Ann Oper Res (2008) 164: 79–95
DOI 10.1007/s10479-008-0320-7

Feedback Stackelberg equilibrium strategies when the private label competes with the national brand

Nawel Amrouche · Guiomar Martín-Herrán · Georges Zaccour

Published online: 31 January 2008

Abstract We consider a noncooperative differential game where a retailer sells her own private label in addition to the manufacturer's brand. We assume that each brand's goodwill evolves according to a modified Nerlove-Arrow dynamics, in such a way that the advertising effort of one brand hurts the competitor's goodwill stock. We characterize Feedback-Stackelberg pricing and advertising strategies and employ simulations to analyze their sensitivity to the main model parameters.

Keywords Marketing channels · Private label · Advertising · Pricing · Differential games · Feedback-Stackelberg equilibrium

1 Introduction

Until recently, the general wisdom was that private labels (store brands (*SB*) or retailers' brands) were essentially meant to serve price-sensitive market segments. Thanks to the improved quality of *SB*s, this positioning has changed over time. Nowadays, retailers' brands are seen by consumers, at least in some markets and some product categories, as valuable substitutes to established manufacturers' or national brands (see, e.g., Hoch and Banerji 1993 and Wilensky 1994). This new status has shifted, to some extent, the balance of power

We thank an anonymous Reviewer for very helpful comments. Research supported by NSERC, Canada, and FQRSC, Quebec. Research completed when the second author was a visiting professor at GERAD, HEC Montréal. The second author's research was partially supported by MEC and JCYL under projects SEJ2005-03858/ECON and VA045A06, co-financed by FEDER funds.

N. Amrouche
Long Island University, Long Island, NY, USA

G. Martín-Herrán
Departamento de Economía Aplicada (Matemáticas), Universidad de Valladolid, Valladolid, Spain

G. Zaccour (✉)
Chair in Game Theory & Management, GERAD, HEC Montréal, Montréal, Canada
e-mail: georges.zaccour@gerad.ca

from manufacturers to retailers. Further, some store brands (and not just the stores themselves) are now advertised nationally, something that is hardly affordable for many brands, and that is not necessary when the main selling attribute is (a low) price. For instance, jeans labels like Arizona (JCPenney), Canyon River Blues (Sears, Roebuck and Co.) and Badge (Federated Department Stores) have become not only acceptable to consumers but even trendy, thanks to clever advertising that develops the product image separately from the store image (Anderson et al. 2001).

The aim of this paper is to characterize pricing and advertising equilibrium strategies of an *NB* manufacturer, and of a retailer who, on top of offering the *NB* to consumers, also sells her own *SB*. This means that the retailer is at once a distributing agent for the manufacturer and her competitor. Following a rather established tradition in the marketing channels literature, we assume that the game is played à la Stackelberg, with the manufacturer acting as leader and the retailer as follower (see, e.g., Raju et al. 1995; Cotterill and Putsis 2001; Sayman et al. 2002; Sayman and Raju 2004; Choi and Coughlan 2006 and Karray and Zaccour 2006 for a static game, and Ailawadi et al. 2005 and Jørgensen et al. 2000, 2001a, 2001b, 2003 for examples of Stackelberg differential games). The argument typically given to explain this informational asymmetry is that the manufacturer has to announce her wholesale price first, and the retailer can then decide on the consumer price.

One main assumption of our model is that advertising plays a dual role: it increases the goodwill (reputation, brand equity) level of the brand, and hence its demand, and negatively affects the other brand's goodwill. This dual role has been generally ignored in the literature on dynamic marketing channels, where the evolution of goodwill levels is assumed to depend only on the brand's own advertising or promotion efforts (see, e.g., Chintagunta and Jain 1992; Jørgensen and Zaccour 1999, 2003; Jørgensen et al. 2000, 2001a, 2001b, 2003). One exception to this is Amrouche et al. (2007) where, however, the game is played à la Nash, meaning that the players are symmetric in terms of the information structure. One result is that the manufacturer's transfer price and advertising strategies are constant (feedback degenerate), which admittedly lacks appeal in practice. The idea that brand goodwill is vulnerable to a competitor's advertising is pursued in Chakrabarti and Haller (2004) and Nair and Narasimhan (2006). The former investigates, in a three-stage game framework, comparative advertising wars in the context of competing *NBs*. The approach in Nair and Narasimhan (2006) is closer to ours in terms of its approach, that is, a differential game incorporating goodwill dynamics, that uses, however, only manufacturer brands. This means that the players are inherently symmetric, which is not the case in our attempt. Indeed, context here involves a manufacturer who receives revenues only from the sales of her brand, whereas the retailer is interested in the fate of a product category that is made up of the manufacturer's brand as well as her own. Note that since advertising exhibits carry-over effects, the use of a dynamic model is rather natural.

The rest of the paper is structured as follows. In Sect. 2, we develop a model for the channel under study. In Sect. 3, we derive the Feedback-Stackelberg Equilibrium. In Sect. 4, we interpret the numerical findings and propose corresponding managerial implications. In Sect. 5, we conclude and suggest directions for future research.

2 The model

We consider a marketing channel made up of the manufacturer of a national brand (player M) and a retailer (player R). The retailer sells the manufacturer's national brand

(*NB*) and her own private label or store brand (*SB*). We assume that the retailer manufactures the store brand, or purchases it from a (dummy player) manufacturer, at a cost d_s. We denote by d_n the constant unit production cost of the *NB*. The retailer controls the price to consumer of the national brand ($p_n(t)$), as well as the retail price of the store brand ($p_s(t)$) at each instant of time $t \in [0, \infty)$. The manufacturer decides the *NB*'s wholesale price to the retailer, $w_n(t)$. (From now on, subscript n stands for national brand and s for store brand.)

The demand for each brand is assumed to be linear and given by

$$D_n(t) = \beta_n + G_n(t) - p_n(t) + \psi p_s(t),$$

$$D_s(t) = \beta_s + G_s(t) - p_s(t) + \psi p_n(t),$$

where $\beta_i, \psi, i \in \{n, s\}$, are positive parameters and G_i denotes brand i's goodwill (or brand equity) for $i \in \{n, s\}$. Parameter ψ measures the degree of substitutability between the two brands. Given the standard assumption in economics that the cross-price effect is lower than the direct one (normalized to 1), we take $\psi \in [0, 1)$. Note that, in the above demand specification, the market potential, given by $\beta_i + G_i(t)$, is not constant, but depends positively on the brand goodwill .

