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Existence and uniqueness of optimal dynamic pricing and advertising controls without concavity

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ABSTRACT

We consider a pricing and advertising dynamic-optimization problem where the goodwill dynamics evolve à la Nerlove–Arrow. The firm maximizes its profit over a finite-planning horizon corresponding to the product's lifespan, and it turns out that the Hamiltonian is non-concave. We show the existence and uniqueness of an optimal solution under some mild conditions.

1. Introduction

An investment in advertising has a long-lasting effect, that is, it not only boosts the firm's demand and revenues today but also its future ones. The main reason for this is that advertising is a main input in building the brand's reputation, which in turn, is a significant driver of current and future sales. This simple observation in itself explains the existence of an extensive literature on *dynamic* advertising models and decisions, which started more than five decades ago and is still ongoing.

In this paper, we consider a continuous-time dynamic-optimization model à la Nerlove–Arrow and determine optimal pricing and advertising decisions. The cornerstone piece in a Nerlove–Arrow (N–A) model is the goodwill stock, a variable that positively affects sales. The firm can raise its goodwill stock, which is also referred to in the literature and in practice as brand reputation or brand equity, by investing in advertising, while part of this goodwill is lost as a result of consumers forgetting of the advertisement messages. In the parlance of dynamic optimization, the goodwill is a state variable that summarizes, in a compact way, the firm's current and past advertising outlays on its sales, which can also depend on other decision variables such as price and quality.

Our contribution to the Nerlove–Arrow class of advertising models is threefold. First, in any monopoly-profit-maximization model, one expects the price to be a decision variable, not a given parameter. Indeed, there is no valid conceptual reason to assume both a monopolistic environment and an exogenously given price during the whole planning horizon. By letting the price be a control variable, we add some realism to the stream of literature that only considered advertising. Second, as any product has a finite lifespan, we believe that the model must also have a finite terminal date. This reasoning especially holds when one assumes away any quality improvements over time for the product, which has been the norm rather than the exception in the dynamic advertising literature (see [13] and [42] for a discussion). The advantage of an infinite planning horizon resides in the fact that the optimal (or equilibrium in a competitive model) solution is stationary, which is typically easier to compute than a time-varying solution. Here, we stick to a finite horizon, and by the same token, provide insights into the firm's advertising and pricing trajectories in a more realistic context, and beyond the steady-state values that are often the focal point of the analysis in infinite-horizon models. The third contribution concerns the existence and uniqueness of an optimal solution. In our case, it turns out that the Hamiltonian function corresponding to the profit-maximization problem is not concave. Consequently, the usual sufficient conditions of optimality cannot be applied. We show, under some mild conditions, that the non-oscillating interior pricing and advertising solution is indeed optimal.

We shall refrain from extensively reviewing the literature and refer the reader to the comprehensive surveys in Feichtinger et al. [17] and Huang et al. [23]. (Other surveys of interest include [14,15,22,30].) Instead, we focus on the literature that is directly relevant to our paper, namely, the contributions that used an N–A

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model, which is referred to as a capital stock advertising model in Feichtinger et al. [17], and highlight our contribution.

Table 1 provides an updated list of the papers covered in the survey in Huang et al. [23]. Each paper is characterized in terms of four features: (i) the price being or not being a decision variable; (ii) the planning horizon (finite or infinite); (iii) the type of strategic interactions; and finally (iv) a brief description of the main topic. A first observation based on Table 1 is that only 7 of the 36 listed papers included price as a decision variable, and in all these cases, the planning horizon was infinite. The conclusion here is that we do not know much about optimal pricing policies in the (probably more realistic) case of a finite terminal date. Note that the previous literature surveys in [43,17] and [14] included 19 papers using N–A dynamics, and only two of them had price as a decision variable, and both retained an infinite planning horizon. As price is clearly profit-relevant, we do believe that including it as a decision variable in a finite horizon setting fills an important gap in the literature. A second observation is that the N–A framework has been used in a wide variety of topics, both in an oligopoly/monopoly settings and in supply chains, which signals the framework's broad appeal. Finally, we note that 19 papers retained an infinite horizon, whereas 17 had a finite terminal date. In the later case, the focus has often been on the introduction of a new product or the management of advertising for a perishable (seasonal) product.

