\partial^{[\alpha]} f}{\partial^{\alpha_1} x_1 \dots \partial^{\alpha_d} x_d}$; δ_{ab} denotes Kronecker delta; $\mathbb{T}^d$ is the unit d-dimensional torus; $\mathbb{C}$ is the set of complex numbers.

Let M be a non-degenerate $d \times d$ integer matrix whose eigenvalues are bigger than 1 in module, M^* is the conjugate matrix to M , I_d denotes the unit $d \times d$ matrix. We say that numbers $k, n \in \mathbb{Z}^d$ are congruent modulo M (write $k \equiv n \pmod{M}$) if $k - n = M\ell$, $\ell \in \mathbb{Z}^d$. The integer lattice $\mathbb{Z}^d$ is splitted into cosets with respect to the introduced relation of congruence. The number of cosets is equal to $|\det M|$ (see, e.g., [3, §2.7]). Let us take an arbitrary representative from each coset, call them digits and denote the set of digits by $D(M)$. Throughout the paper we consider that such a matrix M is fixed, $m = |\det M|$, $D(M) = \{s_0, \dots, s_{m-1}\}$, $s_0 = \mathbf{0}$, $k = 1, \dots, m-1$, $R(M) = \{M^{-1}s_0, \dots, M^{-1}s_{m-1}\}$.

We will consider wavelet frames constructed in the framework of multiresolution analysis. Let a MRA in $L_2(\mathbb{R}^d)$ be generated by a scaling function φ which satisfies the refinement equation

$$\widehat{\varphi}(x) = m_0(M^{*-1}x)\widehat{\varphi}(M^{*-1}x),$$

where $m_0 \in L_2(\mathbb{T}^d)$ is its mask (refinable mask). For any $m_\nu \in L_2(\mathbb{T}^d)$, there exists a unique set of functions $\mu_{\nu k} \in L_2(\mathbb{T}^d)$, $k = 0, \dots, m-1$, (polyphase representatives of m_ν) so that

$$m_\nu(x) = \frac{1}{\sqrt{m}} \sum_{k=0}^{m-1} e^{2\pi i (s_k, x)} \mu_{\nu k}(M^* x). \quad (1)$$

The functions $\mu_{\nu k}$ can be expressed by

$$\mu_{\nu k}(x) = \frac{1}{\sqrt{m}} \sum_{s \in D(M^*)} e^{-2\pi i (M^{-1}s_k, x+s)} m_\nu(M^{*-1}(x+s)).$$

It is clear from these formulas that a function m_ν is differentiable (n times) on $R(M^*)$ if and only if its polyphase representatives $\mu_{\nu k}$, $k = 0, \dots, m-1$, are differentiable (n times) at the origin and m_ν is a trigonometric polynomial if and only if its polyphase representatives, are a trigonometric polynomials.

Now, let another MRA be generated by a scaling function $\widetilde{\varphi}$ with a mask $\widetilde{m}_0$. According to *Unitary Extension Principle* [4], to construct dual wavelet frames one finds wavelet masks $m_\nu, \widetilde{m}_\nu$, $\nu = 1, \dots, r$, $r \geq m-1$, so that the polyphase matrices

$$\mathcal{M} := \begin{pmatrix} \mu_{00} & \dots & \mu_{0,m-1} \\ \vdots & \ddots & \vdots \\ \mu_{r,0} & \dots & \mu_{r,m-1} \end{pmatrix}, \quad \widetilde{\mathcal{M}} := \begin{pmatrix} \widetilde{\mu}_{00} & \dots & \widetilde{\mu}_{0,m-1} \\ \vdots & \ddots & \vdots \\ \widetilde{\mu}_{r,0} & \dots & \widetilde{\mu}_{r,m-1} \end{pmatrix},$$

satisfy

$$\mathcal{M}^{T}\overline{\widetilde{\mathcal{M}}} = I_m, \tag{2}$$

and define wavelet functions by

$$\begin{aligned} \widehat{\psi}^{(\nu)}(x) &= m_\nu(M^{*-1}x)\widehat{\varphi}(M^{*-1}x), \\ \widehat{\widetilde{\psi}^{(\nu)}}(x) &= \widetilde{m}_\nu(M^{*-1}x)\widehat{\widetilde{\varphi}}(M^{*-1}x). \end{aligned}$$

The corresponding dual wavelet systems are $\{\psi_{jk}^{(\nu)}\}$, $\{\widetilde{\psi}_{jk}^{(\nu)}\}$, where
$\psi_{jk}^{(\nu)} := m^{j/2}\psi^{(\nu)}(M^j \cdot + k)$, $\widetilde{\psi}_{jk}^{(\nu)} := m^{j/2}\widetilde{\psi}^{(\nu)}(M^j \cdot + k)$, $j, k \in \mathbb{Z}^d$, $\nu = 0, \dots, r$.

It is known that if $\mathcal{M} = \widetilde{\mathcal{M}}$ then $\{\psi_{jk}^{(\nu)}\}$ is a tight frame in $L_2(\mathbb{R}^d)$. If $\mathcal{M}, \widetilde{\mathcal{M}}$ are arbitrary matrixes satisfying (2), under some additional assumptions on $\varphi, \widetilde{\varphi}, m_\nu, \widetilde{m}_\nu$ (see [5], [6], [3, §2.7]), we can state that $\{\psi_{jk}^{(\nu)}\}$, $\{\widetilde{\psi}_{jk}^{(\nu)}\}$ are dual frames in $L_2(\mathbb{R}^d)$.

Throughout the paper we will consider that wavelet systems $\{\psi_{jk}^{(\nu)}\}$, $\{\widetilde{\psi}_{jk}^{(\nu)}\}$ are constructed by means of Unitary Extension Principle from generating scaling functions $\varphi, \widetilde{\varphi}$ whose masks $m_0, \widetilde{m}_0$ are continuous at the origin and $m_0(\mathbf{0}) = \widetilde{m}_0(\mathbf{0}) = 1$.

Definition 1 *We say that a wavelet system $\{\psi_{jk}^{(\nu)}\}$ has vanishing moments up to order α, $\alpha \in \mathbb{Z}_+^d$, (has VM_α* property *in the sequel), if $D^\beta\widehat{\psi}^{(\nu)}(\mathbf{0}) = 0$, $\nu = 1, \dots, r$, for all $\beta \in \mathbb{Z}_+^d$, $\beta \leq \alpha$.*

Assume that the functions $\widehat{\varphi}, m_1, \dots, m_r$ have derivatives up to order α at the origin. It easily follows from Leibniz formula that VM_α property holds if and only if

$$D^\beta(m_\nu(M^{*-1}x)\Big|_{x=\mathbf{0}} = 0, \nu = 1, \dots, r, \quad \forall \beta \in \mathbb{Z}_+^d, \beta \leq \alpha. \tag{3}$$

In the case $r = m - 1$, there exist different criterions for vanishing moment. It is known [7] how to describe vanishing moment property in terms of linear identities for Fourier coefficients of the dual refinable mask (so-called *sum rule*). Some other descriptions of masks providing VM_α property are found in terms of zero-conditions [7] and in terms of containment in a quotient ideal [8]. The following polyphase criterion was given in [9]: VM_α property is valid for $\{\psi_{jk}^{(\nu)}\}$ if and only if there exist complex numbers λ_γ, $\gamma \in \mathbb{Z}_+^d$, $\gamma \le \alpha$, such that $\lambda_{\mathbf{0}} = 1$,

$$D^\beta \widetilde{\mu}_{0k}(\mathbf{0}) = \frac{1}{\sqrt{m}} \sum_{\mathbf{0} \le \gamma \le \beta} \lambda_\gamma \binom{\beta}{\gamma} (-2\pi i M^{-1} s_k)^{\beta-\gamma} \quad \forall \beta \in \mathbb{Z}_+^d, \beta \le \alpha, \tag{4}$$

for each $k = 0, \dots, m-1$. The set of parameters λ_γ in (4) is unique, and λ_γ does not depend on α due to the following statement.