Denote by $A_i(t)$ the advertising expenditures at time t for brand $i, i \in \{n, s\}$. We assume that the brands' goodwill stocks evolve according to the following dynamics:

$$\dot{G}_n(t) = A_n(t) - \delta G_n(t) - k_n A_s(t), \quad G_n(0) = G_{n0} \geq 0, \tag{1}$$

$$\dot{G}_s(t) = A_s(t) - \delta G_s(t) - k_s A_n(t), \quad G_s(0) = G_{s0} \geq 0, \tag{2}$$

where δ is the decay rate, which is assumed to be the same for both brands. The above equations extend the standard Nerlove and Arrow (1962) dynamics by adding the term $-k_i A_j(t)$, where $k_i \geq 0, i, j \in \{n, s\}, i \neq j$. This term corresponds to the loss in market potential of brand i due to the advertising effort of brand j. The parameter k_i captures the vulnerability of brand i to the advertising done by brand j. We assume that the marginal impact on goodwill of one's own advertising is greater than the marginal effect of the competitor's advertising, and hence $k_i < 1$.

Brand i's advertising cost, $C_i(A_i)$, is assumed convex increasing, as follows:

$$C_i(A_i) = c_{i1} A_i + \frac{c_{i2}}{2} A_i^2, \quad i \in \{n, s\},$$

where c_{i1} and c_{i2} are positive constants. Assuming profit-maximization behavior, the manufacturer and retailer optimization problems read as follows:[1]

$$\max_{w, A_n} \pi_M = \int_0^\infty e^{-\rho t} \left\{ (w - d_n)[\beta_n + G_n - p_n + \psi p_s] - \left(c_{n1} + \frac{c_{n2}}{2} A_n \right) A_n \right\} dt, \tag{3}$$

$$\begin{aligned} \max_{A_s, p_n, p_s} \pi_R = \int_0^\infty e^{-\rho t} \Big\{ & (p_n - w)[\beta_n + G_n - p_n + \psi p_s] + (p_s - d_s)[\beta_s + G_s - p_s + \psi p_n] \\ & - \left(c_{s1} + \frac{c_{s2}}{2} A_s \right) A_s \Big\} dt, \end{aligned} \tag{4}$$

subject to the dynamic constraints describing the time evolution of the brands' goodwill, (1) and (2), and where ρ denotes the common discount rate.

[1] From now on the time argument is eliminated when no confusion may arise.

Expressions (3), (4), (1) and (2) define a two-player infinite-horizon differential game with two state variables, namely, the brand's goodwill stocks, G_n, G_s, and five control variables, namely, the wholesale and retail prices, w_n, p_n, p_s, and the advertising investments, A_n, A_s.

As stated in the introduction, we follow the literature in marketing channels and assume that the game is played à la Stackelberg with the manufacturer acting as leader and the retailer as follower. This means that the sequence of events is as follows. First, the *NB*'s manufacturer announces her advertising (A_n) and wholesale price (w). The retailer reacts to this information and decides on how much to invest on *SB*'s advertising (A_s) and on the retail prices of both brands, i.e., p_s and p_n. Finally, the manufacturer takes into account the retailer's reaction and chooses the optimal wholesale price and advertising investment.

As in Roberts and Samuelson (1988) and Martín-Herrán and Taboubi (2005), we assume that, when optimizing her profit, each brand takes into account only her own brand's dynamics. The reasons for this assumption are tractability and the fact that observing the evolution of a competitor's goodwill is costly. Although the manufacturer and retailer are partially myopic, their strategies, as will be shown later, still depend on both goodwill stocks, i.e., they are nondegenerate feedback strategies. As is customary in infinite-horizon differential games, we suppose that the manufacturer and retailer choose strategies that are stationary feedback (see, for example, Dockner et al. 2000), that is, that pricing and advertising strategies are time independent and only depend on the current levels of the state variables G_n and G_s.

3 Feedback Stackelberg equilibrium

The retailer, as a follower, chooses the retail prices and the advertising investment that maximize her objective functional (4), subject to the state dynamics equation for the brand goodwill of the *SB* given in (2). Since her pricing-decision variables do not affect the dynamics of the *SB*'s goodwill stock, her optimal retail-pricing decisions are the same as if she solved a static optimization problem. However, her advertising investment affects the time evolution of the *SB*'s goodwill stock, and the retailer takes into account its dynamics when choosing her optimal advertising level.

The next proposition characterizes the retailer's reaction functions.

Proposition 1 *Let us denote by $V_R(G_s, G_n)$ the retailer's value function. The retailer's reaction functions read*:

$$p_n(w, G_s, G_n) = \frac{w}{2} + \frac{G_n + \psi G_s + \beta_n + \psi \beta_s}{2(1-\psi^2)}, \tag{5}$$

$$p_s(w, G_s, G_n) = \frac{d_s}{2} + \frac{G_s + \psi G_n + \beta_s + \psi \beta_n}{2(1-\psi^2)}, \tag{6}$$

$$A_s(w, G_s, G_n) = \frac{1}{c_{s2}}\left(\frac{\partial V_R}{\partial G_s}(G_s, G_n) - c_{s1}\right), \tag{7}$$

if there are positive expressions, and zero otherwise.

Proof The Hamilton-Jacobi-Bellman (HJB) equation associated with the retailer's optimization problem is as follows:

$$\rho V_R(G_s, G_n) = \max_{p_n, p_s, A_s} \Big\{ (p_n - w)[\beta_n + G_n - p_n + \psi p_s] + (p_s - d_s)[\beta_s + G_s - p_s + \psi p_n] - \left(c_{s1} + \frac{c_{s2}}{2} A_s \right) A_s + \frac{\partial V_R}{\partial G_s}(A_s - \delta G_s - k_s A_n) \Big\}. \quad (8)$$

Assuming interior solutions, from the necessary conditions for optimality, and taking the partial derivatives of the RHS in (8) with respect to p_n, p_s and A_s and equating to zero, we obtain

$$G_n - 2(p_n - \psi p_s) + w - d_s \psi + \beta_n = 0,$$
$$G_s - 2(p_s - \psi p_n) - w\psi + d_s + \beta_s = 0,$$
$$-c_{s1} - c_{s2} A_s + \frac{\partial V_R}{\partial G_s}(G_s, G_n) = 0.$$

Solving the above equations, the retailer's reaction functions in (5), (6) and (7) are obtained. □

Let us stress that, while the retailer is partially myopic and disregards the dynamics of her competitor's goodwill stock when making her decisions, each pricing and advertising best-reply function depends on both goodwill stocks.

