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OPTIMAL CHOICE

Reducing the Pareto Set Based on Set-Point Information

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1 **Abstract**—An axiomatic approach to reducing the Pareto set in the problem of multicriteria choice based on an information set obtained from the decision maker is considered. The information reflects the degree up to which the decision maker is ready to lose values of one group of criteria to improve values of several other groups of criteria. It is proved that using this information one can eliminate a whole number of elements from the initial Pareto set, simplifying the subsequent choice.

1 *Keywords*: multicriteria choice, reducing the Pareto set, axiomatic approach

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INTRODUCTION

1 Multicriteria problems are common in many technological and economic fields. They form a vast class that is also interesting in terms of theory. In [3], we classified various methods for solving the problems
2 mentioned. The main challenge here is both to axiomatize the very concept of the "best" decision and reasonably use information available in the problem when searching for the "best" decision.

We started developing the so-called axiomatic approach designed to solve problems with numerical criteria as early as the beginning of the 1980s. The approach implies abandoning the formal definition of the concept of the "best" decision and postulating certain axioms of "smart" behavior of the decision maker that help him to use incomplete fragmented information on his preferences to construct upper bounds for the unknown set of chosen decisions [1, 2].

A pair of criteria and two numbers specified by the decision maker is the simplest information on his preferences, with the first number showing the trade-off for one criterion the decision maker can accept to win in terms of another criterion by the value equivalent to the second number. This information is generally easy to elicit and when taken into account based on special rules it helps to reduce the number of feasible alternatives for the "best" decision, which simplifies the final choice.

A more general situation occurs when two groups of criteria and two sets of numbers rather than two criteria and two numbers are considered. In this case, the information obtained from the decision maker characterizes the degree up to which the decision maker is ready to be flexible (to trade-off) when comparing two groups of criteria rather than just two criteria. Here, we also developed the rules [2] for this information to be taken into account.

Finally, in the most general case, there can be several pairs of groups of criteria of a similar type. This means the set of such information on the decision maker's preferences needs to be taken into account. Unfortunately, no rules have been proposed so far for an arbitrary finite set of the stated information to be taken into account (unless the set of initial alternatives is supposed to be finite a priori). 3

In this work, we assume the finite set of information on preferences showing whether the decision maker is ready to trade off when he compares alternatives so that to lose in terms of one group of criteria and to win in terms of some other fixed groups of criteria to be given. This is somewhat symmetric to what we considered in [4] dealing with point-set information.

Being a further move in developing the axiomatic approach, the results we obtain here are the rules for the given set of information to be taken into account. To apply these rules, one needs to form a new vector criterion by certain formulas and then remove the initial alternatives that are out of the Pareto set in terms of the new vector criterion.

1. STATEMENT OF THE PROBLEM. SMART CHOICE AXIOMS

We consider the *multicriteria choice problem* 1
(model) $\langle X, f, \succ_X \rangle$ stated as follows. Given are

X, viz., the set of feasible alternatives (decisions) to be chosen from; this is a non-empty set of arbitrary nature, $f = (f_1, \ldots, f_m)$ $m \geq 2$, viz., the vector criterion given on the set X and taking numerical values in the arithmetic vector space R^m,

$\succ_X$, viz., the asymmetric binary relationship of strict preference of the decision maker given on the set X. Writing $x_1 \succ_X x_2$ for $x_1, x_2 \in X$, we mean that the alternative x_1 is preferable to the alternative x_2 for the decision maker; in other words, choosing from these two alternatives, the decision maker chooses the first one rather than the second one.

The subset of the set of feasible alternatives X called *the set of chosen alternatives* and denoted by $C(X)$ is the
1 solution to the multicriteria choice problem. In the particular case, this set can consist of one element.