Our main results show that the optimal pricing policy follows the goodwill stock and is time-invariant. Further, the advertising trajectory is convex and monotone. Depending on the parameter values, advertising expenditures and the goodwill stock can be increasing or decreasing over time.

The rest of the paper is organized as follows: Section 2 introduces the model. Section 3 presents the optimal non-oscillating interior solution, whose existence and uniqueness are shown in Section 4 under some mild conditions. Section 5 concludes.

2. Model

We consider a planning horizon $[0,T]$, with time t running continuously. The initial date corresponds to the introduction of a new product by the firm, and T to the end of the selling season. After T, the product loses its appeal because of, e.g., a change of season for fashion apparel, or the arrival of a new version for software. Denote by $p(t)$ the product's price at time $t \in [0,T]$, and by $G(t)$ the goodwill stock (brand reputation or brand equity). The demand function is given by

$$q(t) = \alpha G(t) - \beta p(t), \tag{1}$$

where α and β are strictly positive parameters. Following a long tradition in economics and management science, our demand function is affine, however with the additional feature that the market potential at any t is proportional to the brand goodwill.

Remark 1. Our demand function is micro-founded, i.e., it is derived from consumer's utility maximization problem. To show it, let the utility function of the representative consumer be given by the following quadratic function: $U(q,y) = \phi q - \frac{\kappa q^2}{2} + y$, where q is the demand for the firm's product, y is a composite good whose price is normalized to one, and ϕ and κ are positive parameters. The budget constraint is given by $pq + y = I$, where p is the price of the product and I the income. Maximizing $U(q,y)$ subject to the budget constraint yields the demand $q = \frac{\phi - p}{\kappa}$. It suffices to set $\alpha G = \frac{\phi}{\kappa}$ and $\beta = \frac{1}{\kappa}$ to get (1).

The goodwill stock evolves à la Nerlove–Arrow [40], i.e.,

$$\frac{dG}{dt}(t) = \dot{G}(t) = ka(t) - \delta G(t), \; G(0) = G_0 > 0, \tag{2}$$

where $a(t)$ is the advertising investment at time t, $k > 0$ is the marginal efficiency of advertising, and δ is the decay rate. Following a broad literature on dynamic advertising models (see, e.g., [29,17] and [23]), we assume that the advertising cost is convex increasing and given by the quadratic function $C(a) = \frac{\omega}{2}a^2(t)$, where ω is a positive parameter. Without any loss of generality, we suppose that the marginal production cost is constant and set equal to zero. An implication of this assumption is that the price can also be interpreted as a profit margin.

Assuming profit-optimizing behavior, the firm maximizes its stream of profits over the planning horizon, that is,

$$\max_{p(t),a(t)} = \int_0^T \left(p(t)(\alpha G(t) - \beta p(t)) - \frac{\omega}{2}a^2(t)\right)dt + s\,G(T), \tag{3}$$

subject to (2),

where $sG(T)$ is the salvage value of the brand at T, which measures the future profits that the firm can obtain from marketing products under the same brand name. We suppose that $S(G(T))$ can be approximated by a linear function, that is, $S(G(T)) = sG(T)$, where s is a positive parameter.

3. Optimal solution

We will highlight below that, since the optimization problem at hand is non-concave and the control set is unbounded, the existence and uniqueness of an optimal interior solution are far from assured. We will proceed in two steps. First, we present the optimal pricing and advertising decisions assuming the existence of an interior solution. Second, we provide a set of sufficient conditions that guarantees the existence and uniqueness of an optimal interior solution.

