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A Model of Nanosized Thin Film Coating with Sinusoidal Interface

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Abstract. In this paper, we present an approach to the stress concentration analysis of an isotropic ultra-thin film coating with thickness from hundreds to a few nanometers coherently bonded to a substrate through an undulated interphase region. To capture the size dependence of the mechanical properties observed in nanostructured materials, we use Gurtin-Murdoch model in which surface and interphase domains are represented as negligibly thin layers ideally adhering to the bulk phases. This model is characterized by traction discontinuities at the surface and interface where the additional surface and interface stresses appear due to different bond lengths, angles and charge distribution of surface and interface atoms. In the case of plane strain conditions, the elasticity solution for a four-phase system is derived in the terms of the Goursat-Kolosovs complex potentials.

INTRODUCTION

The analysis of pattern formation at the bimaterial interfaces of composites and coatings has been an active field of research in the past decades [1, 2, 3, 4]. It has been shown that mismatch strain and mass transport driven by chemical potential gradient contribute to the morphological evolution of the phase boundary [5]. The interface propagates from one phase into another, corrugates and develops the parallel grooves. Depending on the mechanical and geometrical properties of interphase region, these defects may concentrate large stresses and lead to the crack nucleation. Thus, it is important from both the fundamental and technological point of view to understand how an interface roughness and stiffness influence on the stress distribution along contact area of two joined solids.

As an extension of our previous papers [6, 7], theoretical model for analysis of the stress field around interfacial defects of isotropic nanosized thin film coating is established. It is assumed that the stressed film is coherently bonded to a substrate under plane strain conditions. We formulate a corresponding two-dimensional boundary value problem in terms of the complex variable. The constitutive equations of the surface linear elasticity are written in a simplified form where only tangential components of the surface and interface displacements are taking in the consideration. The conditions of mechanical equilibrium on the planar surface and curved interface are described in the terms of generalized Young-Laplace equations [8] with unknown surface and interface stresses. The additional boundary equations are introduced from the inseparability conditions of bulk, surface and interfase domains [9, 10].

Following to the superposition principle [6, 7, 11, 12], the solution of the original boundary value problem for an infinite strip joined along the curvilinear interfacial boundary to a half-plane is presented as a sum of two auxiliary problems. In the first problem, we suppose that the unknown self-balanced load and surface stress are applied to the rectilinear boundary of the homogeneous half-plane with the elastic properties of the film. The second problem describes a deformation of two joint half-planes with elastic properties of the film and substrate caused by the unknown jumps of stresses and displacements along the curvilinear interface. According to Muskhelishvili's technique [13], the components of stress and strain tensors for each domain are related to the unknown Goursat-Kolosov's complex potentials. We rewrite the boundary equations taking into account these relations. Unfortunately, it is impossible to find complex potentials and express them analytically in terms of the right-hand side of the boundary equation in the case of a curvilinear boundary. However, one can obtain their approximate expressions using boundary perturbation method

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[6, 7, 12, 14, 15] whereby the unknown functions are sought in the form of a power series in the small parameter (the ratio of the amplitude to the wavelength of the interface perturbation). Based on the solution of Riemann-Hilbert problem, we derive the integral dependencies of complex potentials on the surface and interface stresses for the first-order approximation. In this way, the original boundary value problem is reduced to the integral equations system which is solved analytically employing the properties of Cauchy type integrals.

PROBLEM FORMULATION

A two-dimensional boundary value problem is formulated in the terms of complex variable $z = x_1 + ix_2$ for a strip Ω_1 of thickness h_f with a rectilinear external boundary Γ_1 bonded to a half-plain Ω_2 along a curvilinear interface Γ_2:

$$\Gamma_1 = \left\{z : z \equiv z_1 = x_1 + ih_f\right\},\ \Gamma_2 = \{z : z \equiv z_2 = x_1 + i\varepsilon f(x_1)\},\ f(x_1) = -a\cos(k_a x_1),\ k_a = \frac{2\pi}{a}, \tag{1}$$

$$\Omega_1 = \left\{z : \varepsilon f(x_1) < x_2 < h_f\right\},\ \Omega_2 = \{z : x_2 < \varepsilon f(x_1)\}, \tag{2}$$

where a is a wavelength of undulation, $A = a\varepsilon$ is the maximum deviation of the interface profile from the flat shape $x_1 = h_f$ and ε is a small parameter, i.e. $0 < \varepsilon \ll 1$.