Proposition 2 *[9] If (4) is valid for the polyphase representatives of $\widetilde{m}_0$, then*

$$\lambda_\beta = D^\beta \left(\widetilde{m}_0(M^{*-1}x) \right) \Big|_{x=\mathbf{0}} \tag{5}$$

for all $\beta \in \mathbb{Z}_+^d$, $\beta \le \alpha$.

So, in the case $r = m-1$, VM_α property for $\{\psi_{jk}^{(\nu)}\}$ depends only on $\widetilde{m}_0$, i.e. only the first raw of the matrix $\widetilde{\mathcal{M}}$ is responsible for vanishing moments of wavelets generated by the matrix $\mathcal{M}$. In the case $r > m-1$, VM_α property for $\{\psi_{jk}^{(\nu)}\}$ depends also on the way of construction of matrixes $\mathcal{M}$, $\widetilde{\mathcal{M}}$. This may be illustrated by the following example.

Let $d = 1$, $M = m = 2$, $\mu_{00} = \mu_{01} = \widetilde{\mu}_{00} = \widetilde{\mu}_{01} \equiv \frac{1}{\sqrt{2}}$,

$$\mathcal{M} = \widetilde{\mathcal{M}} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}, \quad \mathcal{M}' = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad \widetilde{\mathcal{M}}' = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Either of pairs $\mathcal{M}, \widetilde{\mathcal{M}}$ and $\mathcal{M}', \widetilde{\mathcal{M}}'$ satisfies (2). The matrixes $\mathcal{M}, \widetilde{\mathcal{M}}$ generate wavelet masks $m_1(x) = m_2(x) = \widetilde{m}_1(x) = \widetilde{m}_2(x) = \frac{1}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} e^{2\pi i x}$. It is clear that $m_1(0) = m_2(0) = \widetilde{m}_1(0) = \widetilde{m}_2(0) = 0$, i.e. for the corresponding wavelet systems VM_0 property is valid. The matrixes $\mathcal{M}', \widetilde{\mathcal{M}}'$ generate wavelet masks $m_1'(x) = \frac{1}{2}$, $m_2'(x) = \frac{1}{2} e^{2\pi i x}$, $\widetilde{m}_1'(x) = \frac{1}{2} - \frac{1}{2} e^{2\pi i x}, \widetilde{m}_2'(x) = -\frac{1}{2} + \frac{1}{22} e^{2\pi i x}$, and we have $m_1'(0) \ne 0, m_2'(0) \ne 0$.

Theorem 3 *Let $\alpha \in \mathbb{Z}_+^d$, $r \ge m-1$, the functions $\mu_{\nu,k}, \widetilde{\mu}_{\nu k} \in L_2(\mathbb{T}^d)$, $\nu, k = 0, \dots, r$, have derivatives up to order α at the origin, the matrixes $\mathcal{N} := \{\mu_{\nu k}\}_{\nu,k=0}^r$ and $\widetilde{\mathcal{N}} := \{\overline{\mu}_{\nu k}\}_{\nu,k=0}^r$ satisfy*

$$\mathcal{N}\overline{\widetilde{\mathcal{N}}^T} = I_{r+1}; \tag{6}$$

masks $\widetilde{m}_0, m_1, \dots, m_{m-1}$ are defined by (1). Then condition (3) is valid if and only if

(a) there exist $\lambda_\gamma \in \mathbb{C}$, $\gamma \in \mathbb{Z}_+^d$, $\gamma \le \alpha$, such that $\lambda_{\mathbf{0}} = 1$ and (4) holds for $k = 0, \dots, m-1$;

(b) $D^\gamma \widetilde{\mu}_{0k}(\mathbf{0}) = 0$, $k = m, \dots, r$ for all $\gamma \in \mathbb{Z}_+^d$, $\gamma \le \alpha$.

`Proof.` Suppose that (3) is valid. We will prove (a) and (b) by induction on α. Check the initial step for $\alpha = 0$. Let $m_\nu(\mathbf{0}) = 0$, $\nu = 1, \dots, r$. It follows form (1) that

$$\sum_{k=0}^{m-1} \mu_{\nu k}(\mathbf{0}) = 0, \quad \nu = 1, \dots, r. \tag{7}$$

On the other hand, by (6),

$$\sum_{k=0}^{r} \overline{\widetilde{\mu}_{0k}(\mathbf{0})} \mu_{\nu k}(\mathbf{0}) = 0, \quad \nu = 1, \dots, m-1.$$

Because of linear independence of the vectors $(\mu_{\nu 0}(\mathbf{0}), \dots, \mu_{\nu,r}(\mathbf{0})) \in \mathbb{R}^{r+1}$, $\nu = 1, \dots, r$, there exists λ so that

$$\widetilde{\mu}_{00}(\mathbf{0}) = \ldots = \widetilde{\mu}_{0,m-1}(\mathbf{0}) = \lambda, \quad \widetilde{\mu}_{0m}(\mathbf{0}) = \ldots = \widetilde{\mu}_{0,r}(\mathbf{0}) = 0.$$

Taking into account the condition $\widetilde{m}_0(\mathbf{0}) = 1$ which is equivalent to

$$\frac{1}{\sqrt{m}} \left(\widetilde{\mu}_{\nu 0}(\mathbf{0}) + \ldots + \widetilde{\mu}_{\nu,m-1}(\mathbf{0})\right) = 1,$$

we obtain $\lambda = \frac{1}{\sqrt{m}}$.

For the inductive step we assume that (3) is valid for $\alpha > \mathbf{0}$ and $(a), (b)$ holds for all $\alpha' \in \mathbb{Z}_+^d$, $\alpha' < \alpha$. So, due to Proposition 2, there exist constants

$\lambda_\gamma \in \mathbb{C}$, $\gamma \in \mathbb{Z}_+^d$, $\gamma < \alpha$ such that (4) holds for all $\beta < \alpha$. If $\gamma \in \mathbb{Z}_+^d$, $\gamma < \alpha$, due to (1) and Leibniz formula, we have

$$\frac{1}{\sqrt{m}} \sum_{\mathbf{0}\le\beta\le\alpha-\gamma} \binom{\alpha-\gamma}{\beta} \sum_{k=0}^{m-1} (2\pi i M^{-1} s_k)^{\alpha-\beta-\gamma} D^\beta \mu_{\nu k}(\mathbf{0}) = D^{\alpha-\gamma} m_\nu(M^{*-1}x)\Big|_{x=\mathbf{0}} = 0. \tag{8}$$

It follows from (6) that

$$\sum_{k=0}^{r} \overline{\widetilde{\mu}_{0k}} \mu_{\nu k} = 0, \quad \nu = 1, \dots, m-1.$$

Differentiating this equality α times gives

$$\sum_{\mathbf{0}\le\beta\le\alpha} \binom{\alpha}{\beta} \sum_{k=0}^{r} \overline{D^{\alpha-\beta}\widetilde{\mu}_{0k}(\mathbf{0})} D^\beta \mu_{\nu k}(\mathbf{0}) = 0.$$

Taking into account the inductive hypotheses, we have

$$\sum_{\mathbf{0}\le\beta\le\alpha} \binom{\alpha}{\beta} \sum_{k=0}^{m-1} \overline{D^{\alpha-\beta}\widetilde{\mu}_{0k}(\mathbf{0})} D^\beta \mu_{\nu k}(\mathbf{0}) + \sum_{k=m}^{r} \overline{D^{\alpha}\widetilde{\mu}_{0k}(\mathbf{0})} \mu_{\nu k}(\mathbf{0}) = 0. \tag{9}$$