3.1. An interior solution

We start by making the following assumption, which will imply that the solution is not oscillating (i.e., the eigenvalues of the dynamical system are real numbers).

Assumption: The parameter values satisfy the inequality

$$\delta^2 - \frac{\alpha^2 k^2}{2\beta\omega} > 0. \tag{4}$$

Proposition 1. *Under condition (4), assume that there exists an interior solution; it is then given by*

$$p(t) = \frac{\alpha G(t)}{2\beta}, \tag{5}$$

$$a(t) = \frac{k\left(2\beta s\left((\delta+v)e^{v(T+t)} - (\delta-v)e^{v(T-t)}\right) + \alpha^2 G_0\left(e^{v(2T-t)} - e^{vt}\right)\right)}{2\beta\omega\left((\delta+v)e^{2vT} - (\delta-v)\right)}, \tag{6}$$

and the brand goodwill by

$$G(t) = \frac{2\beta s(\delta^2 - v^2)\left(e^{v(T+t)} - e^{v(T-t)}\right)}{\alpha^2\left((\delta+v)\,e^{2vT} - (\delta-v)\right)} - \frac{\alpha^2 G_0\left((\delta-v)e^{vt} - e^{v(2T-t)}(\delta+v)\right)}{\alpha^2\left((\delta+v)\,e^{2vT} - (\delta-v)\right)}, \tag{7}$$

where $v = \sqrt{\delta^2 - \frac{\alpha^2 k^2}{2\beta\omega}}$.

Proof. Introduce the Hamiltonian

$$\begin{aligned} H(p(t), a(t), G(t), \lambda(t)) = {} & p(t)(\alpha G(t)) \\ & -\beta p(t) - \frac{\omega}{2}a^2(t) \\ & +\lambda(t)(ka(t) - \delta G(t)), \end{aligned} \tag{8}$$

Table 1
Articles using goodwill dynamics.

Papers in chronological order	Price	Horizon	Interaction	Main topic
Jørgensen, Sigué and Zaccour [24]	N	Inf	V	Cooperative advertising
Jørgensen, Sigué and Zaccour [25]	Y	Inf	V	Cooperative advertising
Jørgensen, Taboubi and Zaccour [26]	N	Inf	V	Non-price promotion
Buratto and Viscolani [8]	N	F	M	New product introduction
Cellini and Lambertini [10]	N	Inf	H	Quantity competition
Jørgensen, Taboubi and Zaccour [27]	N	Inf	V	Cooperative advertising
Jørgensen and Zaccour [28]	N	F	V	Non-price promotion
Mosca and Viscolani [38]	N	F	M	New product introduction
Grosset and Viscolani [19]	N	F	M	New product introduction
Lambertini [32]	N	Inf	M	Location choice
Martín-Herrán, Taboubi and Zaccour [37]	N	Inf	H+V	Shelf space allocation
Viscolani and Zaccour [45]	N	Inf	H	Negative advertising
Weber [46]	N	Inf	M	Durable goods
Buratto, Grosset and Viscolani [6]	N	F	M	Advertising by segment
Buratto, Grosset and Viscolani [5]	N	F	M	Launching a new Product
El Ouardighi and Pasin [13]	N	F	H	Quality improvements
Nair and Narasimhan [39]	Y	Inf	H	Quality improvements
Raman [41]	N	F	M	Stochastic goodwill
Bass, Bruce, Majumdar and Murthi [2]	N	F	M	Advertising themes
Buratto, Grosset and Viscolani [7]	N	F	V	Launching a new product
Marinelli [34]	N	F	M	Stochastic goodwill
Amrouche, Martín-Herrán and Zaccour [1]	Y	Inf	V	Private label
Marinelli and Savin [35]	N	F	M	Spacial effects of advertising
Zaccour [47]	Y	Inf	V	Two-part tariff
Buratto and Zaccour [9]	N	F	V	Licensing
Gozzi, Marinelli, and Savin [21]	N	F	M	Lags and stochastic effects
Grosset and Viscolani [20]	N	Inf	M	Passive competition
Karray and Martín-Herrán [31]	Y	Inf	V	Private label
Sigué and Chintagunta [44]	N	Inf	V	Franchising
Bertuzzi and Lambertini [3]	Y	Inf	H	Spatial competition
Favaretto and Viscolani [16]	N	F	M	Seasonal product
De Giovanni [12]	N	Inf	V	Quality vs advertising support
Martín-Herrán, Sigué and Zaccour [36]	N	Inf	V+H	Franchising
Buratto [4]	N	F	V	Licensing
Lambertini and Zaccour [33]	Y	Inf	H	Market power
Reddy, Wrzaczek and Zaccour [42]	N	F	M	Quality improvements