Expression (6) shows that the retail price of the *SB* is independent of the manufacturer's wholesale price. Therefore, expression (6) corresponds to the optimal strategy for the retail price of the *SB*. This strategy depends positively on both brand-goodwill stocks; therefore, an increase in either brand-goodwill stock leads to an increase in the retail price of the *SB*. Moreover, since $\psi \in [0, 1)$, it follows that

$$\frac{\partial p_s}{\partial G_s}(G_n, G_s) > \frac{\partial p_s}{\partial G_n}(G_n, G_s).$$

Thus, this brand's goodwill has a greater impact on the *SB* retail price than does the competing brand's goodwill .

Expression (5) establishes that the retail price of the *NB* depends on the manufacturer's wholesale price. In order to analyze the effect of the brand-goodwill stocks on p_n, the optimal wholesale-pricing strategy has to be computed. However, from (5) one has that the *NB*'s retail price and the manufacturer's wholesale price are strategic complements ($(\partial p_n/\partial w)(w, G_n, G_s) = 1/2$). The retailer augments the price to consumer of the *NB* when she is paying a higher transfer price to the manufacturer.

Once the retailer's reaction functions are computed, the next proposition characterizes the manufacturer's pricing and advertising strategies at equilibrium.

Proposition 2 *Let us denote by $V_M(G_s, G_n)$ the manufacturer's value function. The manufacturer's optimal pricing and advertising strategies are given by*

$$w(G_n, G_s) = \frac{G_n + \beta_n + d_n + \psi d_s}{2}, \quad (9)$$

$$A_n(G_n, G_s) = \frac{1}{c_{n2}} \left(\frac{\partial V_M}{\partial G_n}(G_n, G_s) - c_{n1} \right), \quad (10)$$

if there are positive expressions, and zero otherwise.

Proof The HJB equation associated with the manufacturer's optimization problem is as follows:

$$\rho V_M(G_n, G_s) = \max_{w, A_n}\left\{(w - d_n)[\beta_n + G_n - p_n + \psi p_s] - \left(c_{n1} + \frac{c_{n2}}{2}A_n\right)A_n + \frac{\partial V_M}{\partial G_n}(G_n, G_s)(A_n - \delta G_n - k_n A_s)\right\}. \quad (11)$$

Replacing the reaction functions given in (5), (6) and (7) in the HJB (11), we obtain

$$\rho V_M(G_n, G_s) = \max_{w, A_n}\left\{\frac{(w - d_n)}{2}[G_n - w + \psi d_s + \beta_n] - \left(c_{n1} + \frac{c_{n2}}{2}A_n\right)A_n + \frac{\partial V_M}{\partial G_n}(G_n, G_s)\left(A_n - \delta G_n - \frac{k_n}{c_{s2}}\frac{\partial V_R}{\partial G_s}(G_n, G_s)\right)\right\}.$$

Assuming interior solutions, from the necessary conditions for optimality, and taking the partial derivatives of the RHS of the above expression with respect to w and A_n and equating to zero, we obtain

$$-2w + G_n + \psi d_s + d_n + \beta_n = 0,$$
$$-c_{n1} - c_{n2}A_n + \frac{\partial V_M}{\partial G_n}(G_n, G_s) = 0,$$

and the result in the proposition follows. □

Expression (9) shows that the optimal wholesale-price strategy depends only on the *NB*'s goodwill stock and is independent of the *SB*'s goodwill stock. The higher the *NB*'s goodwill stock, the higher will be the wholesale price charged by the manufacturer to the retailer.

The retailer's optimal pricing and advertising strategies can now be completely characterized.

Corollary 1 *The retailer's optimal strategies are given by*

$$p_n(G_n, G_s) = \frac{3 - \psi^2}{4(1 - \psi^2)}G_n + \frac{\psi}{2(1 - \psi^2)}G_s + \frac{\psi d_s + d_n + \beta_n}{4} + \frac{\beta_n + \psi\beta_s}{2(1 - \psi^2)}, \quad (12)$$

$$p_s(G_n, G_s) = \frac{\psi G_n + G_s - d_s + \beta_s + \psi\beta_n}{2(1 - \psi^2)}, \quad (13)$$

$$A_s(G_s, G_n) = \frac{1}{c_{s2}}\left(\frac{\partial V_R}{\partial G_s}(G_s, G_n) - c_{s1}\right), \quad (14)$$

if there are positive expressions, and zero otherwise.

Proof The retailer's optimal strategies are computed by replacing into the reaction functions given by (5), (6) and (7), the manufacturer's optimal strategies for the manufacturer, given in (9) and (10). □

The sensitivity analysis of the *SB*'s optimal price strategy, p_s, to changes in the goodwill stocks, G_n and G_s has already been analyzed. Let us now carry out this analysis for the *NB*'s optimal price strategy, p_n. Taking into account that ψ belongs to the interval $[0, 1)$, it is easy to see that not only does the optimal retail price of the *NB* increase with both brand-goodwill stocks, but also, the effect of increase in the *NB*'s goodwill stock has a greater impact on its own price than does an increase in the *SB*'s goodwill stock:

$$\frac{\partial p_n}{\partial G_n}(G_n, G_s) > \frac{\partial p_n}{\partial G_s}(G_n, G_s).$$

Therefore, we can conclude that both retail prices behave in a symmetric manner. First, the higher the goodwill stock of either brand, the higher are both retail prices. This finding means that investing in building up goodwill for either brand reduces the amount of price competition between them and enhances their differentiation (market power for both of them as well). Second, one's own brand goodwill affects the brand retail price to a greater extent than does the competitor's brand goodwill.

Thus far, we have completely characterized the different pricing strategies, but the advertising strategies are expressed in terms of the partial derivatives of the retailer's and the manufacturer's value functions, respectively. The next proposition collects the expressions of the optimal value functions, allowing us to fully determine the optimal advertising strategies as well as the outcomes at equilibrium.

Proposition 3 *The following expressions satisfy the HJB equations associated with the retailer's and manufacturer's problems*:

$$V_R(G_n, G_s) = \frac{R_1}{2}G_s^2 + \frac{R_2}{2}G_n^2 + R_3 G_n G_s + R_4 G_s + R_5 G_n + R_6, \tag{15}$$

$$V_M(G_n, G_s) = \frac{M_1}{2}G_n^2 + \frac{M_2}{2}G_s^2 + M_3 G_n G_s + M_4 G_n + M_5 G_s + M_6, \tag{16}$$

where coefficients $R_i, M_i, i = 1, \dots, 6$ *are characterized in the Appendix.*

Proof See the Appendix. □

Corollary 2 *The retailer's and manufacturer's advertising strategies at equilibrium are given by*

$$A_s(G_n, G_s) = \frac{1}{c_{s2}}(R_1 G_s + R_3 G_n + R_4 - c_{s1}), \tag{17}$$

$$A_n(G_n, G_s) = \frac{1}{c_{n2}}(M_1 G_n + M_3 G_s + M_4 - c_{n1}). \tag{18}$$

In the Appendix we establish the conditions leading to bounded brand-goodwill stocks, G_n, G_s, implying the fulfillment of the sufficiency condition for optimality and ensuring that (15), (16), (17), and (18) are the retailer's and manufacturer's value functions and advertising strategies.