The simplified constitutive equations, which take into account only tangential components of the surface/ interface stress and strain tensors, are used to describe the membrane type model for soft surface and interphase layers [1, 10, 15, 16, 17]:

$$\begin{aligned} \sigma^s_{tt}(z_j) &= \gamma^0_j + (\lambda^s_j + 2\mu^s_j)\varepsilon^s_{tt}(z_j), \\ \sigma^s_{33}(z_j) &= \gamma^0_j + \left(\gamma^0_j + \lambda^s_j\right)\varepsilon^s_{tt}(z_j),\ z_j \in \Gamma_j,\ j = \{1, 2\}, \end{aligned} \tag{3}$$

where ε^s_{tt} and σ^s_{tt} are the non-vanishing components of the surface strain and the Piola–Kirchhoff surface stress tensors, respectively; λ^s_j and μ^s_j are the surface Lamé constants, and γ^0_j is the residual surface/interface stress for surface/interphace region Γ_j.

Hooke's law for bulk materials in the case of plane strain can be written as

$$\begin{aligned} \sigma_{nn}(z) &= (\lambda_j + 2\mu_j)\varepsilon_{nn}(z) + \lambda_j\varepsilon_{tt}(z), \\ \sigma_{tt}(z) &= (\lambda_j + 2\mu_j)\varepsilon_{tt}(z) + \lambda_j\varepsilon_{nn}(z),\ \sigma_{nt}(z) = 2\mu_j\varepsilon_{nt}(z), \\ \sigma_{33}(z) &= \frac{\lambda_j}{\lambda_j + \mu_j}\left[\sigma_{tt}(z) + \sigma_{nn}(z)\right],\ z \in \Omega_j,\ j = \{1, 2\}, \end{aligned} \tag{4}$$

where σ_{nn}, σ_{tt}, σ_{nt} and ε_{nn}, ε_{tt}, ε_{nt} are the components of bulk stress and strain tensors, respectively, defined in the local Cartesian coordinate system n, t, and λ_j, μ_j are the Lamé constants for phase Ω_j. For cubic crystals, the surface elastic constants are obtained by atomistic simulations in [18, 19].

The conditions of mechanical equilibrium on planar surface Γ_1 and undulated interface Γ_2 are described in terms of generalized Young–Laplace equations [8]:

$$\sigma(z_1) = -i\frac{d\sigma_s(x_1)}{dx_1},\ z_1 \in \Gamma_1, \tag{5}$$

$$\Delta\sigma(z_2) = \sigma^+(z_2) - \sigma^-(z_2) = -T^s\tau_s(x_1),\ z_2 \in \Gamma_2. \tag{6}$$

Here and below, we use the following notations $\sigma_s(x_1) \equiv \sigma^s_{tt}(z_1)$, $\tau_s(x_1) \equiv \sigma^s_{tt}(z_2)$, $T^s(\cdot) = \kappa(x_1)(\cdot) - i\dfrac{1}{h(x_1)}\dfrac{d(\cdot)}{dx_1}$, $\sigma = \sigma_{nn} + i\sigma_{nt}$, $\sigma^\pm(z_2) = \lim\limits_{z \to z_2 \pm i0}\sigma(z)$, κ and h are the local principal curvature and the metric coefficient, respectively.

Since we assumed that the surface phases and the bulk materials are coherent, the inseparability conditions can be defined as it follows:

$$\varepsilon^s_{tt}(z_j) = \varepsilon_{tt}(z_j),\ \Delta u(z_2) = u^+(z_2) - u^-(z_2) = 0,\ z_j \in \Gamma_j, \tag{7}$$

where $u^{\pm}(z_2) = \lim_{z \to z_2 \pm i0} u(z)$, $u = u_1 + iu_2$; u_1 and u_2 are the displacements along the corresponding coordinate axes x_1 and x_2.

At infinity, the stresses $\sigma_{\alpha\beta}$ $(\alpha, \beta = \{1, 2\})$ in coordinates x_1, x_2 and the rotation angle ω are specified as:

$$\lim_{x_2 \to -\infty} (\sigma_{22} - i\sigma_{12}) = \lim_{x_2 \to -\infty} \omega = 0, \; \lim_{x_2 \to -\infty} \sigma_{11} = T_2. \tag{8}$$

A common reason for the appearance of longitudinal stress T_2 is a mismatch between the crystal lattice parameters of a film layer and a substrate [20].