F (finite); Inf (infinite); V (vertical interaction); H (horizontal interaction); M (monopoly, no interaction).

where $\lambda(t)$ is the adjoint variable appended to the state dynamics. Assuming an interior solution, the first-order optimality conditions are

$$\frac{\partial H}{\partial p} = 0 \iff p(t) = \frac{\alpha G(t)}{2\beta}, \tag{9}$$

$$\frac{\partial H}{\partial a} = 0 \iff a(t) = \frac{\lambda(t) k}{\omega}, \tag{10}$$

$$\dot{\lambda}(t) = -\frac{\partial H}{\partial G} = -\alpha p(t) + \lambda(t)\delta, \quad \lambda(T) = s, \tag{11}$$

$$\dot{G}(t) = \frac{\partial H}{\partial \lambda} = ka(t) - \delta G(t), \quad G(0) = G_0. \tag{12}$$

Substituting for the controls from (9)–(10) in the state and adjoint equations, we obtain the following system:

$$\dot{\lambda} = \delta\lambda - \frac{\alpha^2}{2\beta}G, \quad \lambda(T) = s,$$

$$\dot{G} = \frac{k^2}{\omega}\lambda - \delta G, \qquad G(0) = G_0,$$

whose solution is given by

$$\lambda(t) = r_1 e^{vt} + r_2 e^{-vt},$$

$$G(t) = r_1 \frac{2\beta(\delta - v)}{\alpha^2} e^{vt} + r_2 \frac{2\beta(\delta + v)}{\alpha^2} e^{-vt}.$$

where $v = \sqrt{\delta^2 - \frac{\alpha^2 k^2}{2\beta\omega}}$ and

$$r_1 = \frac{2\beta(\delta + v)se^{vT} - \alpha^2 G_0}{2\beta((\delta + v)e^{2vT} - (\delta - v))}, \tag{13}$$

$$r_2 = \frac{e^{2vT}\alpha^2 G_0 - 2\beta(\delta - v)se^{vT}}{2\beta\left((\delta + v)e^{2vT} - (\delta - v)\right)}. \tag{14}$$

We make the following comments:

1. The pricing policy follows the goodwill: the higher the goodwill, the higher the price. This result is empirically observable where well-known brands do indeed command higher prices, and it can be explained by the fact that the market potential is increasing in the brand's reputation. Note that although we have a finite-horizon model, the price does not depend explicitly on time, which is likely a by-product of not having the price in the state dynamics.
2. From (10), we note that the advertising investment is dictated by the familiar rule equating marginal cost (given by wa) to marginal revenue, which is measured by $k\lambda$, that is, the marginal efficiency of advertising in raising the goodwill times its shadow price. Observe that the advertising investment is time-varying and independent of G. Recall that advertising only indirectly affects the demand, i.e., through the brand's reputation.
3. Specializing (7) for $t = T$, we obtain the following terminal value for the brand's reputation:

$$G(T) = \frac{sk^2\left(e^{2vT} - 1\right) + 2ve^{vT}\omega G_0}{\omega\left(\delta\left(e^{2vT} - 1\right) + v(e^{2vT} + 1)\right)},$$

which is, as expected, increasing in the marginal valuation s of the goodwill stock at T, and in the initial goodwill G_0.