In what follows, we also use the set of feasible vectors (outcoms) $Y = f(X) \subset R^m$ and the set of chosen vectors $C(Y) = f(C(X))$. We assume the strict preference relationship $\succ_Y$ (which is generally unknown in practice) to be given on the set of feasible vectors Y; it is matched to the relationship $\succ_X$ as follows

$$x1 \succ_X x_2 \Leftrightarrow f(x_1) \succ_Y f(x_2)$$

$$\text{for all } x_1 \in \tilde{x}_1,\ x_2 \in \tilde{x}_1;\ \tilde{x}_1, \tilde{x}_2 \in \tilde{X},$$

where $\tilde{X}$ is the family of equivalence classes generated by the equivalence relationship $x_1 \sim x_2 \Leftrightarrow f(x_1) = f(x_2)$ on the set X.

1 In terms of outcomes, the multicriteria choice model $\langle Y, \succ_Y \rangle$ includes the set of feasible vectors Y and the strict preference relationship $\succ_Y$ given on the set Y, with the set of chosen vectors $C(Y)$ being the solution
1 to the multicriteria choice problem. The choice problems in terms of alternatives and in terms of outcomes are obviously connected, which makes it feasible to restate every statement made in one set of terms in other terms.

We define the set of Pareto optimal vectors

$$P_f(X) = \{x^* \in Y) | \text{there is no } x \in Y \text{ such that } y \geq y^*\}$$

and the set of Pareto optimal alternatives

$$P_f(X) = \{x^* \in Y) | \text{there is no } x \in X \text{ such that } f(x) \geq f(y^*)\}$$

Here, $y' \geq y''$ means that each component of the first vector is greater than or equivalent to the respective component of the second vector, with $y' \neq y''$, i.e., at least one component of the first vector is strictly greater than the respective component of the second vector.

We recall "smart" requirements on the preference relations and the set of chosen vectors stated previously [2].

Axiom 1 (axiom on exclusion of dominated vectors). For any pair of vectors $y', y'' \in Y$ that satisfy the relationship $y' \succ_Y y''$, $y'' \notin C(Y)$ holds.

Axiom 1 means that any vector that is not chosen in the pair should not be chosen from the entire set of vectors either.

Axiom 2 (axiom on existence of transitive extension). For the relationship $\succ_Y$, there exists the irreflexive and transitive extension on the Cartesian product $\hat{Y} = f_1(X) \times \ldots \times f_m(X) \subset R^m$ denoted by $\succ$ in what follows.

By axiom 2, the relationship $\succ_Y$ is a restriction of the relationship $\succ$ to the set Y and is irreflexive and transitive (and, hence, asymmetric). This axiom means that the decision maker can compare any (not only feasible) vector bounds, behaving in a "smart" (i.e., transitive) way. The indicated extension can turn out to be non-unique; by axiom 2, it does not matter what extension to choose.

Axiom 3 (matching axiom). Each criterion $f_1, f_2, \ldots, f_m$ is matched with the preference relationship $\succ$.

We remember that the criterion f_i is called matched with the preference relationship $\succ$ if $y', y'' \in \hat{Y}$ holds for any two vectors $y', y'' \in \hat{Y}$ such that

$$y' = (y'_1, \ldots, y'_{i-1}, y'_i, y'_{i+1}, \ldots, y'_m),$$

$$y'' = (y'_1, \ldots, y'_{i-1}, y''_i, y'_{i+1}, \ldots y'_m), y'_i > y''_i,$$

In other words, the decision maker is interested in increasing the value of the criterion if this criterion is matched with the preference relationship.

Edgeworth–Pareto principle [1, 2]. Let axioms 1–3 hold. Then, the inclusion $C(Y) \subset P(Y)$ holds for any set of chosen vectors $C(Y)$.

This fundamental principle assumes that the decision maker relying on axioms 1–3 should search for the "best" alternatives only within the Pareto set. In other words, the Pareto set is an upper bound for the unknown set of chosen vectors. This bound is proved to be accurate and if we abandon at least one of the above given axioms the chosen ("best") vector may be out of the Pareto set. It is worth adding that the Pareto principle holds for non-transitive relations as well when only the exclusion axiom and Pareto axiom are assumed to hold.