BOUNDARY PERTURBATION METHOD

In accordance with superposition principle [6, 7, 12], the solution of boundary value problem (1)-(8), specifically, the complex stress and displacement vectors $\sigma(z) = \sigma_{nn}(z) + i\sigma_{nt}(z)$ and $u(z) = u_1(z) + iu_2(z)$, respectively, are presented as a sum of two auxiliary problems:

$$G(z, \eta_j) = \begin{cases} G_1^1(z, \eta_1) + G_1^2(z, \eta_1), & z \in \Omega_1, \\ G_2^2(z, \eta_2), & z \in \Omega_2, \end{cases} \tag{9}$$

where the functions G_j^k are defined as follows:

$$G_j^k(z, \eta_j) = \begin{cases} \sigma^k(z), & \eta_j = 1, \\ -2\mu_j \upsilon^k(z), & \eta_j = 4\nu_j - 3, \end{cases} \; , \; z \in \Omega_j, \tag{10}$$

In Eq. (10), $\upsilon^k = \dfrac{du^k}{dz}$ where the derivative is taken in the direction of the axis t, ν_j is Poisson's ratio of the phase Ω_j $(j, k = \{1, 2\})$.

In the first problem, we suppose that the unknown self-balanced load p and surface stress ϑ_s are applied to the rectilinear boundary Γ_1 of the homogeneous half-plane $D_1^1 = \{z : x_2 < h_f\}$ with the elastic properties of the film Ω_1. So, the boundary condition in the terms of the stress vector σ^1 related to this problem can be written as:

$$\sigma^1(z_1) = p(z_1) - i\frac{d\vartheta_s(z_1)}{dx_1}, \; \int_{-\infty}^{+\infty} p(\zeta) d\zeta = 0, \; z_1 \in \Gamma_1. \tag{11}$$

The stresses σ_{jk}^1 $(j, k = \{1, 2\})$ and the rotation angle ω^1 at infinity are equal to zero:

$$\lim_{x_2 \to -\infty} (\sigma_{22} - i\sigma_{12}) = \lim_{x_2 \to -\infty} \sigma_{11} = \lim_{x_2 \to -\infty} \omega = 0. \tag{12}$$

The second problem describes a coupled deformation of two joint half-planes $D_1^2 = \{z : x_2 > \varepsilon f(x_1)\}$ and $D_2^2 = \{z : x_2 < \varepsilon f(x_1)\}$ with the elastic properties of the film and the substrate, respectively, caused by the unknown jumps of stresses $\Delta\sigma^2$ and displacements Δu^2 along the undulated interface profile and the longitudinal stresses T_j acting in D_j^2 $(j = \{1, 2\})$:

$$\Delta\sigma^2(z_2) = \sigma^{2+}(z_2) - \sigma^{2-}(z_2), \; \Delta u^2(z_2) = u^{2+}(z_2) - u^{2-}(z_2), \; z_2 \in \Gamma_2, \tag{13}$$

$$\begin{gathered} \lim_{x_2 \to \pm\infty} \left(\sigma_{22}^2 - i\sigma_{12}^2\right) = \lim_{x_2 \to \pm\infty} \omega^2 = 0, \lim_{x_2 \to +\infty} \sigma_{11}^2 = T_1, \; \lim_{x_2 \to -\infty} \sigma_{11}^2 = T_2, \\ (1 + \Pi)T_1 = (1 - \Pi)T_2, \; \Pi = \frac{\mu_1(\varkappa_2 + 1) - \mu_2(\varkappa_1 + 1)}{\mu_1(\varkappa_2 + 1) + \mu_2(\varkappa_1 + 1)}, \end{gathered} \tag{14}$$

where $u^{2\pm}(z_2) = \lim_{z\to z_2\pm i0} u^2(z)$, $\sigma^{2\pm}(z_2) = \lim_{z\to z_2\pm i0} \sigma^2(z)$.