4. We assume in this proposition that an interior solution exists. We shall show in Proposition 4 that under mild conditions there exists a unique solution, and that this solution is interior.

3.2. Monotonicity of the optimal solution

Proposition 2. *Advertising is monotonically decreasing over time for all $t \in [0, T]$ if and only if*

$$s \le \tilde{s} = \frac{\alpha^2 G_0}{\beta\left((\delta+v)\,e^{vT} + (\delta - v)\,e^{-vT}\right)}, \qquad (15)$$

Proof. Using the fact that $a(t) = \frac{\lambda(t)k}{\omega}$ we get

$$a(t) = \frac{(r_1 e^{vt} + r_2 e^{-vt})k}{\omega},$$

which is strictly positive for all $t \in [0, T]$.

Differentiating twice $a(t)$ with respect to time, we get

$$\dot{a}(t) = \frac{vk}{\omega}\left(r_1 e^{vt} - r_2 e^{-vt}\right),$$

$$\ddot{a}(t) = \frac{v^2 k}{\omega}\left(r_1 e^{vt} + r_2 e^{-vt}\right).$$

Clearly, $\ddot{a}(t) = v^2 a(t) > 0$ all $t \in [0, T]$. Therefore, $a(t)$ is convex in t and $\dot{a}(t)$ is increasing on $[0, T]$. Substituting for r_1 and r_2 in $\dot{a}(t)$ we obtain

$$\dot{a}(t) = \frac{vk\left(2\beta s e^{vT}\left((\delta+v)e^{vt} + e^{-vt}(\delta - v)\right) - \alpha^2 G_0\left(e^{vt} + e^{v(2T-t)}\right)\right)}{2\omega\beta\left((\delta+v)e^{2vT} - (\delta - v)\right)}. \qquad (16)$$

The value of $\dot{a}$ at T is given by

$$\dot{a}(T) = \frac{vk\left(2\beta s e^{vT}\left((\delta+v)e^{vT} + e^{-vT}(\delta - v)\right) - 2\alpha^2 G_0 e^{vT}\right)}{2\omega\beta\left((\delta+v)\,e^{2vT} - (\delta - v)\right)}.$$

Straightforward computations lead to

$$\dot{a}(T) \le 0 \Leftrightarrow s \le \frac{\alpha^2 G_0}{\beta\left((\delta+v)\,e^{vT} + (\delta - v)\,e^{-vT}\right)}.$$

Since $\dot{a}(t)$ is increasing on $[0, T]$, $\dot{a}(t) \le \dot{a}(T)$, $\forall t \in [0, T]$.

Hence, result follows.

The above result is intuitive. Indeed, if s is sufficiently low, then the firm should start by advertising at a relatively high level and decrease it over time. The earlier the advertising investment is made, the longer the period during which the firm enjoys a high goodwill. Two particular cases are worth mentioning. First, if $s = 0$, that is, if the firm does not benefit from its goodwill after the planning horizon, then the condition in the proposition will always be satisfied. Second, if the brand is new on the market, i.e., $G_0 = 0$, then from (16) we have

$$\dot{a}(t) = \frac{vk\left(2\beta s e^{vT}\left((\delta+v)e^{vt} + e^{-vt}(\delta - v)\right)\right)}{2\omega\beta\left((\delta+v)\,e^{2vT} - (\delta - v)\right)},$$

which is clearly positive, and therefore, advertising is increasing over time for any positive value of s.

Proposition 3. *The price and goodwill are monotonically decreasing over time for all $t \in [0, T]$ if and only if*

$$s \le \hat{s} = \frac{2\omega R_0 \delta}{k^2\left(e^{vT} + e^{-vT}\right)}.$$

Proof. The argument is similar to that used in the proof of the preceding Proposition.