Multiply (8) by $\binom{\alpha}{\alpha-\gamma} \overline{\lambda_\gamma}$ and subtract from (9). After the same manipulation with each $\gamma \in \mathbb{Z}_+^d$, $\gamma < \alpha$, we obtain

$$0 = \sum_{k=m}^{r} \overline{D^{\alpha}\widetilde{\mu}_{0k}(\mathbf{0})} \mu_{\nu k}(\mathbf{0}) + \sum_{\mathbf{0}\le\beta\le\alpha} \binom{\alpha}{\beta} \sum_{k=0}^{m-1} \overline{D^{\alpha-\beta}\widetilde{\mu}_{0k}(\mathbf{0})} D^\beta \mu_{\nu k}(\mathbf{0}) -$$

$$\frac{1}{\sqrt{m}} \sum_{\mathbf{0}\le\gamma<\alpha} \binom{\alpha}{\alpha-\gamma} \overline{\lambda_\gamma} \sum_{\mathbf{0}\le\beta\le\alpha-\gamma} \binom{\alpha-\gamma}{\beta} \sum_{k=0}^{m-1} (2\pi i M^{-1} s_k)^{\alpha-\beta-\gamma} D^\beta \mu_{\nu k}(\mathbf{0}) =$$

$$\sum_{k=m}^{r} \overline{D^{\alpha}\widetilde{\mu}_{0k}(\mathbf{0})} \mu_{\nu k}(\mathbf{0}) + \sum_{\mathbf{0}\le\beta\le\alpha} \binom{\alpha}{\beta} \sum_{k=0}^{m-1} \Bigg(\overline{D^{\alpha-\beta}\widetilde{\mu}_{0k}(\mathbf{0})} -$$

$$\frac{1}{\sqrt{m}} \sum_{\substack{\gamma\ne\alpha \\ \mathbf{0}\le\gamma<\alpha-\beta}} \overline{\binom{\alpha-\gamma}{\beta} \binom{\alpha}{\alpha-\gamma} \binom{\alpha}{\beta}^{-1} \lambda_\gamma (-2\pi i M^{-1} s_k)^{\alpha-\beta-\gamma}} \Bigg) D^\beta \mu_{\nu k}(\mathbf{0}).$$

From this, taking into account that

$$\binom{\alpha-\gamma}{\beta}\binom{\alpha}{\alpha-\gamma}\binom{\alpha}{\beta}^{-1} = \frac{(\alpha-\beta)!}{\gamma!(\alpha-\beta-\gamma)!} = \binom{\alpha-\beta}{\gamma}, \tag{10}$$

and using the inductive hypotheses, the sum over β is deduced to

$$\sum_{\mathbf{0}\le\beta\le\alpha}\binom{\alpha}{\beta}\sum_{k=0}^{m-1}\Bigg(\overline{D^{\alpha-\beta}\widetilde{\mu}_{0k}(\mathbf{0})-}$$

$$\overline{\frac{1}{\sqrt{m}}\sum_{\substack{\gamma\neq\alpha\\ \mathbf{0}\le\gamma\le\alpha-\beta}}\binom{\alpha-\beta}{\gamma}\lambda_\gamma(-2\pi i M^{-1}s_k)^{\alpha-\beta-\gamma}}\Bigg) D^\beta\mu_{\nu k}(\mathbf{0}) =$$

$$\sum_{k=0}^{m-1}\left(\overline{D^\alpha\widetilde{\mu}_{0k}(\mathbf{0}) - \frac{1}{\sqrt{m}}\sum_{\mathbf{0}\le\gamma<\alpha}\binom{\alpha}{\gamma}\lambda_\gamma(-2\pi i M^{-1}s_k)^{\alpha-\gamma}}\right)\mu_{\nu k}(\mathbf{0}).$$

So, we have

$$\sum_{k=0}^{m-1}\left(\overline{D^\alpha\widetilde{\mu}_{0k}(\mathbf{0}) - \frac{1}{\sqrt{m}}\sum_{\mathbf{0}\le\gamma<\alpha}\binom{\alpha}{\gamma}\lambda_\gamma(-2\pi i M^{-1}s_k)^{\alpha-\gamma}}\right)\mu_{\nu k}(\mathbf{0}) +$$

$$\sum_{k=m}^{r}\overline{D^\alpha\widetilde{\mu}_{0k}(\mathbf{0})}\mu_{\nu k}(\mathbf{0}) = 0.$$

Similarly to the arguments for the initial step, it follows from (7) that there exists λ_α such that

$$D^\alpha\widetilde{\mu}_{0k}(\mathbf{0}) - \frac{1}{\sqrt{m}}\sum_{\mathbf{0}\le\gamma<\alpha}\binom{\alpha}{\gamma}\lambda_\gamma(-2\pi i M^{-1}s_k)^{\alpha-\gamma} = \frac{\lambda_\alpha}{\sqrt{m}}, \quad k=0,\dots,m-1,$$

$$D^\alpha\widetilde{\mu}_{0k}(\mathbf{0}) = 0, \quad k=m,\dots,r.$$

Thus, (4) is valid for $\beta=\alpha$ as was to be proved.

Now we assume that $(a),(b)$ are valid. We will prove (3) by induction on α. If (4) is valid for $\alpha=\mathbf{0}$, then $\widetilde{\mu}_{0k}(\mathbf{0}) = 1/\sqrt{m}$, $k=0,\dots,m-1$. It follows from (6) and (b) that

$$\mu_{\nu 0}(\mathbf{0}) + \ldots + \mu_{\nu,m-1}(\mathbf{0}) = 0, \quad \nu = 1,\dots,r.$$

Hence, on the basis of (1), $m_\nu(\mathbf{0}) = 0$, $\nu = 1, \dots, r$, what proves the initial step.

For the inductive step, we assume that $(a), (b)$ is valid for $\alpha > \mathbf{0}$ and (3) holds for all $\alpha' \in \mathbb{Z}_+^d$, $\alpha' < \alpha$, i.e.

$$D^{\alpha-\gamma} m_\nu(M^{*-1}x)\Big|_{x=\mathbf{0}} = 0, \quad \gamma \in \mathbb{Z}_+^d, \quad \gamma \neq \mathbf{0}, \quad \gamma \leq \alpha.$$

This yields (8) for $\gamma \neq \mathbf{0}$. Multiply (8) by $\binom{\alpha}{\alpha-\gamma}\overline{\lambda_\gamma}$ and add to (1) differentiated α times. After the same manipulation with each $\gamma \in \mathbb{Z}_+^d$, $\gamma < \alpha$, we obtain

$$D^\alpha m_\nu(M^{*-1}x)\Big|_{x=\mathbf{0}} = \frac{1}{\sqrt{m}} \sum_{\mathbf{0}\leq\beta\leq\alpha} \binom{\alpha}{\beta} \sum_{k=0}^{m-1} (2\pi i M^{-1}s_k)^{\alpha-\beta} D^\beta \mu_{\nu k}(\mathbf{0}) +$$

$$\frac{1}{\sqrt{m}} \sum_{\mathbf{0}<\gamma\leq\alpha} \binom{\alpha}{\alpha-\gamma} \overline{\lambda_\gamma} \sum_{\mathbf{0}\leq\beta\leq\alpha-\gamma} \binom{\alpha-\gamma}{\beta} \sum_{k=0}^{m-1} (2\pi i M^{-1}s_k)^{\alpha-\beta-\gamma} D^\beta \mu_{\nu k}(\mathbf{0}) =$$

$$\frac{1}{\sqrt{m}} \sum_{\mathbf{0}\leq\beta\leq\alpha} \binom{\alpha}{\beta} \sum_{\mathbf{0}\leq\gamma\leq\alpha-\beta} \overline{\lambda_\gamma} \binom{\alpha}{\alpha-\gamma} \binom{\alpha-\gamma}{\beta} \binom{\alpha}{\beta}^{-1} \cdot$$

$$\sum_{k=0}^{m-1} (2\pi i M^{-1}s_k)^{\alpha-\beta-\gamma} D^\beta \mu_{\nu k}(\mathbf{0}).$$