Taking into account Eqs. (9) and (10), boundary conditions (5)-(7) and (11) as well as constitutive equations (3)–(4), we obtain the system of the boundary equations for the unknown functions p, ϑ_s, σ_s and τ_s:

$$\sigma^1(z_1) = p(z_1) - i\vartheta_s'(z_1), \tag{15}$$

$$\Delta\sigma^2(z_2) = -T^s\tau_s(z_2) - \sigma^1(z_2),\ \Delta u^2(z_2) = -u^1(z_2), \tag{16}$$

$$\sigma^1(z_1) + \sigma^2(z_1) = -i\sigma_s'(z_1), \tag{17}$$

$$\vartheta_s(z_1) = \gamma_0^1 + \left(\lambda_1^s + 2\mu_1^s\right)\varepsilon_{tt}^1(z_1), \tag{18}$$

$$\sigma_s(z_1) = \gamma_0^1 + \left(\lambda_1^s + 2\mu_1^s\right)\left[\varepsilon_{tt}^1(z_1) + \varepsilon_{tt}^2(z_1)\right], \tag{19}$$

$$\tau_s(z_2) = \gamma_0^2 + \left(\lambda_2^s + 2\mu_2^s\right)\varepsilon_{tt}^2(z_2), \tag{20}$$

where a prime denotes the derivative with respect to the argument x_1.

To solve the plane elasticity problems (11)-(14), we use the Muskhelishvili's representations of stress σ^k and the displacement u^k complex vectors as the functionals of Goursat-Kolosov's complex potentials Φ_j^k and Υ_j^k ($j,k = \{1,2\}$) [13]:

$$G_j^k(z,\eta_j) = \eta_j\Phi_j^k(w_j) + \overline{\Phi_j^k(w_j)} - \left[\Upsilon_j^k(\overline{w_j}) + \overline{\Phi_j^k(w_j)} - \left(w_j - \overline{w_j}\right)\overline{\Phi_j^{k\prime}(w_j)}\right]e^{-2i\alpha},\ w_1 = z - ih_f,\ w_2 = z,\ z \in \Omega_j, \tag{21}$$

where α is the angle between the t-axis and the x_1-axis, the bar over a symbol denotes the complex conjugation, and a prime denotes the derivative with respect to the argument; Φ_1^1 and Υ_1^1 are the functions holomorphic, respectively, in D_1^1 and $\widetilde{D_1^1} = \{z : x_2 > h_f\}$; the functions Φ_j^2 and Υ_j^2 are holomorphic in D_j^2 and $\widetilde{D_1^2} = \{z : x_2 > -\varepsilon f(x_1)\}$, $\widetilde{D_2^2} = \{z : x_2 < -\varepsilon f(x_1)\}$, respectively.

Assuming $\alpha = \{0;\pi/2\}$ in Eq. (21) and taking the conditions (12) and (14) into account, we derive the values of Φ_j^k and Υ_j^k at infinity:

$$\lim_{x_2\to-\infty}\Phi_1^1(z) = \lim_{x_2\to-\infty}\Upsilon_1^1(z) = 0,\ \lim_{|x_2|\to\infty}\Phi_j^2(z) = \lim_{|x_2|\to\infty}\Upsilon_j^2(z) = T_j/4. \tag{22}$$

Unfortunately, it's impossible to derive the exact solution of the boundary equations (15)-(20) due to the curvature of the interface boundary Γ_2. However, using the boundary perturbation method [6, 7, 12, 14, 15], we could find the unknown functions Φ_j^k, Υ_j^k, p and ϑ_s in the first-order approximation:

$$\Psi(z) = \Psi_{(0)}(z) + \varepsilon\Psi_{(1)}(z), \tag{23}$$

where Ψ could be any of the listed functions.

The values of the functions $\Psi_{(m)}$ ($m = \{0,1\}$) on the curvilinear boundary Γ_2 can be presented by the linear Taylor polynomial in the vicinity of the line $x_2 = 0$, treating the real variable x_1 as a parameter:

$$\Psi_{(m)}(z_2) = \Psi_{(m)}(x_1) + i\varepsilon f(x_1)\Psi_{(m)}'(x_1). \tag{24}$$

The linearization in the space of the parameter ε for the subsequent functions from Eqs. (6) and (21) can be written as:

$$e^{-2i\alpha} = 1 - 2i\varepsilon f'(x_1),\ \kappa(x_1) = \varepsilon f''(x_1),\ h(x_1) = 1. \tag{25}$$