If the marginal salvage value is small enough, then the firm will reduce its advertising investment over time and, as a result, the goodwill decreases over time, and as does the market potential, given by $\alpha G(t)$. In turn, this brings down the price. As for the previous proposition, we note that if $s = 0$, then the condition in the proposition will be clearly always satisfied.

Remark 2. We observe that $\tilde{s} < \hat{s}$. Compute the difference

$$\tilde{s} - \hat{s} = G_0 \frac{\left(k^2\alpha^2 - 2\omega\beta\delta^2\right)\left(e^{vT} + e^{-vT}\right) - 2\omega\delta\beta v\left(e^{vT} - e^{-vT}\right)}{\beta\left((\delta+v)\,e^{vT} + (\delta - v)\,e^{-vT}\right)k^2\left(e^{vT} + e^{-vT}\right)}.$$

As the denominator is positive, the sign of the above expression is the same as the sign of the numerator. By (4), $k^2\alpha^2 - 2\omega\beta\delta^2$ is negative and hence the result.

The behavior over time of the reputation stock is ambiguous. Indeed, differentiating $G(t)$ with respect to time, we get

$$\dot{G}(t) = \frac{v\left(k^2 s\left(e^{v(T+t)} + e^{v(T-t)}\right) - \omega G_0\left((\delta - v)e^{vt} + e^{v(2T-t)}(\delta + v)\right)\right)}{\omega\left((\delta+v)\,e^{2vT} - (\delta - v)\right)},$$

which shows that the sign of $\dot{G}(t)$ depends on all the model's parameter values. If $s = 0$ (respectively, $G_0 = 0$), then $\dot{G}(t)$ is negative (respectively, positive) for all t.

4. Existence and uniqueness of the solution

Proposition 1 is stated assuming that a solution exists and is interior. This proposition provides a candidate, but it is not granted in our case that this candidate is indeed a solution, because the Hamiltonian in (8) is not concave. Indeed, the Hessian of the Hamiltonian function is given by

$$\textit{Hessian}\; H(p, a, G) = \begin{pmatrix} -2\beta & 0 & \alpha \\ 0 & -\omega & 0 \\ \alpha & 0 & 0 \end{pmatrix},$$

which is not negative semidefinite. Consequently, classical sufficient conditions of optimality cannot be invoked. Now, we show that our candidate is indeed a solution. To do so, consider the following optimal-control constrained problem (I):

$$\max_{p(t),\,a(t)} \int_0^T \left(p(t)\left(\alpha G(t) - \beta p(t)\right) - \frac{\omega}{2}a^2(t)\right)dt + s.G(T)$$

$$\dot{G}(t) = ka(t) - \delta G(t), \; G(0) = G_0,$$
$$0 \le a(t) \le n, \text{ and } \; 0 \le p(t) \le b,$$

where n and b are positive real numbers.

Proposition 4. *Assume that*

$$\omega > \frac{k\alpha^2}{2\beta\delta}\left(\frac{G_0}{n} + \frac{k}{\delta}\right) + \frac{ks}{n}, \qquad (17)$$

$$b > \frac{\alpha}{2\beta}\left(G_0 + \frac{nk}{\delta}\right). \qquad (18)$$

Then, (5)–(7) is indeed the unique solution.

Proof. We shall prove that (5)–(7) is the unique optimal solution. The proof proceeds in three steps.

Step 1. We first show that there exists a solution to problem (I). Set $F(G, a, p) = p(\alpha R - \beta p_l) - \frac{\omega}{2}a^2$, $g(G(T)) = sG(T)$ and $f(G, a, p) = ka - \delta G$. Set $K = [0, n] \times [0, b]$. It is clear that $G(t)$ is bounded, that F and g are continuous functions, and that f is Lipschitz in the state variable and continuous in the control variable, such that the set of admissible solutions is non-empty. Consider the set

$$Q(G) = \{(v, z) \;/\; v \le F(G, a, p),\; z = f(G, a, p),\; (a, p) \in K\}.$$

It is convex since F is concave with respect to the control (a, p) and f is linear with respect to the control. Thus, by Filippov [18] existence theorem, there exists a solution to the problem (see also the survey in Cesari [11]).