Unfortunately, we cannot obtain closed-form expressions of the coefficients of the retailer's and manufacturer's value functions $R_i, M_i, i = 1, \dots, 6$. However, in all the numerical simulations we carried out using MATLAB 7.0.1, coefficients R_1, R_3, M_1 and M_3 are positive.[2] Therefore, we can establish the following conjecture.

Conjecture 1 *At equilibrium, the retailer's and manufacturer's advertising investments are each positively related to her own brand-goodwill stock and to her competitor's goodwill stock.*

The literature on advertising expenditures in dynamic oligopolies refers to this behavior as informative advertising (see, for example, Deal 1979; Roberts and Samuelson 1988; Erickson 1991, and Espinosa and Mariel 2001). In accordance with these authors, the advertising investment of any channel member leads to higher market sales, since advertising for any of the brands leads to a higher goodwill stock for the brand (via the brand-goodwill -stock dynamics), but also to a higher advertising investment from the competitor (since her feedback strategy indicates that her reaction to an increase in her rival's brand-goodwill stock must be an increase in her own advertising effort). Hence the competitor's brand-goodwill stock also increases (via the brand-goodwill-stock dynamics). The rise of both brand-goodwill stocks leads to higher sales for both brands, and to a greater total market size.

4 Numerical results

As mentioned above, one cannot find analytical solutions to the Ricatti equations that characterize the coefficients of the retailer's and manufacturer's value functions. Therefore, we cannot find closed-form expressions of the advertising strategies or the total payoff for each player. Hence, we resort to simulations to gain some insight into the behavior of such equilibrium strategies and outcomes.[3]

We are interested in the long-term behavior of the decision variables, prices and advertising efforts, the goodwill stocks, as well as the brands' demands and profits. Therefore, the values of different variables are at the steady state. (The superscript ss stands for steady state, and π_k^{ss}, $k = n, s$, denotes the instantaneous profit at the steady state for brand k.)

We run two series of sensitivity analyses on key model parameters β_i, k_i, d_i and $c_{ij}, i \in \{n, s\}$, $j = 1, 2$. In the first series, the players are assumed symmetric in all respects,[4] i.e., in terms of their advertising and purchasing costs, baseline sales, and vulnerability to competitor advertising. Here we assess the impact of varying the values of each of these key parameters. In the next series, we do the same but taking into account an asymmetry between the players, in each of these parameters in turn. As a benchmark case, we assume the following values for the parameters:

Demand parameters: $\beta_n = \beta_s = 1.1, \psi = 0.25,$
Cost parameters: $d_n = d_s = 0.01, c_{n1} = c_{s1} = 1, c_{n2} = c_{s2} = 0.01,$

[2] The Appendix presents a detailed description of the procedure used to reduce the solution of a system of 12 non-linear equations into the solution of a system of two non-linear equations. This latter system is numerically solved using MATLAB 7.0.1.

[3] The MATLAB code for generating numerical results is available from the authors upon request.

[4] Note that symmetry here refers to the parameters' values, and not the structure of the game (player R is selling the two brands, and player M is only interested in her own), or the structure of the information.

Table 1 Variations in steady-state values in the symmetric case

	p_n^{ss}	p_s^{ss}	w^{ss}	A_n^{ss}	A_s^{ss}	G_n^{ss}	G_s^{ss}	D_n^{ss}	D_s^{ss}	π_n^{ss}	π_s^{ss}
$c_{n1} = c_{s1}(1; 1.5; 2)$	−	−	−	−	−	−	−	−	−	−	−
$c_{n2} = c_{s2}(0.01; 0.02; 0.03)$	−	−	−	−	−	−	−	−	−	−	−
$k_n = k_s(0.1; 0.15; 0.2)$	+	+	+	+	+	+	+	+	+	+	+
$\beta_n = \beta_s(1.1; 1.15; 1.2)$	+	+	+	+	+	+	+	+	+	+	+
$d_n = d_s(0.01; 0.02; 0.03)$	−	−	−	−	−	−	−	−	−	−	−

Table 2 Variations in steady-state values in the asymmetric case

	p_n^{ss}	p_s^{ss}	w^{ss}	A_n^{ss}	A_s^{ss}	G_n^{ss}	G_s^{ss}	D_n^{ss}	D_s^{ss}	π_n^{ss}	π_s^{ss}
$c_{n1}(1; 1.5; 2)$	−	−	−	−	−	−	−	−	−	−	−
$c_{s1}(1; 1.5; 2)$	−	−	−	−	−	−	−	−	−	−	−
$c_{n2}(0.01; 0.03; 0.05)$	−	−	−	−	−	−	−	−	−	−	−
$c_{s2}(0.01; 0.03; 0.05)$	−	−	−	−	−	−	−	−	−	−	−
$k_n(0.1; 0.15; 0.2)$	+	+	+	+	+	+	+	+	+	+	+
$k_s(0.1; 0.15; 0.2)$	+	+	+	+	+	+	+	+	+	+	+
$\beta_n(1.1; 1.15; 1.2)$	+	+	+	+	+	+	+	+	+	+	+
$\beta_s(1.1; 1.15; 1.2)$	+	+	+	+	+	+	−	+	+	+	+
$d_n(0.01; 0.02; 0.03)$	−	−	+	−	−	−	−	−	−	−	−
$d_s(0.01; 0.02; 0.03)$	−	−	−	−	+	−	+	−	−	−	−

Dynamics parameters: $k_n = k_s = 0.1, \delta = 0.1,$
Discount Rate: $\rho = 0.03.$

The same qualitative effects as those given here were obtained for the numerical simulations carried out for different values of the model parameters.

Table 1 reports the sensitivity analyses conducted in the symmetric case. For each parameter, we consider three values, e.g., 1.10, 1.15 and 1.2 for β_i. A plus sign means that the value taken by the variable in question increases when the parameter changes from any value to the next one. For instance, the plus sign in cell (p_n, β_i) means that p_n increases when β_i goes from 1.10 to 1.15 *and* from 1.15 to 1.20. Table 2 reports the results for the asymmetric scenario. The values considered for each parameter are written below each parameter. In both scenarios, when a parameter is varied, all others remain at their benchmark values.