After that, we reduce the boundary conditions (15) and (16) to the Riemann-Hilbert problems on the boundary values of functions $\Phi^k_{j(m)}$ and $\Upsilon^k_{j(m)}$ and derive the integral dependencies of Goursat-Kolosov complex potentials on the unknown functions $p_{(m)}$, $\vartheta'_{s(m)}$ and $\tau'_{s(m)}$ for zero- and first-order approximations ($m = 0, 1$, respectively):

$$\begin{cases} \Upsilon^1_{1(m)}(z) = \int\limits_{-\infty}^{+\infty} K^1_1\left(p_{(m)}(\zeta), \vartheta'_{s(m)}(\zeta), z\right) d\zeta, \ \operatorname{Im} z > 0, \\ \Phi^1_{1(m)}(z) = \int\limits_{-\infty}^{+\infty} K^1_1\left(p_{(m)}(\zeta), \vartheta'_{s(m)}(\zeta), z\right) d\zeta, \ \operatorname{Im} z < 0 \end{cases} \tag{26}$$

$$\begin{cases} \Phi^2_{1(m)}(z) = \int\limits_{-\infty}^{+\infty} K^2_{11}\left(\tau'_{s(m)}(\zeta), p_{(m)}(\zeta), \vartheta'_{s(m)}(\zeta), z\right) d\zeta, \ \operatorname{Im} z > 0, \\ \Upsilon^2_{1(m)}(z) = \int\limits_{-\infty}^{+\infty} K^2_{12}\left(\tau'_{s(m)}(\zeta), p_{(m)}(\zeta), \vartheta'_{s(m)}(\zeta), z\right) d\zeta, \ \operatorname{Im} z < 0, \end{cases} \tag{27}$$

$$\begin{cases} \Upsilon^2_{2(m)}(z) = \int\limits_{-\infty}^{+\infty} K^2_{21}\left(\tau'_{s(m)}(\zeta), p_{(m)}(\zeta), \vartheta'_{s(m)}(\zeta), z\right) d\zeta, \ \operatorname{Im} z > 0, \\ \Phi^2_{2(m)}(z) = \int\limits_{-\infty}^{+\infty} K^2_{22}\left(\tau'_{s(m)}(\zeta), p_{(m)}(\zeta), \vartheta'_{s(m)}(\zeta), z\right) d\zeta, \ \operatorname{Im} z < 0. \end{cases} \tag{28}$$

Using Eqs. (17)-(20), we construct the integral equations in surface traction $p_{(m)}$, and derivatives of surface and interface stresses $\sigma'_{s(m)}$, $\vartheta'_{s(m)}$ and $\tau'_{s(m)}$, respectively. First of all, taking into account the constitutive equations (3)-(4) and inseparability conditions (7), we couple the surface and interface stresses with the tractions on the surface Γ_1 and interface Γ_2:

$$\vartheta_s(z_1) = \frac{\lambda^s_1 + 2\mu^s_1}{2(\lambda_1 + \mu_1)}\left[(\lambda_1 + 2\mu_1)\sigma^1_{tt}(z_1) - \lambda_1\sigma^1_{nn}(z_1)\right] + \gamma^0_1, \ z_1 \in \Gamma_1, \tag{29}$$

$$\sigma_s(z_1) = \frac{\lambda^s_1 + 2\mu^s_1}{2(\lambda_1 + \mu_1)}\left[(\lambda_1 + 2\mu_1)\left\{\sigma^1_{tt}(z_1) + \sigma^2_{tt}(z_1)\right\} - \lambda_1\left\{\sigma^1_{nn}(z_1) + \sigma^1_{nn}(z_1)\right\}\right] + \gamma^0_1, \ z_1 \in \Gamma_1, \tag{30}$$

$$\tau_s(z_2) = \frac{\lambda^s_2 + 2\mu^s_2}{2(\lambda_2 + \mu_2)}\left[(\lambda_2 + 2\mu_2)\sigma^2_{tt}(z_2) - \lambda_2\sigma^2_{nn}(z_2)\right] + \gamma^0_2, \ z_2 \in \Gamma_2. \tag{31}$$