Due to (10) and (4), this yields

$$D^\alpha m_\nu(M^{*-1}x)\Big|_{x=\mathbf{0}} =$$

$$\frac{1}{\sqrt{m}} \sum_{\mathbf{0}\leq\beta\leq\alpha} \binom{\alpha}{\beta} \sum_{k=0}^{m-1} \overline{\sum_{\mathbf{0}\leq\gamma\leq\alpha-\beta} \lambda_\gamma \binom{\alpha-\beta}{\gamma} (-2\pi i M^{-1}s_k)^{\alpha-\beta-\gamma}} D^\beta \mu_{\nu k}(\mathbf{0}) =$$

$$\sum_{\mathbf{0}\leq\beta\leq\alpha} \binom{\alpha}{\beta} \sum_{k=0}^{m-1} \overline{D^{\alpha-\beta}\widetilde{\mu}_{0k}(\mathbf{0})} D^\beta \mu_{\nu k}(\mathbf{0}) = D^\alpha \left(\sum_{k=0}^{m-1} \overline{\widetilde{\mu}_{0k}(x)} \mu_{\nu k}(x) \right) \Bigg|_{x=\mathbf{0}} =$$

$$D^\alpha \left(\sum_{k=0}^{r} \overline{\widetilde{\mu}_{0k}(x)} \mu_{\nu k}(x) \right) \Bigg|_{x=\mathbf{0}}.$$

It follows from (6) that $D^\alpha m_\nu(M^{*-1}x)\Big|_{x=\mathbf{0}} = 0$ as was to be proved. $\Diamond$

Usually it is more useful to control univariate order of vanishing moment property (for example, to apply Taylor formula).

Definition 4 *We say that a wavelet system* $\{\psi_{jk}^{(\nu)}\}$ *has vanishing moments up to order* n, $n \in \mathbb{Z}_+$, *(has* VM^n property *in the sequel) if* $D^\beta \widehat{\psi}^{(\nu)}(\mathbf{0}) = 0$, $\nu = 1, \dots, m-1$, *for all* $\beta \in \mathbb{Z}_+^d$, $[\beta] \le n$.

Theorem 5 *Let* $n \in \mathbb{Z}_+$, $\quad r \ge m-1$, *the functions* $\mu_{\nu,k}$, $\widetilde{\mu}_{\nu k} \in L_2(\mathbb{T}^d)$, $\nu, k = 0, \dots, r$, *have derivatives up to order* n *at the origin, the matrixes* $\mathcal{N} := \{\mu_{\nu k}\}_{\nu,k=0}^r$ *and* $\widetilde{\mathcal{N}} := \{\widetilde{\mu}_{\nu k}\}_{\nu,k=0}^r$ *satisfy (6); masks* $\widetilde{m}_0, m_1, \dots, m_{m-1}$ *are defined by (1). Then condition (3) is valid for all* $\alpha \in \mathbb{Z}^d$, $[\alpha] \le n$ *if and only if*

(a) there exist $\lambda_\gamma \in \mathbb{C}$, $\gamma \in \mathbb{Z}_+^d$, $[\gamma] \le n$, *such that* $\lambda_{\mathbf{0}} = 1$ *and (4) holds for* $k = 0, \dots, m-1$;

(b) $D^\gamma \widetilde{\mu}_{0k}(\mathbf{0}) = 0$, $k = m, \dots, r$ *for all* $\gamma \in \mathbb{Z}_+^d$, $[\gamma] \le n$.

Proof of this theorem follows immediately from Theorem 3 and Proposition 2.

Let $n \in \mathbb{Z}_+$, we will denote by $L_\infty^{(n)}$ the class of complex-valued functions which are in $L_\infty(\mathbb{T}^d)$ and have continuous derivatives up to order n at the origin.

Lemma 6 *Let* $\mu_{\nu 0}, \widetilde{\mu}_{\nu 0} \in L_\infty^{(n)}$, $\nu = 0, \dots, r$, *and*

$$\sum_{\nu=0}^{r} \mu_{\nu 0} \overline{\widetilde{\mu}_{\nu 0}} = 1. \tag{11}$$

Then there exist functions $\mu_{\nu k}, \widetilde{\mu}_{\nu k} \in L_\infty^{(n)}$, $\nu = 0, \dots, r$, $k = 1, \dots, r$, *such that*

$$\sum_{\nu=0}^{r} \mu_{\nu l} \overline{\widetilde{\mu}_{\nu k}} = \delta_{kl}, \quad k, l = 0, \dots, r. \tag{12}$$

`Proof.` Set

$$\mu'_{\nu 0} := \frac{\mu_{\nu 0}}{\sqrt{\sum\limits_{l=0}^{r} |\mu_{l0}|^2}}, \quad \nu = 0, \dots, r.$$

It is clear that the functions $\mu'_{\nu 0}$ are essentially bounded and

$$\sum_{\nu=0}^{r} |\mu'_{\nu 0}|^2 = 1. \tag{13}$$

It follows from (11) that $\sum\limits_{l=0}^{r}|\mu_{l0}(\mathbf{0})|^2 \neq 0$. So, $\mu'_{\nu 0} \in L_\infty^{(n)}$, $\nu = 0,\dots,r$. Let us extend the unit vector $\mu'_{00},\dots,\mu'_{r0}$ to a unitary matrix. Due to (13), there exist ν_0 so that $\mu'_{\nu_0 0}(\mathbf{0}) \neq 1$. We may consider that $\nu_0 = 0$ (else we will interchange $\mu'_{\nu_0 0}$ and μ'_{00}, extend this new vector to a unitary matrix and interchange its 0-th and ν_0-th rows). Due to Householder transform, an extension to a unitary matrix may be realized by:

$$\mu'_{0k} = \overline{\mu'_{k0}}\frac{1-\mu'_{00}}{1-\overline{\mu'_{00}}}, \quad \mu'_{\nu k} = \delta_{lk} - \frac{\mu'_{\nu 0}\overline{\mu'_{k0}}}{1-\overline{\mu'_{00}}}, \quad \nu, k = 1,\dots,r.$$

Because of (13), we have $|\mu'_{\nu 0}| \leq \sqrt{1-|\mu'_{00}|^2}$, $\nu = 1,\dots,r$. This yields essential boundedness of the the functions $\mu'_{\nu k}$. Since $1-\mu'_{00}(\mathbf{0}) \neq 0$, it follows that $\mu'_{\nu k} \in L_\infty^{(n)}$, $\nu, k = 0,\dots,r$. Set

$$\begin{aligned}
&\widetilde{\mu}_{\nu k} := \mu'_{\nu k}, \quad \nu = 0,\dots,r,\ k = 1,\dots,,r,\\
&\widetilde{Q}_k := (\widetilde{\mu}_{0k},\dots,\widetilde{\mu}_{rk}), k = 0,\dots,r,\\
&Q_0 := (\mu_{00},\dots,\mu_{r0}), Q_k := \widetilde{Q}_k - \widetilde{Q}_k\widetilde{Q}_0^T Q_0, \quad k = 1,\dots,r.
\end{aligned}$$

It is not difficult to see that the entries of Q_k are in $L_\infty^{(n)}$ and $Q_k\overline{\widetilde{Q}_l^T} = \delta_{kl}$, $k, l = 0,\dots,r$. It remains to denote by $\mu_{\nu k}$ the ν-th component of Q_k.$\Diamond$

Lemma 7 *Let A be a class of complex-valued functions such that*

(i) if $f, g \in A$, $a, b \in \mathbb{C}$ then $af + bg \in A$,

(ii) if $f, g \in A$, then $fg \in A$,

and let $\mathcal{A}$ be a class of matrixes whose entries are in A. If any two $n\times 1$ matrixes Q, $\widetilde{Q} \in \mathcal{A}$ satisfying $Q^T\overline{\widetilde{Q}} = 1$ can be extended to $n\times n$ matrixes $\mathcal{N}, \widetilde{\mathcal{N}} \in \mathcal{A}$ satisfying $\mathcal{N}^T\overline{\widetilde{\mathcal{N}}} = I_n$, then any two $n\times j$ matrixes $\mathcal{M}$, $\widetilde{\mathcal{M}} \in \mathcal{A}$, $1 < j < n$, satisfying $\mathcal{M}^T\overline{\widetilde{\mathcal{M}}} = I_j$. can be extended to $n\times n$ matrixes $\mathcal{N}, \widetilde{\mathcal{N}} \in \mathcal{A}$ satisfying $\mathcal{N}^T\overline{\widetilde{\mathcal{N}}} = I_n$,