The results in Tables 1 and 2 allow for the following comments:

- In both the symmetric and asymmetric scenarios, increasing any advertising-cost parameter, i.e., $c_{ij}, i \in \{n, s\}, j = 1, 2$, leads to lower advertising levels. This is conceivable. From there, the chain of effects seems to be as follows: less advertising means lower goodwill stocks, and hence, lower demand, which forces the players to lower their prices. Ultimately, this translates into a lower steady-state profit for each player.
- The impact of varying either the vulnerability parameter, k_n or k_s, in both scenarios, is an upward shift in all steady-state values. The fact that increased vulnerability induces the players to escalate their advertising expenditures is intuitive. Indeed, each player is pursuing her own interest and, as she does not observe—due to partial myopia—the im-

pact of her advertising on the evolution of her competitor's goodwill stock, she increases her advertising to counteract her loss of goodwill due to the competitor's advertising. This defensive action actually has another impact, which is that it attacks the competitor's goodwill, and hence the escalation. What is rather difficult to explain is that this somehow involuntary "advertising war" induced by higher vulnerability is Pareto-improving in terms of payoff. That is, both players achieve higher steady-state profits than those achieved with a lower value for any of the vulnerability parameters.

- Turning to the purchasing cost, note first that all players' margins—that is, $w^{ss} - d_n$ for the manufacturer, and $p_n^{ss} - w^{ss}$ and $p_s^{ss} - d_s$ for the retailer—decrease in both scenarios when d_n or d_s is increased. Basically, this means that the players are not able to transfer to the consumer the increase in manufacturing cost. The lack of funds seems to lead to less advertising or to an advertising level that is not sufficient to boost goodwill, and hence, boost the size of the market and prices.
- A higher $\beta_i, i \in \{n, s\}$ means that the brand becomes more attractive (higher demand). In the symmetric scenario where $\beta_n = \beta_s$, both brands increase their prices following this surge in attraction. This results in higher revenues, and hence, in additional funds being available for advertising, in higher steady-state goodwill stocks, and in more profits. In the asymmetric case, the results are slightly different. To see why, first recall that the manufacturer is interested in the fate of her own brand, whereas the retailer has a stake in both brands. When β_n is increased but β_s is left at its benchmark level, we have the same result as in the symmetric case. This is probably due to the following chain of events. Increasing β_n allows the manufacturer to increase her transfer price, which in turn, leads to a higher retail price. The store brand takes stock of this and increases its price. Thanks to additional revenues, the two players increase their advertising expenditures and we get the same effect as the one obtained by varying k_i. Put differently, a higher β_n is in the best interest of the two players, because both players have the national brand in their portfolios. Now, increasing β_s while keeping β_n at its benchmark value expands the market potential of the store brand. This allows the retailer to increase the price of this brand, without decreasing the demand. The manufacturer reacts by increasing the transfer price to get more revenue to cover the cost of the advertising required to "attack" the market potential of the competing brand and raise its own brand's goodwill. The retailer responds without "too much" escalation, i.e., increasing advertising expenditures but not enough to increase its goodwill. The retailer is surfing on the increase of β_s, as well as on the benefits resulting from the manufacturer's reaction.

5 Conclusion

In this paper, we characterized feedback Stackelberg equilibrium strategies in a marketing channel where a store brand competes with a national one. We carried out a sensitivity analysis to assess the impact of key parameters on strategies, goodwill stocks and payoffs at steady state. One main contribution made by our model is that it accounts for the dual role of advertising and for the related result that an increase in vulnerability leads to higher payoffs for both players. These results have been obtained under some assumptions that we now discuss:

Feedback strategies We have assumed that the players use stationary feedback strategies. The stationarity assumption is rather standard in differential games played over an infinite horizon (see Dockner et al. 2000). One could have considered the easier-to-compute open-loop Stackelberg strategies. However, one would then need to show that the resulting equilibrium is time consistent, which is a rare instance in such case. (See Martín-Herrán et al. 2005 for an example.)

Myopia A crucial assumption in this work is the one of partial myopia. It is clearly of interest to check whether the results obtained stand, qualitatively speaking, if this assumption is relaxed. Note that doing so would come at the cost of developing an adequate numerical algorithm to solve the system of twelve heavily coupled algebraic Ricatti equations. In such a case, the system could not be reduced to a two-equation system as we have done in this paper and the solution would require the development of an appropriate algorithm to numerically determine all the advertising strategies.

Continous time The game is played in continuous time. An alternative formulation is to consider a discrete-time formulation where, at each stage, the leader plays first and next the follower chooses her action, knowing the choice of the leader. This approach may actually yield different results and intuition into this problem. For examples of such discrete-time formulation in dynamic marketing games, the interested reader may refer to Breton et al. (1997, 2006).

Other extensions to this work can be envisioned. First, one could introduce another national brand to capture the competition, not only between store and national brand, but also between manufacturers of national brands. Second, one could reformulate the demand models to account for some saturation effects in goodwill levels, or to put a limit on the total market expansion. Finally, one could also allow the retailer to implement promotional activities, which have a different impact than advertising, e.g., to boost short-term demand without necessarily affecting it in the long term.

Appendix

Proof of Proposition 3

Inserting the retailer's optimal strategies, (12), (13), (14), and the manufacturer's optimal strategies, (9) and (10) in (8) and (11), the coefficients of the value functions $R_i, M_i, i = 1, \ldots, 6$ are determined by identification and solve the following twelve algebraic Ricatti equations (the first six correspond to the retailer, whereas the second six correspond to the manufacturer):

$$2b_1R_1[2b_2M_3 - c_{n2}R_1 + b_7] + b_6 = 0, \tag{19}$$

$$8b_1R_3[4b_2M_1 - c_{n2}R_3] + 8b_1b_5R_2 + b_6(3\psi^2 + 1) = 0, \tag{20}$$

$$2b_1b_2(M_1R_1 + M_3R_3) - 2b_1R_3(c_{n2}R_1 - b_3) + b_6\psi = 0, \tag{21}$$

$$b_2(M_4R_3 + M_1R_4) + R_3[c_{n2}(c_{s1} - R_4) - b_2c_{n1}] + b_5R_5 + \Upsilon_1 = 0, \tag{22}$$

$$b_2(M_4R_1 + M_3R_4) + R_1[c_{n2}(c_{s1} - R_4) - b_2c_{n1}] + b_3R_4 + \Upsilon_2 = 0, \tag{23}$$

$$R_4[2b_2(c_{n1} - M_4) + c_{n2}(R_4 - 2c_{s1})] + c_{n2}c_{s1}^2 - 2b_5R_6 + \Upsilon_3 = 0, \tag{24}$$