The expressions for the stress tensor's components σ^k_{tt} and σ^k_{nn} of the first and second problem ($k = 1, 2$, respectively), can be found from Eq. (21), where one can sum the results for angles between the t-axis and the x_1-axis equal to α and $\alpha + \pi/2$:

$$\begin{aligned} &\sigma^k_{nn}(z) + i\sigma^k_{nt}(z) = \Phi^k_j(z) + \overline{\Phi^k_j(z)} - \left(\Upsilon^k_j(\overline{z}) + \overline{\Phi^k_j(z)} - (z - \overline{z})\,\overline{\Phi^{k\prime}_j(z)}\right)e^{-2i\alpha}, \\ &\sigma^k_{tt}(z) + \sigma^k_{nn}(z) = 4\operatorname{Re}\Phi^k_j(z), \ z \in \Omega_j. \end{aligned} \tag{32}$$

Then, employing the expressions (23) and (24), we rewrite boundary equations (17), (29)-(31) in terms of complex potentials $\Phi^k_{j(m)}$ and $\Upsilon^k_{j(m)}$ for zero- and first-order approximations ($m = 0, 1$, respectively) of boundary perturbation method:

$$p_{(m)}(x_1) - i\vartheta'_{s(m)}(x_1) + i\sigma'_{s(m)}(x_1) = \Upsilon^2_{1(m)}(x_1) - \Phi^2_{1(m)}(x_1) - 2ih\overline{\Phi^{2\prime}_{1(m)}(x_1)}, \tag{33}$$

$$\vartheta_{s(m)}(x_1) = M_1\operatorname{Re}\left[\varkappa_1\Phi^1_{1(m)}(x_1) + \Upsilon^1_{1(m)}(x_1)\right] + W^1_m, \tag{34}$$

$$\sigma_{s(m)}(x_1) = M_1 \mathrm{Re}\left[\varkappa_1\left\{\Phi^1_{1(m)}(x_1) + \Phi^2_{1(m)}(x_1)\right\} + \left\{\Upsilon^1_{1(m)}(x_1) + \Upsilon^2_{1(m)}(x_1)\right\} - 2ih\overline{\Phi^{2\prime}_{1(m)}(x_1)}\right] + W^2_m, \quad (35)$$

$$\tau_{s(m)}(x_1) = M_2 \,\mathrm{Re}\left[\varkappa_2\, \Phi^2_{2(m)}(x_1) + \Upsilon^2_{2(m)}(x_1)\right] + W^3_m(x_1), \quad (36)$$

where $W^1_0 = W^2_0 = \gamma^0_1$, $W^1_1 = W^2_1 = 0$, $W^3_0(x_1) = \gamma^0_2$, $M_1 = (\lambda^1_s + 2\mu^1_s)/2\mu_1$, $M_2 = (\lambda^2_s + 2\mu^2_s)/2\mu_2$, $W^3_1(x_1) = M\,\mathrm{Re}\left[if(x_1)\left(\varkappa_2\Phi^{2\prime}_{2(0)}(x_1) - \Upsilon^{2\prime}_{2(0)}(x_1) - 2\overline{\Phi^{2\prime}_{2(0)}(x_1)}\right) - 2if'(x_1)\left(\overline{\Phi^2_{2(0)}(x_1)} + \Upsilon^2_{2(0)}(x_1)\right)\right]$.

Thus, in view of Eqs. (26)-(28), we arrive to integro-differential equations in functions $p_{(m)}$, $\sigma_{s(m)}$, $\vartheta_{s(m)}$ and $\tau_{s(m)}$. For convenience, we transform Eqs. (34)-(36) to hypersingular integral equations taking into account the Sokhotski-Plemelj formulas for the limiting values of the Cauchy-type integrals [13] and differentiating with respect to the argument x_1. Finally, the system of the boundary equations (15)-(20) takes the form of the integral equations system in the unknown functions $p_{(m)}$, $\vartheta'_{(m)}$, $\sigma'_{s\,(m)}$ and $\tau'_{s\,(m)}$ that consists of one singular and three hypersingular equations. The kernels of these integral equations are the same for each step of approximation. The right-hand sides are the known continuos functions.