Step 2. Let us show that a solution of (I) is interior, i.e., it must satisfy $0 < p(t) < b$ and $0 < a(t) < n$ for all t.

The solution to problem (I) maximizes the following Hamiltonian at each date t:

$$H = p(t)(\alpha G(t) - \beta p_l(t)) - \frac{\omega}{2}a^2(t) + \lambda(t)(ka(t) - \delta G(t)) + \epsilon(t)p(t) + \eta(t)(b - p(t)) + \alpha(t)a(t) + \psi(t)(n - a(t)),$$

where $\epsilon(t)$, $\eta(t)$, $\alpha(t)$ and $\psi(t)$ are Lagrange multipliers. Therefore, the following first-order conditions must be satisfied:

$$\frac{\partial H}{\partial p} = 0 \Leftrightarrow \alpha G(t) - 2\beta p(t) + \epsilon(t) - \eta(t) = 0,$$
$$\frac{\partial H}{\partial a} = 0 \Leftrightarrow -\omega a(t) + k\lambda(t) + \alpha(t) - \psi(t) = 0,$$
$$\dot{\lambda}(t) = -\frac{\partial H}{\partial G} = -\alpha p(t) + \lambda(t)\delta, \quad \lambda(T) = s,$$
$$\dot{G}(t) = \frac{\partial H}{\partial \lambda} = ka(t) - \delta G(t), \quad G(0) = G_0,$$
$$p(t) \geq 0, \ \epsilon(t) \geq 0, \ \epsilon(t)p(t) = 0,$$
$$b - p_l(t) \geq 0, \ \eta(t) \geq 0, \ \eta(t)(b - p_l(t)) = 0,$$
$$a(t) \geq 0, \ \alpha(t) \geq 0, \ \alpha(t)a(t) = 0,$$
$$n - a(t) \geq 0, \ \psi(t) \geq 0, \ \psi(t)(n - a(t)) = 0.$$

Solving for $G(t)$, we have

$$G(t) = G_0 e^{-\delta t} + k\int_0^t e^{\delta(u-t)}a(u)du > G_0 e^{-\delta t} > 0. \tag{19}$$

Therefore, $G(t)$ is always positive.

Let us show that $0 < p(t) < b$, for all t. Assume that $p(t) = 0$. Then, $\eta(t) = 0$ and the first optimality condition becomes $\alpha G(t) + \epsilon(t) = 0$. Since $\epsilon(t) \geq 0$ and $G(t)$ is positive, we get a contradiction. Now, assume that $p(t) = b$ for some t. Then $\epsilon(t) = 0$, and from the first optimality condition, we have $p(t) = b = \frac{\alpha G(t) - \eta(t)}{2\beta}$. Since $\eta(t) \geq 0$, it follows that $b < \frac{\alpha G(t)}{2\beta}$. From (19) and the fact that $a(t) \leq n$, we have

$$b < \frac{\alpha}{2\beta}\left\{G_0 e^{-\delta t} + kn\int_0^t e^{\delta(u-t)}du\right\} < \frac{\alpha}{2\beta}\left(G_0 + \frac{kn}{\delta}\right). \tag{20}$$

But this contradicts assumption (18) in the statement of the proposition. We thus have $p(t) < b$ and $p(t) = \frac{\alpha_l G(t)}{2\tilde{\beta}_l}$.