$$4M_1[2b_4R_3 - c_{s2}M_1 + b_7] - b_6 = 0, \tag{25}$$

$$M_3[2b_4R_1 - c_{s2}M_3] + b_5M_2 = 0, \tag{26}$$

$$b_4(M_1R_1 + M_3R_3) - M_3(c_{s2}M_1 - b_3) = 0, \tag{27}$$

$$b_4(M_1R_4 + M_4R_3) + M_1[c_{s2}(c_{n1} - M_4) - b_4c_{s1}] + b_3M_4 + \Upsilon_4 = 0, \tag{28}$$

$$b_4(M_4R_1 + M_3R_4) + M_3[c_{s2}(c_{n1} - M_4) - b_4c_{s1}] + b_5M_5 = 0, \tag{29}$$

$$2[-c_{n1}c_{s2} + b_4(c_{s1} - R_4)]M_4 + c_{s2}(M_4^2 + c_{n1}^2) - 2b_5M_6 + \Upsilon_5 = 0, \tag{30}$$

where

$$b_1 = \psi^2 - 1, \qquad b_2 = c_{s2}k_s, \qquad b_3 = c_{n2}c_{s2}(\delta + \rho), \qquad b_4 = c_{n2}k_n,$$
$$b_5 = c_{n2}c_{s2}\rho, \qquad b_6 = c_{n2}c_{s2}, \qquad b_7 = c_{n2}c_{s2}(2\delta + \rho),$$

and

$$\Upsilon_1 = \frac{b_1(d_n + 3\psi d_s) + \beta_n(1 + 3\psi^2) + 4\psi\beta_s}{8b_1},$$
$$\Upsilon_2 = \frac{d_sb_1 + \beta_s + \psi\beta_n}{2b_1},$$
$$\Upsilon_3 = \frac{[d_n(-d_n + 2\psi d_s + 2\beta_n) + 2d_s(4\beta_s + 3\psi\beta_n) + d_s^2(3b_1 - 1)]b_1 + [2(\beta_s + \psi\beta_n)]^2 - b_1\beta_n^2}{16b_1},$$
$$\Upsilon_4 = \frac{d_n - \psi d_s - \beta_n}{4},$$
$$\Upsilon_5 = -\frac{[\psi d_s - d_n + \beta_n]^2}{3}.$$

From (19), we can obtain M_3 as a function of R_1, and from (25), R_3 as a function of M_1:

$$M_3 = f_1(R_1), \tag{31}$$

$$R_3 = f_2(M_1), \tag{32}$$

where

$$f_1(R_1) = \frac{\Omega_2(R_1)}{8b_2(4b_1)^2b_6R_1}, \qquad f_2(M_1) = \frac{\Omega_1(M_1)}{4b_4(4b_1)^2b_6M_1},$$

with

$$\Omega_1(M_1) = 4b_1b_6[b_6 + 8b_1M_1(-b_7 + c_{s2}M_1)],$$
$$\Omega_2(R_1) = -32b_6b_1[b_6 + 2b_1R_1(b_7 - c_{n2}R_1)].$$

Replacing the expressions of R_3 and M_3 in (20) and (26), we can rewrite R_2 and M_2 as a function of M_1 and R_1, respectively:

$$R_2 = f_3(M_1), \tag{33}$$

$$M_2 = f_4(R_1), \tag{34}$$

where

$$f_3(M_1) = \frac{c_{n2}\Omega_1^2(M_1) - 2b_4M_1^2\Theta_1(M_1)}{4b_4^2b_7(32b_1^2b_6)^2M_1^2}, \qquad f_4(R_1) = \frac{c_{s2}\Omega_2^2(R_1) - 2b_2R_1^2\Theta_2(R_1)}{4b_2^2b_7(64b_1^2b_6)^2R_1^2},$$

with

$$\Theta_1(M_1) = 64b_1^2 b_6[b_2\Omega_1(M_1) + 4b_1 b_4 b_6^2(1 + 3\psi^2)],$$
$$\Theta_2(R_1) = 128b_1^2 b_6 b_4 \Omega_2(R_1).$$

From (24) and (30), we obtain R_6 and M_6 as functions of R_4 and M_4:

$$R_6 = f_5(R_4, M_4), \qquad M_6 = f_6(R_4, M_4), \tag{35}$$

where

$$f_5(R_4, M_4) = \frac{c_{n2}(c_{s1} - R_4)^2 + 2b_2(c_{n1} - M_4)R_4 + \Upsilon_3}{2b_5},$$
$$f_6(R_4, M_4) = \frac{c_{s2}(c_{n1} - M_4)^2 + 2b_4(c_{s1} - R_4)M_4 + \Upsilon_6}{2b_5}.$$

Replacing the expressions of R_3 and M_3 in (22) and (28), we can rewrite R_5 and M_5 as a function of M_1, M_4, R_4 and R_1, M_4, R_4, respectively:

$$R_5 = f_7(M_1, M_4, R_4), \tag{36}$$
$$M_5 = f_8(R_1, M_4, R_4), \tag{37}$$

where

$$f_7(M_1, M_4, R_4) = \frac{[b_2(c_{n1} - M_4) - c_{n2}(c_{s1} - R_4)]\Omega_1(M_1)}{b_4 b_5 (8b_1)^2 b_6 M_1} - \frac{\Upsilon_1 + b_2 M_1 R_4}{b_5 b_6},$$
$$f_8(R_1, M_4, R_4) = \frac{[b_4(c_{s1} - R_4) - c_{s2}(c_{n1} - M_4)]\Omega_2(R_1)}{2b_2 b_5 (8b_1)^2 b_6 R_1} - \frac{c_{n1} k_n M_4 R_1}{b_5 b_6}.$$