`Proof.` We will prove by induction on j. The base for $j = 1$ is given. Let us check the inductive step $j-1 \to j$. Let $j\times n$ matrixes $\mathcal{M}, \widetilde{\mathcal{M}} \in \mathcal{A}$ satisfy $\mathcal{M}^T\overline{\widetilde{\mathcal{M}}} = I_j$. Denote by Q_k, $\widetilde{Q}_k$ the k-th columns respectively of $\mathcal{M}$, $\widetilde{\mathcal{M}}$. Due to the statement of the base, the matrixes Q, $\widetilde{Q} \in \mathcal{A}$ can be extended to $n\times n$ matrixes $\mathcal{N}', \widetilde{\mathcal{N}'} \in \mathcal{A}$ satisfying $\mathcal{N}'^T\overline{\widetilde{\mathcal{N}'}} = I_n$, Let Q'_k, $\widetilde{Q}'_k$, $k = 2,\dots,n$, denote the k-th columns respectively of $\mathcal{N}', \widetilde{\mathcal{N}'} \in \mathcal{A}$. Fix a point x for which $Q_l'(x)^T \overline{\widetilde{Q}'}_k(x) = \delta_{kl}$. Since the vectors $Q_2'(x), \ldots, Q_n'(x)$ form a basis for the orthogonal complement to $\widetilde{Q}_1'(x)$ in $\mathbb{R}^n$, we have

$$Q_k(x) = \sum_{l=2}^{n} \alpha_{lk}(x) Q_l'(x), \quad k = 2, \ldots, j.$$

Similarly,

$$\widetilde{Q}_k(x) = \sum_{l=2}^{n} \widetilde{\alpha}_{lk}(x) \widetilde{Q}_l'(x), \quad k = 2, \ldots, j.$$

It is clear that $\alpha_{lk}, \widetilde{\alpha}_{lk} \in A$ and $\sum\limits_{l=2}^{n} \alpha_{lk} \overline{\widetilde{\alpha}}_{lk'} = \delta_{kk'}$, $k, k' = 2, \ldots, j$. Due to the inductive hypotheses, there exist functions $\alpha_{lk}, \widetilde{\alpha}_{lk} \in A$, $l = 2, \ldots, n$, $k = j+1, \ldots, n$, such that

$$\sum_{l=2}^{n} \alpha_{lk} \overline{\widetilde{\alpha}}_{lk'} = \delta_{kk'}, \quad k, k' = 2, \ldots, n. \tag{14}$$

Set

$$Q_k := \sum_{l=2}^{n} \alpha_{lk} Q_l', \quad \widetilde{Q}_k := \sum_{l=2}^{n} \widetilde{\alpha}_{lk} \widetilde{Q}_l', \quad k = j+1, \ldots, n.$$

Because of (14) and biorthogonality of the systems $Q_1^T, {Q_2'}^T, \ldots, {Q_n'}^T$ and $\overline{\widetilde{Q}}_1, \overline{\widetilde{Q}'_2}, \ldots, \overline{\widetilde{Q}'_n}$, we obtain

$$Q_l(x)^T \overline{\widetilde{Q}}_k(x) = \delta_{kl}, \quad k, l = 1, \ldots, n.$$

To complete the proof it remains to introduce matrixes $\mathcal{N}$ and $\widetilde{\mathcal{N}}$ whose columns are respectively $Q_1, \ldots, Q_n$ and $\widetilde{Q}_1, \ldots, \widetilde{Q}_n$. $\Diamond$

Now we are ready to give a necessary condition for VM^n property.

Theorem 8 *Let dual wavelet systems $\{\psi_{jk}^{(\nu)}\}$, $\{\widetilde{\psi}_{jk}^{(\nu)}\}$ with VM^n property be generated by refinable functions $\varphi, \widetilde{\varphi}$ whose Fourier transforms have derivatives up to order n at the origin, and let the entries $\mu_{\nu k}, \widetilde{\mu}_{\nu k}$ of the corresponding polyphase matrixes $\mathcal{M}, \widetilde{\mathcal{M}}$ be in $L_\infty^{(n)}$. Then there exist complex numbers*

$\lambda_\gamma, \widetilde{\lambda}_\gamma$, $\gamma \in \mathbb{Z}_+^d$, $[\gamma] \le n$, such that

$$\widetilde{\lambda}_{\mathbf{0}} = \lambda_{\mathbf{0}} = 1, \qquad \sum_{\mathbf{0}\le\gamma\le\alpha} \binom{\alpha}{\gamma} \lambda_\gamma \overline{\widetilde{\lambda}_{\alpha-\gamma}} = 0 \quad \forall \alpha \in \mathbb{Z}_+^d, 0 < [\alpha] \le n; \quad (15)$$

$$D^\beta \mu_{0k}(\mathbf{0}) = \frac{1}{\sqrt{m}} \sum_{\mathbf{0}\le\gamma\le\beta} \lambda_\gamma \binom{\beta}{\gamma} (-2\pi i M^{-1} s_k)^{\beta-\gamma} \quad \forall \beta \in \mathbb{Z}_+^d, [\beta] \le n, \quad (16)$$

$$D^\beta \widetilde{\mu}_{0k}(\mathbf{0}) = \frac{1}{\sqrt{m}} \sum_{\mathbf{0}\le\gamma\le\beta} \widetilde{\lambda}_\gamma \binom{\beta}{\gamma} (-2\pi i M^{-1} s_k)^{\beta-\gamma} \quad \forall \beta \in \mathbb{Z}_+^d, [\beta] \le n, \quad (17)$$

for $k = 0, \dots, m-1$.

`Proof.` Due to Lemmas 6, 7, the $(r+1)\times m$ matrixes $\mathcal{M}, \widetilde{\mathcal{M}}$ can be extended to $(r+1)\times(r+1)$ matrixes $\mathcal{N}, \widetilde{\mathcal{N}}$ such that their entries are in $L_\infty^{(n)}$ and $\mathcal{N}^T \overline{\widetilde{\mathcal{N}}} = I_{r+1}$. It follows from conditions (a) of Theorem 5 that there exist complex numbers $\lambda_\gamma, \widetilde{\lambda}_\gamma$, $\gamma \in \mathbb{Z}_+^d$, $[\gamma] \le n$, such that $\widetilde{\lambda}_{\mathbf{0}} = \lambda_{\mathbf{0}} = 1$ and (16), (17) is valid for $k = 0, \dots, m-1$. It remains to check (15).