Applied problems frequently involve the set of information of the following type: the vectors $u^i, v^i \in R^m$ are given (these are generally found directly from the decision maker when he is asked about his preferences) and not compared by the relationship $\geq$ yet satisfy the conditions $u^i \succ v^i$, $i = 1, \ldots, k$. Such information evokes two questions, viz., whether it is consistent and if it is, how one can use it to reduce the Pareto set.

As for consistency, this issue was resolved completely in [2], where the strict definition for a consistent set was given and consistency criteria were obtained that can be used to solve particular problems. The second question is still waiting for its final answer. In the general case, only certain sets of vectors are determined such that the respective upper bounds are found for the unknown set C as a Pareto set with the new vector criterion [2].

Below, we state propositions that are further steps in this direction. For the Pareto set to be more or less significantly reduced, taking into account the relationships $u^i \succ v^i$, we add another axiom to the ones stated above.

Axiom 4 (invariance axiom). The preference relationship $\succ$ is invariant with respect to a linear positive transformation that is:

homogeneous, i.e., if $u^i \succ v^i$ is met, $\lambda u^i \succ \lambda v^i$ is always met for any positive λ, additive, i.e., if $u^i \succ v^i$ is met, $(u^i + c) \succ (v^i + c)$ is always met for any vector $c \in R^m$.

Note that the relationship $u^i \succ v^i$ in the hypotheses of axiom 4 is equipotent to $u^i - v^i \succ 0$, where 0_m stands for the zero vector (0, 0, …, 0). Then, the vector v_i in the relationship $u^i \succ v^i$ can be also considered to be zero, which is assumed below. However, it would be fair to say that it is sometimes more convenient to assume that the vector v^i consists of unities only when performing comparisons. Such an assumption can be also always considered true if needed due to invariance. One can make similar remarks regarding the vector u^i.

Since the vectors u_i and 0_m are not compatible with respect to the relationship ≥, there will always be at least one strictly positive and at least one strictly negative component in the vector u^i. The vector u^i can also have zero components. Thus, if the relationship $u^i \succ 0_m$ holds, the vector u^i is "better" (i.e., greater) than the zero vector 0_m with respect to some components while it will be necessarily "worse" (i.e., less) than the zero vector with respect to other components. The relationship u^i met can be interpreted as the decision maker being able to trade off, i.e., to sacrifice with respect to some criteria in order to get more with respect to other criteria (with values for the rest of criteria preserved).

2. TAKING SET-POINT INFORMATION INTO ACCOUNT

First, we consider the case when, according to the available information on preferences, the decision maker is ready to sacrifice the values in the third group C to increase the values of criteria in the two groups A and B. According to the following theorem, such information is always consistent; and to take it into account, one needs to form a vector criterion by replacing all components from the group C in the "old" vector criterion by "new" components calculated by certain formulas. This can only increase the total number of components in the newly formed vector criterion as compared to the dimension of the "old" vector criterion. Then, one needs to construct the Pareto set for the formed vector criterion. This set yields the sought upper bound for the unknown set of chosen vectors, i.e., all vectors outside it should not be chosen since it is not consistent with the information available on the decision maker's preferences.

Theorem 1. Suppose axioms 1–4 hold, three pairwise non-overlapping subsets of numbers of criteria A, B, $V \subset I$ are given and information is available that $y' \succ 0_m$ and y', where the vectors y'' and y'' have the following components

$$y'_i = w'_i \text{ for all } i \in A;\ y'_k = -w'_k \text{ for all } k \in C$$

$$y'_s = 0 \text{ for all } s \in I \backslash (A \cup C);$$

$$y''_j = w''_j \text{ for all } j \in B;\ y''_k = -w''_k \text{ for all } k \in C$$

$$y''_s = 0 \text{ for all } s \in I \backslash (B \cup C)$$

and all w'_i, w'_k, w''_j, w''_k are fixed positive values.