Now, we show that $a(t) > 0$ for all t. Assume by way of contradiction that there is some t at which $a(t) = 0$. Then $\psi(t) = 0$, and the second optimality condition becomes $k\lambda(t) + \alpha(t) = 0$. Since $\alpha(t) \geq 0$, we therefore have $\lambda(t) = -\frac{\alpha(t)}{k} \leq 0$. It is immediate to see that $t \neq T$. Now solving for $\lambda(t)$ and using $p(t) = \frac{\alpha G(t)}{2\beta}$, we get

$$\lambda(t) = e^{\delta t}\left\{\int_t^T e^{-\delta u}\frac{\alpha^2}{2\beta}G(u)du + e^{-\delta T}s\right\}. \tag{21}$$

It follows that $\lambda(t) > 0$, which is a contradiction.

Let us now show that $a(t) < n$ for all t. Assume by way of contradiction that there is a date t at which $a(t) = n$. Then $\alpha(t) = 0$, and we must have $-\omega n + k\lambda(t) - \psi(t) = 0$. Thus, $k\lambda(t) = \omega n + \psi(t)$, or $\lambda(t) \geq \frac{\omega n}{k}$. Now from (19), we have

$$G(t) = G_0 e^{-\delta t} + k\int_0^t e^{\delta(u-t)}a(u)du, \tag{22}$$
$$\leq G_0 e^{-\delta t} + kn\int_0^t e^{\delta(u-t)}du \tag{23}$$
$$= G_0 e^{-\delta t} + \frac{kn}{\delta}(1 - e^{-\delta t}) \leq G_0 + \frac{kn}{\delta}. \tag{24}$$

Using (21), we get

$$\frac{\omega n}{k} \leq \lambda(t) \leq e^{\delta t}\left\{\int_t^T e^{-\delta u}\frac{\alpha^2}{2\beta}(G_0 + \frac{kn}{\delta})du + e^{-\delta T}s\right\} \tag{25}$$
$$= \frac{\alpha^2}{2\beta\delta}(G_0 + \frac{kn}{\delta})(1 - e^{\delta(t-T)}) + e^{\delta(t-T)}s \tag{26}$$
$$< \frac{\alpha^2}{2\beta\delta}(G_0 + \frac{kn}{\delta}) + s. \tag{27}$$

But this last inequation contradicts assumption (17) made in the statement of the proposition.

Step 3. Now we can check that (5)–(7) is the unique solution of the first-order conditions of Problem (I) (where $\epsilon(t) = \eta(t) = \alpha(t) = \psi(t) = 0$). Since the optimal solution of Problem (I) exists, it is this unique solution. Hence the result.

The interpretation of the above proposition is straightforward. The firm's instantaneous profit is linearly increasing in G. To avoid having an infinite value for G, which would be the result of large investments in advertising, condition (17) requires that the advertising cost to be large enough. Further, if the upper bound (18) on the instantaneous price is large enough, then it is never optimal to charge this upper bound (because demand would otherwise be too low). Naturally, G_0 must also not be too high; otherwise, it would be profitable to increase G in order to obtain higher profits. Note, however, that the higher the bound on instantaneous advertising, the lower ω can be, but the higher b must be. This is intuitive. The higher is n, the higher the reputation can be, and the higher the price can be set. And, to avoid being constrained, b must also be higher. Likewise, it is less necessary that ω be high to avoid choosing $a(t) = n$. Finally, also notice that the higher is the efficiency of advertising, that is, the higher is k, then the higher must be the cost ω of advertising be to get an interior solution. The influence of the other parameters, s, δ, can be understood in the same way.

5. Concluding remarks

We provided sufficiency conditions to show the existence and uniqueness of an interior solution of a dynamic pricing and advertising non-concave problem. Our results show that the pricing trajectory follows the goodwill, and that the advertising trajectory is time-varying and monotone. The behavior over time of the control and state trajectories depends on the parameter values. Interesting extensions of our work include the case where the price also influences the dynamics and the case where horizontal and/or vertical strategic interactions are present.

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