Let $\alpha \in \mathbb{Z}_+^d$, $0 < [\alpha] \le n$, $k = 0, \dots, m-1$, $\rho := 2\pi M^{-1} s_k$. Due to (16), (17), we have

$$m D^\alpha \left(\mu_{0k}(x) \overline{\widetilde{\mu}_{0k}(x)} \right) \Big|_{x=\mathbf{0}} = \sum_{\mathbf{0}\le\beta\le\alpha} \binom{\alpha}{\beta} D^\beta \mu_{0k}(\mathbf{0}) \overline{D^{\alpha-\beta} \widetilde{\mu}_{0k}(\mathbf{0})} =$$

$$\sum_{\mathbf{0}\le\beta\le\alpha} \binom{\alpha}{\beta} \sum_{\mathbf{0}\le\gamma\le\beta} \binom{\beta}{\gamma} \lambda_{\beta-\gamma} (-i\rho)^\gamma \sum_{\mathbf{0}\le\delta\le\alpha-\beta} \binom{\alpha-\beta}{\delta} \overline{\widetilde{\lambda}_{\alpha-\beta-\delta}} (i\rho)^\delta =$$

$$\sum_{\mathbf{0}\le\gamma\le\alpha} \sum_{\gamma\le\beta\le\alpha} \sum_{\mathbf{0}\le\delta\le\alpha-\beta} \binom{\alpha}{\beta} \binom{\beta}{\gamma} \binom{\alpha-\beta}{\delta} (-\mathbf{1})^\gamma (i\rho)^{\gamma+\delta} \lambda_{\beta-\gamma} \overline{\widetilde{\lambda}_{\alpha-\beta-\delta}} =$$

$$\sum_{\mathbf{0}\le\gamma\le\alpha} \sum_{\mathbf{0}\le\delta\le\alpha-\gamma} \sum_{\gamma\le\beta\le\alpha-\delta} \binom{\alpha}{\beta} \binom{\beta}{\gamma} \binom{\alpha-\beta}{\delta} (-\mathbf{1})^\gamma (i\rho)^{\gamma+\delta} \lambda_{\beta-\gamma} \overline{\widetilde{\lambda}_{\alpha-\beta-\delta}} =$$

$$\sum_{\mathbf{0}\le\gamma\le\alpha} \sum_{\gamma\le\epsilon\le\alpha} \sum_{\gamma\le\beta\le\alpha-\epsilon+\gamma} \binom{\alpha}{\beta} \binom{\beta}{\gamma} \binom{\alpha-\beta}{\epsilon-\gamma} (-\mathbf{1})^\gamma (i\rho)^{\epsilon} \lambda_{\beta-\gamma} \overline{\widetilde{\lambda}_{\alpha-\beta-\epsilon+\gamma}} =$$

$$\sum_{\mathbf{0}\le\epsilon\le\alpha} (i\rho)^\epsilon \sum_{\mathbf{0}\le\gamma\le\epsilon} \sum_{\gamma\le\beta\le\alpha-\epsilon+\gamma} \binom{\alpha}{\beta} \binom{\beta}{\gamma} \binom{\alpha-\beta}{\epsilon-\gamma} (-\mathbf{1})^\gamma \lambda_{\beta-\gamma} \overline{\widetilde{\lambda}_{\alpha-\beta-\epsilon+\gamma}} =$$

$$\sum_{\mathbf{0}\le\epsilon\le\alpha} (i\rho)^\epsilon \sum_{\mathbf{0}\le\gamma\le\epsilon} \sum_{\mathbf{0}\le\kappa\le\alpha-\epsilon} \binom{\alpha}{\kappa+\gamma} \binom{\kappa+\gamma}{\gamma} \binom{\alpha-\kappa-\gamma}{\epsilon-\gamma} (-\mathbf{1})^\gamma \lambda_\kappa \overline{\widetilde{\lambda}_{\alpha-\kappa-\epsilon}} =$$

$$\sum_{\mathbf{0}\le\epsilon\le\alpha}(i\rho)^{\epsilon}\sum_{\mathbf{0}\le\kappa\le\alpha-\epsilon}\lambda_{\kappa}\overline{\widetilde{\lambda}_{\alpha-\kappa-\epsilon}}\sum_{\mathbf{0}\le\gamma\le\epsilon}\frac{\alpha!}{\kappa!\gamma!(\epsilon-\gamma)!(\alpha-\kappa-\epsilon)!}(-\mathbf{1})^{\gamma}=$$

$$\sum_{\mathbf{0}\le\epsilon\le\alpha}(i\rho)^{\epsilon}\sum_{\mathbf{0}\le\kappa\le\alpha-\epsilon}\lambda_{\kappa}\overline{\widetilde{\lambda}_{\alpha-\kappa-\epsilon}}\begin{pmatrix}\alpha\\ \kappa+\epsilon\end{pmatrix}\sum_{\mathbf{0}\le\gamma\le\epsilon}\begin{pmatrix}\epsilon\\ \gamma\end{pmatrix}(-\mathbf{1})^{\gamma}=$$

$$\sum_{\mathbf{0}\le\epsilon\le\alpha}(i\rho)^{\epsilon}\sum_{\mathbf{0}\le\kappa\le\alpha-\epsilon}\lambda_{\kappa}\overline{\widetilde{\lambda}_{\alpha-\kappa-\epsilon}}\begin{pmatrix}\alpha\\ \kappa+\epsilon\end{pmatrix}(\mathbf{1}-\mathbf{1})^{\epsilon}=\sum_{\mathbf{0}\le\kappa\le\alpha}\lambda_{\kappa}\overline{\widetilde{\lambda}_{\alpha-\kappa}}. \quad (18)$$

So, $D^{\alpha}\left(\mu_{0k}(x)\overline{\widetilde{\mu}_{0k}(x)}\right)\Big|_{x=\mathbf{0}}$ does not depend on k and

$$\sum_{\mathbf{0}\le\kappa\le\alpha}\lambda_{\kappa}\overline{\widetilde{\lambda}_{\alpha-\kappa}}=\sum_{k=0}^{m-1}D^{\alpha}\left(\mu_{0k}(x)\overline{\widetilde{\mu}_{0k}(x)}\right)\Big|_{x=\mathbf{0}}=D^{\alpha}\left(\sum_{k=0}^{m-1}\mu_{0k}(x)\overline{\widetilde{\mu}_{0k}(x)}\right)\Big|_{x=\mathbf{0}}.$$

Due to conditions (b) of Theorem 5,

$$D^{\gamma}\mu_{0k}(\mathbf{0})=0, D^{\gamma}\widetilde{\mu}_{0k}(\mathbf{0})=0,\ \ k=m,\dots,r,\quad \forall\gamma\in\mathbb{Z}_{+}^{d},[\gamma]\le n.$$

From this, taking into account that $\sum_{k=0}^{r}\mu_{0k}(x)\overline{\widetilde{\mu}_{0k}(x)}=1$, we obtain

$$D^{\alpha}\left(\sum_{k=0}^{m-1}\mu_{0k}(x)\overline{\widetilde{\mu}_{0k}(x)}\right)\Big|_{x=\mathbf{0}}=D^{\alpha}\left(\sum_{k=0}^{r}\mu_{0k}(x)\overline{\widetilde{\mu}_{0k}(x)}\right)\Big|_{x=\mathbf{0}}=0.\Diamond$$

Corollary 9 *Let $\mu_{0k},\widetilde{\mu}_{0k}\in L_{\infty}^{(n)}$, $\nu=0,\dots,m-1$. If there exist complex numbers $\lambda_{\gamma},\widetilde{\lambda}_{\gamma}$, $\gamma\in\mathbb{Z}_{+}^{d}$, $[\gamma]\le n$, such that (15), (16), (17) are fulfilled, then*

$$D^{\alpha}\left(\overline{\mu_{0k}(x)}\widetilde{\mu}_{0k}(x)\right)\Big|_{x=\mathbf{0}}=0,\quad k=0,\dots,m-1,$$

for all $\alpha\in\mathbb{Z}_{+}^{d}$, $0<[\alpha]\le n$.

The proof of Corollary 9 follows from (18).