Then, the given information is consistent, and for any set of chosen vectors $C(Y)$, the inclusions hold

$$C(Y) \subset \hat{P}(Y) \subset P(Y), \quad (1)$$

where $\hat{P}(Y) = f(P_g(X))$ is the subset of feasible vectors that correspond to the set of Pareto optimal alternatives in the multicriteria problem with the set of feasi- 1
ble alternatives X and new $(m - |C| + |A| \cdot |B| \cdot |C|)$-dimensional vector criterion g with the components

$$g_{ijk} = w'_i w''_j f_k + w''_j w'_k f_i + w'_i w''_k f_j$$
$$\text{for all } i \in A, j \in B, k \in C; \quad (2)$$
$$g_s = f_s \text{ for all } s \in I \backslash C.$$

By theorem 1, to take into account the set of information on the decision maker's preferences obtained as two relationships $y' \succ 0_m$ and $y'' \succ 0_m$ and reduce the Pareto set, one needs to form the new vector criterion g by formulas given in the theorem hypotheses and then construct the Pareto set $P_g(X)$ for the formed criterion. The set of vectors corresponding to the new Pareto set, i.e., $\hat{P}(Y) = f(P_g(X))$, will be the sought upper bound for the set of chosen vectors. Note that theorem 1 is universal, holding for any criteria f and arbitrary sets of feasible decisions X without any restrictions since no requirements are imposed on the objects given in its statement.

Following the scheme of proof of theorem 1, we can use a similar line of reasoning to prove a more general proposition, when, for the sake of increasing the values with respect to criteria of the group A, the decision maker can afford to decrease the criteria of the entire set of groups B, rather than the criteria of three groups. The following result holds.

Theorem 2. Suppose axioms 1–4 hold and the finite set of $k + 1$ pairwise non-overlapping subsets of numbers of criteria $A_1, A_2, \ldots, A_k, B \subset I$ is given and information is available that $y^s \succ 0_m$, $s = 1, \ldots, k$ where the vectors y^s (for $s = 1, \ldots, k$) have the components

$$y^s_i = w^s_i \text{ for all } i \in A_s;$$

$$y^s_j = w^s_j \text{ for all } i \in B;$$

$$y^s_i = 0 \text{ for all } t \in I \backslash (A_1 \cup \ldots \cup A_s \cup B),$$

and all w^s_i and w^s_j are fixed positive values.

Then, the given set-point information is consistent, and inclusions (1) hold for any set of chosen vectors $C(Y)$, where $\hat{P}(Y) = f(P_g(X))$ is the subset of feasible vectors corresponding to the set of Pareto optimal

1 alternatives in the multicriteria problem with the set of feasible alternatives X and the new $(m - |B| + |B| \cdot \prod_{s=1}^{k} |A_s|)$-dimensional vector criterion g with the components

$$g^s_{ji_1\ldots i_k} = w^s_{i_1} \cdot \ldots \cdot w^s_{i_s} f_j + w^s_j \cdot w^s_{i_2} \cdot \ldots \cdot w^s_{i_k} f_{i_1}$$

$$+ \ldots + w^s_{i_1} \cdot \ldots \cdot w^s_{i_{k-1}} \cdot w^s_j f_{i_k} \text{ for all } i_s \in A_s, j \in B,$$

$$s = 1, \ldots, k, g_s = f_s \text{ for all } s \in I \backslash B.$$

In the particular case, $|A_s| = |B| = 1$, $s = 1, 2$; $k = 2$, theorem 2 becomes theorem 4.4 from [3].

CONCLUSIONS

In this work, we develop the axiomatic approach to solving the problem of reducing the Pareto set based on the set of information on the preference relationship of the decision maker; we have been developing this approach for the last 30 years [3]. What makes this approach stand out is that it adopts four "smart" axioms that regulate the behavior of the decision maker when he makes his choice and allow using the well-elaborated apparatus of convex analysis. No restrictions are imposed on the set of feasible alternatives on which the choice is made and on the vector criterion.

Theorem 1 and its more general version, theorem 2, is the principal result of this work, showing how we should "reconstruct" the initial vector criterion so that the Pareto set for the new criterion served as the upper bound for any unknown set of chosen bounds after the decision maker provides the information on his set-point preferences. This new Pareto set is a part of the "old" one; the difference is formed by those Pareto optimal bounds we manage to eliminate from consideration based on the indicated information.