Substituting the expressions of M_3 and R_3 in (23) and (28), we can rewrite these equations in terms of variables R_1, M_4, R_4 and M_1, M_4, R_4, respectively. Solving these two equations, we obtain coefficients R_4 and M_4 as functions of R_1, M_1:

$$R_4 = \frac{\mathrm{Num}(R_4)}{\mathrm{Den}(R_4)},$$
$$M_4 = \frac{(8b_1)^2 b_6 M_1(\Upsilon_4 + (c_{n1}c_{s2} + b_4(R_4 - c_{s1})M_1)}{(8b_1)^2 b_6(c_{s2}M_1 - b_3)M_1 - \Omega_1(M_1)},$$

where

$$\mathrm{Num}(R_4) = 2b_4 b_2 b_6 (8b_1)^2 R_1[C_1 M_1^2 R_1 - \Sigma_3(M_1, R_1)M_1 - \Sigma_5(R_1)\Omega_1(M_1)],$$
$$\begin{aligned}\mathrm{Den}(R_4) = {} & 8b_4 b_6 b_2 (32b_1^2 b_6)^2 M_1 R_1\{b_6[b_6 - b_4 b_2]M_1 R_1 + b_3(\Sigma_1(M_1) + \Sigma_2(R_1) - b_3 b_6)\} \\ & + b_4 b_2[b_6^2\Omega_1(M_1)\Omega_2(R_1) + 2\Sigma_4(M_1, R_1)],\end{aligned}$$

with

$$C_1 = (8b_1)^2 b_6^3 c_{s1}(b_6 - b_4 b_2),$$

$$\Sigma_1(M_1) = (b_3 - c_{s2}M_1)a_8,$$

$$\Sigma_2(R_1) = (b_3 - c_{n2}R_1)b_6,$$

$$\Sigma_3(M_1, R_1) = (8b_6b_1)^2[\Upsilon_2\Sigma_1(M_1) + (b_3c_{n2}(c_{s1}b_6 - c_{n1}c_{s2}b_2) - b_2b_6\Upsilon_4)R_1],$$

$$\Sigma_4(M_1, R_1) = 2(4b_6b_1)^2[2R_1\Sigma_2(R_1)\Omega_1(M_1) + M_1\Sigma_1(M_1)\Omega_2(R_1)],$$

$$\Sigma_5(R_1) = b_6^2[c_{n2}c_{s1}R_1 + \Upsilon_2 - c_{n1}b_2R_1].$$

Finally, replacing the expressions of R_3 and M_3 in (21) and (27), we obtain a two equations system with two unknowns R_1, M_1:

$$4b_2b_4(32b_1^2b_6)^3(M_1R_1)^2 + \Xi_1(M_1)\Omega_2(R_1) = 0,$$

$$2b_4b_1b_6(64b_1^2b_6)^3M_1R_1(b_6\psi + 2b_1b_2M_1R_1) + \Xi_2(R_1)\Omega_1(M_1) = 0,$$

where

$$\Xi_1(M_1) = (4b_1)^2b_6[\Omega_1(M_1) + (8b_1)^2b_6M_1(b_3 - c_{s2}M_1)],$$

$$\Xi_2(R_1) = 2(4b_1)^2b_6[\Omega_2(R_1) + 2(8b_1)^2b_6R_1(b_3 - c_{n2}R_1)].$$

Unfortunately, the system above is heavily non-linear and we could not solve it analytically. This system is numerically solved using the MATLAB function ‘fsolve’. By default ‘fsolve’ chooses the medium-scale algorithm and uses the trust-region dogleg method (see the MATLAB documentation, The MathWorks, Inc.). The MATLAB code for generating numerical results is available from the authors upon request.

Proof of the sufficient conditions for optimality

A sufficient condition guaranteeing that the expressions in (15), (16), (14), and (10) are retailer’s and manufacturer’s value functions and advertising strategies is given by:

$$\lim_{t\to\infty} e^{-\rho t}V_R(G_n(t), G_s(t)) = 0, \qquad \lim_{t\to\infty} e^{-\rho t}V_M(G_n(t), G_s(t)) = 0, \tag{38}$$

where $(G_n(t), G_s(t))$ is the solution of the closed-loop dynamics obtained after substitution of the advertising strategies (14) and (10) into the goodwill dynamics given by (1) and (2). This solution can be written as

$$G_n(t) = X_1^n e^{\lambda_1 t} + X_2^n e^{\lambda_2 t} + G_n^{ss}, \tag{39}$$

$$G_s(t) = X_1^s e^{\lambda_1 t} + X_2^s e^{\lambda_2 t} + G_s^{ss}, \tag{40}$$

where $X_1^i, X_2^i, i = n, s$, are constants, G_n^{ss} and G_s^{ss} refer to the steady states of the goodwill variables and $\lambda_i, i = 1, 2$ are the real eigenvalues of the matrix associated to the system of linear differential equations.

The steady-state levels of the goodwill stocks, G_n^{ss}, G_s^{ss} are given by:

$$G_n^{ss} = \frac{\mathrm{Num}(G_n^{ss})}{\mathrm{Den}(G_n^{ss}, G_s^{ss})}, \qquad G_s^{ss} = \frac{\mathrm{Num}(G_s^{ss})}{\mathrm{Den}(G_n^{ss}, G_s^{ss})},$$

where

$$\begin{aligned}
\mathrm{Num}(G_n^{ss}) &= (R_4 - c_{s1})(M_3(1 - k_n k_s) - c_{n2} k_n \delta) - (M_4 - c_{n1})(R_1(1 - k_n k_s) - c_{s2}\delta), \\
\mathrm{Num}(G_s^{ss}) &= (M_4 - c_{n1})(R_3(1 - k_n k_s) - c_{s2} k_s \delta) - (R_4 - c_{s1})(M_1(1 - k_n k_s) - c_{n2}\delta), \\
\mathrm{Den}(G_n^{ss}, G_s^{ss}) &= R_1(M_1(1 - k_n k_s) - c_{n2}\delta) - R_3(M_3(1 - k_n k_s) - c_{n2} k_n \delta) \\
&\quad + c_{s2}\delta(k_s M_3 - M_1 + c_{n2}\delta).
\end{aligned}$$

The computation of the state variable trajectories allows the identification of the paths for the pricing and advertising strategies, and for the evolution of the demand function for both brands. The results can be obtained after replacing $G_n(t)$ and $G_s(t)$ by their respective values from (39) and (40) into (12), (13), (9), (14) and (10).

The quadratic functional specification in (15) and (16) allows condition (38) to be satisfied whenever the goodwill stocks are bounded.

The next proposition states the conditions for G_n, G_s to be bounded.

Proposition 4 *The goodwill stocks are bounded if the conditions listed below apply*:

1. *B given in (43) is negative and the initial goodwill stocks are related according to the following expression*

$$G_{s0} = \frac{\frac{k_n R_3}{c_{s2}} - \frac{M_1}{c_{n2}} + \delta + \lambda_1}{\frac{M_3}{c_{n2}} - \frac{k_n R_1}{c_{s2}}}(G_{n0} - G_n^{ss}) + G_s^{ss}, \quad \lambda_1 < 0, \tag{41}$$

2. *B and A given in (42) and (43) are positive and $A^2 - 4B$ is greater or equal to zero.*

Proof The eigenvalues λ_1 and λ_2 of the matrix associated to the closed-loop dynamics can be written as:

$$\lambda_{1,2} = \frac{-A \pm \sqrt{A^2 - 4B}}{2},$$

where

$$A = \frac{k_s M_3 - M_1}{c_{n2}} + \frac{k_n R_3 - R_1}{c_{s2}} + 2\delta, \tag{42}$$

$$B = \left(\frac{R_1}{c_{s2}} - \frac{k_s M_3}{c_{n2}} - \delta\right)\left(\frac{M_1}{c_{n2}} - \frac{k_n R_3}{c_{s2}} - \delta\right) - \left(\frac{M_3}{c_{n2}} - \frac{k_n R_1}{c_{s2}}\right)\left(\frac{R_3}{c_{s2}} - \frac{k_s M_1}{c_{n2}}\right). \tag{43}$$

It is straightforward to see that if B is negative, then one eigenvalue (the one affected by the negative sign before the square root) is negative, while the other one is positive. Under this assumption, the steady state (G_n^{ss}, G_s^{ss}) is a saddle point. The initial conditions on the goodwill lying on the stable subspace associated to the negative eigenvalue λ_1, given by (41), allow the system to converge to the steady state as time approaches infinity.