It is not difficult to see that conditions (15)-(17) are not sufficient for refinable functions $\varphi,\widetilde{\varphi}$ to generate dual wavelet systems $\{\psi_{jk}^{(\nu)}\}$, $\{\widetilde{\psi}_{jk}^{(\nu)}\}$ with VM^{n} property. Really, let $d=1$, $M=m=2$, $s_0=0, s_1=1$, $\lambda_0=\widetilde{\lambda}_0=1$, $\lambda_1=\widetilde{\lambda}_1=0$, $\mu_{00}=\widetilde{\mu}_{00}\equiv\frac{1}{\sqrt{2}}$, $\quad\mu_{01}=\widetilde{\mu}_{01}=\frac{1}{\sqrt{2}}\left(1-\frac{i}{2}\sin 2\pi x\right)$. It is clear that (16), (17) hold for $n=1$, and (15) is fulfilled. Assume that dual wavelet systems with VM^1 property are generated by these functions. This means that polyphase matrixes $\mathcal{M}, \widetilde{\mathcal{M}}$ whose first rows are (μ_{00}, μ_{01}), $(\widetilde{\mu}_{00}, \widetilde{\mu}_{01})$ respectively satisfy (2). and such that (3) holds for the corresponding wavelet masks. Due to Lemmas 6, 7, the matrixes $\mathcal{M}, \widetilde{\mathcal{M}}$ can be extended to matrixes $\mathcal{N}, \widetilde{\mathcal{N}}$ such that their entries $\mu_{\nu k}$, $\widetilde{\mu}_{\nu k}$, $\nu, k = 0, \dots, r$, are in $L_\infty^{(1)}$ and $\mathcal{N}^T, \overline{\widetilde{\mathcal{N}}}$ are mutually inverse. It follows from Theorem 5 that

$$\frac{d^2}{dx^2}\left(\sum_{k=2}^{r} \mu_{0k}(x)\overline{\widetilde{\mu}_{0k}(x)}\right)\Bigg|_{x=0} = 0. \tag{19}$$

But

$$\frac{d^2}{dx^2}\left(\sum_{k=2}^{r} \mu_{0k}(x)\overline{\widetilde{\mu}_{0k}(x)}\right) = \frac{d^2}{dx^2}\left(1 - \sum_{k=0}^{1} \mu_{0k}(x)\overline{\widetilde{\mu}_{0k}(x)}\right) =$$

$$\frac{d^2}{dx^2}\left(1 - \frac{1}{2} - \frac{1}{2}\left(1 + \frac{1}{4}\sin^2 2\pi x\right)\right) = \frac{d^2}{dx^2}\left(-\frac{1}{8}\sin^2 2\pi x\right) = -\pi^2 \cos 4\pi x.$$

So, we see that (19) is not true.

A sufficient condition is given in the following statement.

Theorem 10 *Let $\varphi, \widetilde{\varphi}$ be refinable functions, their Fourier transforms $\widehat{\varphi}, \widehat{\widetilde{\varphi}}$ have derivatives up to order n at the origin, $\widehat{\varphi}_0(\mathbf{0}) = \widehat{\widetilde{\varphi}}_0(\mathbf{0}) = 1$, and let $\mu_{00}, \dots, \mu_{0,m-1}$, $\widetilde{\mu}_{00}, \dots, \widetilde{\mu}_{0,m-1} \in L_\infty^{(n)}$ be the polyphase representatives of their masks. If there exist complex numbers $\lambda_\gamma, \widetilde{\lambda}_\gamma$, $\gamma \in \mathbb{Z}_+^d$, $[\gamma] \le n$, such that $\lambda_{\mathbf{0}} = \widetilde{\lambda}_{\mathbf{0}} = 1$, (16), (17) holds for $k = 0, \dots, m-1$ and there exist functions μ_{0k}, $\widetilde{\mu}_{0k} \in L_\infty^{(n)}$, $k = m, \dots, r$, such that*

$$\sum_{k=0}^{r} \mu_{0k}\overline{\widetilde{\mu}_{0k}} = 1,$$

$$D^\beta \mu_{0k}(\mathbf{0}) = D^\beta \widetilde{\mu}_{0k}(\mathbf{0}) = 0, \ k = m, \dots, r, \quad \forall \beta \in \mathbb{Z}_+^d, [\beta] \le n.$$

then the functions $\varphi, \widetilde{\varphi}$ generate dual wavelet systems $\{\psi_{jk}^{(\nu)}\}$, $\{\widetilde{\psi}_{jk}^{(\nu)}\}$ with VM^n property.

`Proof.` Set $Q = (\mu_{00}, \dots, \mu_{0r})$, $\widetilde{Q} = (\widetilde{\mu}_{00}, \dots, \widetilde{\mu}_{0r})$. Due to Lemma 6, the $1 \times (r+1)$ matrixes $Q, \widetilde{Q}$ can be extended to $(r+1) \times (r+1)$ matrixes

$\mathcal{N} = \{\mu_{\nu k}\}_{\nu,k=0}^{r}, \widetilde{\mathcal{N}} = \{\overline{\mu}_{\nu k}\}_{\nu,k=0}^{r}$ such that their entries are in $L_\infty^{(n)}$ and $\mathcal{N}\overline{\widetilde{\mathcal{N}}^T} = I_{r+1}$. So, the matrixes

$$\mathcal{M} := \begin{pmatrix} \mu_{00} & \cdots & \mu_{0,m-1} \\ \vdots & \ddots & \vdots \\ \mu_{r,0} & \cdots & \mu_{r,m-1} \end{pmatrix}, \quad \widetilde{\mathcal{M}} := \begin{pmatrix} \widetilde{\mu}_{00} & \cdots & \widetilde{\mu}_{0,m-1} \\ \vdots & \ddots & \vdots \\ \widetilde{\mu}_{r,0} & \cdots & \widetilde{\mu}_{r,m-1} \end{pmatrix},$$

satisfy (2). It follows from Theorem 5 that the corresponding wavelet masks $m_1, \dots, m_{m-1}, \widetilde{m}_1, \dots, \widetilde{m}_{m-1}$ satisfy (3) for all $\alpha \in \mathbb{Z}^d$, $[\alpha] \le n$, what was to be proved. $\Diamond$

Applied mathematicians and engineers are especially interested in construction of compactly supported wavelet systems. To provide this property generating refinable functions should be compactly supported and wavelet masks should be trigonometric polynomials.

Theorem 11 *Let $\varphi, \widetilde{\varphi}$ be compactly supported refinable functions with polynomial masks, $\widehat{\varphi}_0(\mathbf{0}) = \widehat{\widetilde{\varphi}}_0(\mathbf{0}) = 1$, and let $\mu_{00}, \dots, \mu_{0,m-1}$, $\widetilde{\mu}_{00}, \dots, \widetilde{\mu}_{0,m-1}$ be the polyphase representatives of their masks. If there exist complex numbers $\lambda_\gamma, \widetilde{\lambda}_\gamma$, $\gamma \in \mathbb{Z}_+^d$, $[\gamma] \le n$, such that (16), (17) holds for $k = 0, \dots, m-1$ and there exist trigonometric polynomials μ_{0k}, $\widetilde{\mu}_{0k}$, $k = m, \dots, r$, such that*

$$\sum_{k=0}^{r} \mu_{0k}\overline{\widetilde{\mu}_{0k}} = 1,$$
$$D^\beta \mu_{0k}(\mathbf{0}) = D^\beta \widetilde{\mu}_{0k}(\mathbf{0}) = 0, \;\; k = m, \dots, r, \quad \forall \beta \in \mathbb{Z}_+^d, [\beta] \le n.$$

then the functions $\varphi, \widetilde{\varphi}$ generate dual compactly supported wavelet systems $\{\psi_{jk}^{(\nu)}\}$, $\{\widetilde{\psi}_{jk}^{(\nu)}\}$ with VM^n property.

`Proof.` Set $Q = (\mu_{00}, \dots, \mu_{0r})$, $\widetilde{Q} = (\widetilde{\mu}_{00}, \dots, \widetilde{\mu}_{0r})$. Due to Suslin's solution of a generalized Serre conjecture[12], the row Q can be extended to a unimodular matrix with polynomial entries. After this it is not difficult to find $(r+1) \times (r+1)$ matrices $\mathcal{N}, \widetilde{\mathcal{N}}$ extending $Q, \widetilde{Q}$. such that their entries are trigonometric polynomials and $\mathcal{N}\overline{\widetilde{\mathcal{N}}^T} = I_{r+1}$ (see [10], [11], [3, §2.6]). Next we repeat the arguments of the previous proof. $\Diamond$

Finally, we will describe a method for construction compactly supported wavelet frames with VM^n property.