APPENDIX

The proof of theorem 1 consists of four stages.

(1) First, we prove that the available information is consistent. By theorem 4.7 [2], this information is consistent if and only if the system of linear algebraic equations

$$\sum_{i=1}^{m} \lambda_i e^i + \lambda_{m+1} y' + \lambda_{m+2} y'' = 0_m \qquad (7)$$

with respect to $\lambda_1, \ldots, \lambda_{m+2}$ does not have any nonzero non-negative solutions. Here, e^i stands for an m-dimensional vector, with its i-th component being unity and all others being zero. In the unfolded form, system (7) is

$$\lambda_i e^i + \lambda_{m+1} w'_i = 0 \text{ for all } i \in A,$$

$$\lambda_j e^i + \lambda_{m+2} w''_j = 0 \text{ for all } i \in B,$$

$$\lambda_k e^i - \lambda_{m+1} w'_k - \lambda_{m+2} w''_k = 0 \text{ for all } k \in C,$$

$$\lambda_l e^l = 0 \text{ for all } l \in I \backslash (A \cup B \cup C).$$

Since λ_i, λ_j, λ_{m+1}, λ_{m+2} are non-negative, the equations written in the first and second rows lead to the equalities $\lambda_i = \lambda_{m+1} = \lambda_{m+2} = 0$ for all $i \in A$. In this case, the rest of the equations yield the equalities $\lambda_j = 0$ for all other numbers $j \in I \backslash (A \cup B)$. Hence, the only non-negative solution of system (7) is zero. This proves that information $y' \succ 0_m$, $y'' \succ 0_m$ is consistent.

(2) We use K to denote the sharp convex cone (without zero) of the cone relationship $\succ$. The fact that K is sharp and convex follows from axioms 2–4 and corollary 2.1 [2].

We use M to denote the sharp convex cone (without zero) generated by the vectors $e^1, \ldots, e^m, y'y''$, and y. The cone M is sharp by theorem 4.6 [2] and the above-proved consistency of the given information.

There can be four cases: $|A| > 1$ and $|B| > 1$; $|A| = 1$ and $|B| > 1$; $|A| > 1$and $|B| = 1$; and $|A| = |B| = 1$. In the first case, all vectors $e^1, e^2, \ldots, e^m, y', y''$ are generators of the cone M, since each of them cannot be represented as a linear non-negative combination of the rest of the vectors of this set (since the cone M they generate is sharp). In the second case (i.e., when $A = \{i\}$), obviously the vector e^i can be represented as a linear positive combination of the vector y' and all vectors e^s for $s \in B$. Hence, the vectors $e_1, e_2, \ldots, e^m, y'$, and y'' without the vector e^i are the cone generators in the second case. Similarly, the vectors $e^1, e^2, \ldots, e^m, y'$, and y'' without the vector e^j are generators of the cone M in the third case, whereas we should eliminate the vectors e^i and e^j from the generators simultaneously in the fourth case.

We start from the first case. We introduce a cone (without zero) dual to the polyhedral cone M

$$C = \{y \in R^m | \langle u, y \rangle \geqq 0 \text{ for all } u \in M\} \backslash \{0_m\}.$$

By the duality theory of convex analysis ([5], p. 175), internal normals to $(m-1)$-dimensional faces of the cone M serve as generators of the polyhedral cone C. Vice versa, internal normals to $(m-1)$-dimensional faces of the cone C serve as generators of the cone M.

Since the vectors $e^1, e^2, \ldots, e^m, y'$, and y'' are generators of the cone M, the set of nonzero solutions of the system of linear inequalities

$$\langle e^i, y \rangle \geqq 0 \text{ for all } i \in I \quad \langle y', y \rangle \geqq 0, \quad \langle y'', y \rangle \geqq 0 \quad (8)$$

coincides with the dual cone C.