If both B and A are positive, then both eigenvalues λ_1 and λ_2 are negative. Under this assumption, the steady state (G_n^{ss}, G_s^{ss}) is globally asymptotically stable. □

Item 1 characterizes the situation where the steady state is stable in the saddle-point sense. The saddle-point property means that given the initial goodwill level G_{n0}, we can find values of G_{s0} (that satisfy (41)) such that the closed-loop system converges to the steady state (G_n^{ss}, G_s^{ss}) as time approaches infinity.

Item 2 guarantees a globally asymptotically stable equilibrium. In this case, any initial level of the goodwill stocks converges to the steady state as time approaches infinity and leads to a bounded time path.

References

Ailawadi, K. L., Kopalle, P. K., & Neslin, S. A. (2005). Predicting competitive response to a major policy change: Combining game-theoretic and empirical analysis. *Marketing Science*, *24*(1), 12–24.

Amrouche, N., Martín-Herrán, G., & Zaccour, G. (2007). *Pricing and advertising of private and national brands in a dynamic marketing channel* (Working Paper).

Anderson, E., Coughlan, A. T., El-Ansary, A., & Stern, L. W. (2001). *Marketing channels* (6th edn.). New York: Prentice Hall.

Breton, M., Chauny, F., & Zaccour, G. (1997). A leader-follower dynamic game of new product diffusion. *Journal of Optimization Theory and Applications*, *92*(1), 77–98.

Breton, M., Jarrar, R., & Zaccour, G. (2006). A note on feedback Stackelberg equilibria in a Lanchester model with empirical application. *Management Science*, *52*(5), 804–811.

Chakrabarti, S., & Haller, H. (2004). *An analysis of advertising wars* (Working paper). Virginia Polytechnic Institute, USA.

Chintagunta, P. K., & Jain, D. (1992). A dynamic model of channel member strategies for marketing expenditures. *Marketing Science*, *11*(2), 168–188.

Choi, S. C., & Coughlan, A. T. (2006). Private label positioning: Quality versus feature differentiation from the national brand. *Journal of Retailing*, *82*(2), 79–93.

Cotterill, R. W., & Putsis, Jr., W. P. (2001). Do models of vertical strategic interaction for national and store brands meet the market test? *Journal of Retailing*, *77*, 83–109.

Deal, K. R. (1979). Optimizing advertising expenditures in a dynamic duopoly. *Operations Research*, *27*(4), 682–692.

Dockner, E., Jørgensen, S., Van Long, N., & Sorger, G. (2000). *Differential games in economics and management science*. Cambridge University Press: Cambridge.

Erickson, G. M. (1991). *Dynamic models of advertising competition*. Kluwer Academic: Dordrecht.

Espinosa, M., & Mariel, P. (2001). A model of optimal advertising expenditures in a dynamic duopoly. *Atlantic Economic Journal*, *29*(2), 135–161.

Hoch, S. J., & Banerji, S. (1993). When do private labels succeed? *Sloan Management Review*, *34*, 57–67.

Jørgensen, S., & Zaccour, G. (1999). Equilibrium pricing and advertising strategies in a marketing channel. *Journal of Optimization Theory and Applications*, *102*(1), 111–125.

Jørgensen, S., & Zaccour, G. (2003). A differential game of retailer promotions. *Automatica*, *39*(7), 1145–1155.

Jørgensen, S., Sigué, S. P., & Zaccour, G. (2000). Dynamic cooperative advertising in a channel. *Journal of Retailing*, *76*(1), 71–92.

Jørgensen, S., Sigué, S. P., & Zaccour, G. (2001a). Stackelberg leadership in a marketing channel. *International Game Theory Review*, *3*(1), 13–26.

Jørgensen, S., Taboubi, S., & Zaccour, G. (2001b). Cooperative advertising in a marketing channel. *Journal of Optimization Theory and Applications*, *110*(1), 145–158.

Jørgensen, S., Taboubi, S., & Zaccour, G. (2003). Retail promotions with negative brand image effects: Is cooperation possible? *European Journal of Operational Research*, *150*(2), 395–405.

Karray, S., & Zaccour, G. (2006). Could co-op advertising be a manufacturer's counter strategy to store brands? *Journal of Business Research*, *59*, 1008–1015.

Martín-Herrán, G., & Taboubi, S. (2005). Shelf-space allocation and advertising decisions in the marketing channel: A differential game approach. *International Game Theory Review*, *7*(3), 1–18.

Martín-Herrán, G., Taboubi, S., & Zaccour, G. (2005). A time-consistent open-loop Stackelberg equilibrium of shelf-space allocation. *Automatica*, *41*(6), 971–982.

Nair, A., & Narasimhan, R. (2006). Dynamics of competing with quality and advertising based goodwill. *European Journal of Operational Research*, *175*, 462–474.

Nerlove, M., & Arrow, L. (1962). Optimal advertising policy under dynamic considerations. *Economica*, *29*(114), 129–142.

Raju, J. S., Sethuraman, R., & Dhar, S. K. (1995). The introduction and performance of store brands. *Management Science*, *41*(6), 957–978.

Roberts, M. J., & Samuelson, L. (1988). An empirical analysis of dynamic nonprice competition in an oligopolistic industry. *RAND Journal of Economics*, *19*(2), 200–220.

Sayman, S., & Raju, J. S. (2004). How category characteristics affect the number of store brands offered by the retailer: A model and empirical analysis. *Journal of Retailing*, *41*(6), 957–978.

Sayman, S., Hoch, S. J., & Raju, J. S. (2002). Positioning of store brands. *Marketing Science*, *21*(4), 378–397.

Wilensky, D. (1994). Private label success no secret as discount. *Discount Store News*, *33*(5), 23–44.