Step 1. Given $n \in \mathbb{Z}^d$ and given set of parameters $\lambda_\beta \in \mathbb{C}$, $\beta \in \mathbb{Z}_+^d$, $[\beta] \le n$, $\lambda_{\mathbf{0}} = 1$, find a dual set of parameters $\widetilde{\lambda}_\beta \in \mathbb{C}$ satisfying (15) by the following recursive formulas

$$\widetilde{\lambda}_{\mathbf{0}} = 1, \quad \widetilde{\lambda}_\alpha = -\overline{\lambda_\alpha} - \sum_{\mathbf{0}<\beta\le\alpha} \binom{\alpha}{\beta} \overline{\lambda_\beta} \widetilde{\lambda}_{\alpha-\beta}.$$

Step 2. Chose functions $\mu'_{00}, \dots, \mu'_{0\,m-1}$ and $\widetilde{\mu}_{00}, \dots, \widetilde{\mu}_{0\,m-1}$ defined by

$$\mu'_{0k}(x) = \frac{1}{\sqrt{m}} \sum_{[\alpha]\le n} g_\alpha(x) \sum_{\mathbf{0}\le\beta\le\alpha} \binom{\alpha}{\beta} \left(-2\pi i M^{-1} s_k)\right)^\beta \lambda_{\alpha-\beta} +$$

$$\sum_{[\alpha]=n+1} T_\alpha(x) \prod_{j=1}^{\alpha_j} \left(1 - e^{2\pi i x_j}\right)^{\alpha_j},$$

$$\widetilde{\mu}_{0k}(x) = \frac{1}{\sqrt{m}} \sum_{[\alpha]\le n} g_\alpha(x) \sum_{\mathbf{0}\le\beta\le\alpha} \binom{\alpha}{\beta} \left(-2\pi i M^{-1} s_k)\right)^\beta \widetilde{\lambda}_{\alpha-\beta} +$$

$$\sum_{[\alpha]=n+1} \widetilde{T}_\alpha(x) \prod_{j=1}^{\alpha_j} \left(1 - e^{2\pi i x_j}\right)^{\alpha_j},$$

where $T_\alpha, \widetilde{T}_\alpha$ are arbitrary trigonometric polynomials, g_α are trigonometric polynomials such that $D^\alpha g_\alpha(\mathbf{0}) = 1$, $D^\beta g_\alpha(\mathbf{0}) = 0$ for all $\beta \in \mathbb{Z}_+^d$, $\beta \ne \alpha$, $[\beta] \le n$ (recursive formulas for computing g_α are given in [9]). It is clear that (17) are fulfilled

Step 3. Set $\sigma := \sum_{l=0}^{m-1} \overline{\mu'_{0l}}\, \widetilde{\mu}_{0l}$, $\mu_{0k} := (2-\sigma)\mu'_{0k}, k = 0, \dots, m-1$.

Due to Corollary 9, we have $D^\beta \sigma(\mathbf{0}) = 0$ for all $\beta \in \mathbb{Z}_+^d$, $0 < [\beta] \le n$. It follows that $D^\beta \mu_{0k}(\mathbf{0}) = D^\beta \mu'_{0k}(\mathbf{0})$ for all $\beta \in \mathbb{Z}_+^d$, $[\beta] \le n$. It is not difficult to see that (16) holds and

$$1 - \sum_{k=0}^{m-1} \overline{\mu_{0k}(x)} \widetilde{\mu}_{0k} = (1-\sigma)^2.$$

Set $\mu_{0m} := 1 - \sigma, \widetilde{\mu}_{0m} := 1 - \overline{1-\sigma}$.

Step 4. Find matrixes

$$\mathcal{M} = \begin{pmatrix} \mu_{00} & \dots & \mu_{0,m-1} & \mu_{0,m} \\ \mu_{10} & \dots & \mu_{1,m-1} & * \\ \dots & \dots & \dots & \dots \\ \mu_{m,0} & \dots & \mu_{m,m-1} & * \end{pmatrix}, \quad \widetilde{\mathcal{M}} = \begin{pmatrix} \widetilde{\mu}_{00} & \dots & \widetilde{\mu}_{0,m-1} & \widetilde{\mu}_{0,m} \\ \widetilde{\mu}_{10} & \dots & \widetilde{\mu}_{1,m-1} & * \\ \dots & \dots & \dots & \dots \\ \widetilde{\mu}_{m,0} & \dots & \widetilde{\mu}_{m,m-1} & * \end{pmatrix}$$

such that their entries are trigonometric polynomials and $\mathcal{M}\overline{\widetilde{\mathcal{M}}^T} = I_{m+1}$.

Though matrixes $\mathcal{M}, \widetilde{\mathcal{M}}$ can be constructed theoretically (see the proof of Theorem 11), it is very complicate to implement the algorithm in practice. Instead, we suggest the following explicit way (the payment of simplicity of this way is increasing of the redundancy of the frames).

Set $\mu_{0,m+1} := 0, \widetilde{\mu}_{0,m+1} := 0$. For each $\nu = 1, \dots, m+1$, define

$$\widetilde{\mu}_{\nu,m+1} := \overline{\mu_{0,m+1-\nu}}, \quad \mu_{\nu,m+1} := \overline{\widetilde{\mu}_{0,m+1-\nu}},$$
$$\mu_{\nu k} := \delta_{m+1-\nu,k} - \mu_{0k}\overline{\widetilde{\mu}_{0,m+1-\nu}}, \quad \widetilde{\mu}_{\nu k} := \delta_{m+1-\nu,k} - \widetilde{\mu}_{0k}\overline{\mu_{0,m+1-\nu}}, \quad k = 0, \dots, m.$$

It is not difficult to check that the matrixes $\mathcal{M} := \{\mu_{\nu k}\}_{\nu,k=0}^{m+1}$, $\widetilde{\mathcal{M}} := \{\widetilde{\mu}_{\nu k}\}_{\nu,k=0}^{m+1}$ satisfy $\mathcal{M}\overline{\widetilde{\mathcal{M}}^T} = I_{m+2}$.

References

[1] Ingrid Daubechies, Bin Han, Amos Ron, and Zuowei Shen, Framelets: MRA-based constructions of wavelet frames, ACHA, Vol. 14 (2003), No. 1, 1–46.

[2] Ming-Jun Lai and Petukhov A. *Method of Virtual Components for Constructing Redundant Filter and Wavelet Frames* (to appear).

[3] Novikov I., Protassov V., Skopina M. *Wavelet Theory.* Moscow: Fizmatlit.

[4] Ron A. and Shen Z., *Affine systems in $\mathbb{L}_2(R^d)$: the analysis of the analysis operator*, J. Func. Anal. **148** (1997), 408–447.

[5] Bin Han. *On dual wavelet tight frames* ACHA. 1997. V. 4. P. 380-413.

[6] Hernandes E. and Weis G.A. *A first cours of wavalets* CRC Press, Boca Raton, FL. 1996.

[7] Jia R.Q. *Approximation properties of multivariate wavelets*, Math. Comp. **67** (1998), 647-655.

[8] Möller H.M. and Sauer T. *Multivariate refinable functions of high approximation order via quotient ideals of Laurent polynomials*, Adv. Comput. Math. **20** (2004), No.1-3, 205-228.

[9] Skopina M. *On Construction of Multivariate Wavelets with Vanishing Moments*, ACHA, (to appear).

[10] Riemenschneider S.D. and Shen Z.W. *Construction of compactly supported biorthogonal wavelets in* $L_2(\mathbb{R}^s)$, Preprint, 1997.

[11] Ji H., Riemenschneider S.D. and Shen Z. *Multivariate compactly supported fundamental refinable functions, dual and biorthogonal wavelets*, Studies of Applied Mathematics **102** (1999), 173-204.

[12] Suslin A. *The structure of the special linear group over rings of polynomials*, Izv. Akad. Nauk SSSR. Ser. Mat. **41** (1977), No 2, 235-252 (in Russian)

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