(3) We find the fundamental totality of solutions of system of linear inequalities (8). This should be a minimal (with respect to the number) system of vectors, such that the set of its linear non-negative combinations coincides exactly with the set of solutions of sys-

tem (8). No vector of the fundamental totality can be represented as a linear non-negative combination of other vectors of this totality.

We give a set of solutions of system of linear inequalities (8). One can easily verify that the vectors $e^{ijk} = w_j'' w_k' e^i + w_i' w_k'' e^j + w_j'' w_i' e^k$ for all $i \in A, j \in B, k \in C$ satisfy system (8), with inequalities from the second and third rows for these vectors met as equalities.

Thus, the set consisting of vectors e^s for all $s \in I \backslash C$, vectors e^{ijk} for all $i \in A, j \in B, k \in C$ (we denote this set by (*)) belongs to the dual cone C. One can easily see that no vector of this totality can be represented as a linear non-negative combination of other vectors. The total number p of all vectors of the set is $p = m - |C| + |A| \cdot |B| \cdot |C|$.

To verify that this set of vectors forms a fundamental totality of solutions of system (8), we still need to prove that the system of linear inequalities (8) has no other (up to a positive factor) solutions but all possible linear non-negative combinations of vectors of the set (*) given above. To do this, along with system (8), we consider its respective system of $m + 2$ linear equations

$$\langle e^i, y \rangle = 0 \text{ for all } i \in I \quad \langle y', y \rangle = 0, \quad \langle y'', y \rangle = 0 \quad (9)$$

We can calculate the ranks of the respective matrices to verify that any subsystem of $m - 1$ vectors of the system $e^1, e^2, \ldots, e^m, y', y''$ is linearly independent. Hence, the sought fundamental totality of solutions of system of linear inequalities (8) is among (one-dimensional) non-zero solutions of subsystems of $m - 1$ equations of system of linear equations (9).

We eliminate three equations from system (9) at a time and write the solutions of the resulting subsystems that also satisfy system of inequalities (8). Vectors found in such a way form the fundamental totality of solutions of system of inequalities (8).

When we remove the last two equations of system (9), the unit basis vectors $e^1, e^2, \ldots, e^m$ serve as nonzero solutions of the resulting subsystems (up to the accuracy of positive factor). However, one can easily see that only the vectors e^s from this set, such that $s \in I \backslash C$, satisfy the system of inequalities (8).

If the last equation of the system of linear equations (9) is not removed, whereas the last but one is, the vectors that do not satisfy system of inequalities (8) serve as non-zero solutions of the resulting ("reduced") subsystems (up to the accuracy of a positive factor).

Similarly, if the last but one equation from (9) is left and the last one is removed, there will be no vectors that satisfy system of inequalities (8) among the solutions of the "reduced" subsystems either.

When both of the last equations are left in the "reduced" subsystem, we arrive at all possible solutions of the form e^{ijk} for all $i \in A, j \in B, k \in C$.

Since we have considered all possible variants of removing triples of equations from system of linear equations (9), there are no other (up to the accuracy of the positive factor) solutions of subsystems consisting of $m - 2$ equations of system (9) that satisfy system of linear inequalities (8). There are no other one-dimensional solutions of system (9) that satisfy system of inequalities (8). This means that the system of vectors (*) forms the fundamental totality of solutions of system of linear inequalities (8). Hence, any solution of system of inequalities (8) can be represented as a non-negative linear combination of this totality of vectors. In what follows, for the sake of convenience, we denote this totality as $a^1, a^2, \ldots, a^p$.

If $|A| = 1$ (i.e., $A = \{i\}$), the line of reasoning is similar yet slightly simpler than those given above. In this case, we should consider the system of $m + 1$ equations that differs from (9) by the fact it has no equation $\langle e^i, y \rangle = 0$. Here, we should remove only two equations at a time from the system to obtain the same fundamental totality of solutions of system of linear inequalities (8). Similar reasoning is applicable to all other cases. We will not discuss them.

(4) The final stage of the proof is similar to [4], taking into account the end of the proof of theorem 3.5 [2].

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SPELL: 1. Multicriteria, 2. axiomatize, 3. priori