In the case of the zero-order approximation, we arrive at the homogeneous integral equations which have only zero solution following from the physical considerations. Taking into account Eq. (22), the complex potentials of the zero-order approximation can be written as:

$$\begin{aligned} &\Phi^1_{1\,(0)}(z) = \Upsilon^1_{1\,(0)}(z) = 0, \;\; z \in \Omega_1, \\ &\Phi^2_{j\,(0)}(z) = \Upsilon^2_{j\,(0)}(z) = T_j/4, \;\; z \in \Omega_j. \end{aligned} \quad (37)$$

As follows from Eqs. (9), (10) and (21), they correspond to the piecewise uniform stress state of the film coating with flat interface:

$$\begin{aligned} &\sigma_{11(0)}(z) = T_j, \;\; z \in \Omega_j, \\ &\sigma_{s(0)}(z_1) = \gamma^0_1 + \frac{M_1(1+\varkappa_1)}{4}T_1, \;\; \tau_{s(0)}(z_2) = \gamma^0_2 + \frac{M_2(1+\varkappa_2)}{4}T_2. \end{aligned} \quad (38)$$

In the case of the first-order approximation, we obtain the solution in the terms of trigonometric polynomials of degree $N = 1$ with the unknown coefficients A_l and B_l ($l = \{1, 4\}$):

$$\begin{aligned} &p_{(1)}(x_1) = A_1 \sin(k_a x_1) + B_1 \cos(k_a x_1), \;\; \vartheta'_{s(1)}(x_1) = A_2 \sin(k_a x_1) + B_2 \cos(k_a x_1), \\ &\sigma'_{s(1)}(x_1) = A_3 \sin(k_a x_1) + B_3 \cos(k_a x_1), \;\; \tau'_{s(1)}(x_1) = A_4 \sin(k_a x_1) + B_4 \cos(k_a x_1). \end{aligned} \quad (39)$$

Based on the properties of the Cauchy-type integrals, the system of the integral equations (33)-(36) is reduced to the linear system of algebraic equations for the unknown coefficients A_l and B_l. After finding these coefficients, one can define the complex potentials $\Phi^k_{j(1)}$ and $\Upsilon^k_{j(1)}$ (j,k={1,2}) from Eqs. (26)-(28) and, as a consequence, the solution of the original boundary value problem from Eq. (21).

RESULTS AND CONCLUSIONS

In order to study the size-effect caused by a nanoscale roughness of interface between an ultra-thin film and a substrate, we consider the dependence of stress concentration factor $S = \max \sigma^-_{tt}/T_2$, where $\sigma^-_{tt}(z_2) = \lim_{z \to z_2 - i0} \sigma_{tt}(z)$, on the perturbation wavelength a for the different film thicknesses $h_f = 1.5A$, $5A$ (Fig. 1 (a) and (b), respectively) and stiffness ratios $\mu_1/\mu_2 = 0.1$, 0.3 (red and green lines, respectively) in the case of $A = 0.1a$, $\gamma^0_1 = \gamma^0_2 = 0$, $\nu_1 = \nu_2 = 0.3$. As it was shown in [6], the hoop stresses reach their maximum values at the bottom of asperities in the stiffer material (i.e. σ^-_{tt} when $\mu_1/\mu_2 < 1$ and $\sigma^+_{tt} = \lim_{z \to z_2 + i0} \sigma_{tt}(z)$ when $\mu_1/\mu_2 > 1$) and the corresponding stresses at the same point on the other side of interface boundary (i.e. σ^+_{tt} when $\mu_1/\mu_2 < 1$ and σ^-_{tt} when $\mu_1/\mu_2 > 1$) are approximately as many times smaller as Young's modulus of the softer material is smaller than that of the stiffer material. So, here, we only consider the case when $\mu_1/\mu_2 < 1$. For comparison, we present the classical solution without surface and interface stresses when $M_1 = M_2 = 0$ (dashed lines) [6], and the solution based on Gurtin-Murdoch model [21, 22] with $M_1 = M_2 = 0.117$ nm (continuous lines) [18, 19].

One can see from Fig. 1 that consideration of surface and interface stresses reduce the stress concentration factor S^-, but their combined effect decreases when the wavelength a increases and the solution approaches the classical one (i.e. the dashed lines). However, the influence of the length scale parameter a decreases with the stiffening of the film. It can be seen that the stress concentration factor S decreases when the stiffness ratio μ_1/μ_2 as well as film thickness h_f increases. Note that when the film is sufficiently thick (i.e. $h_f > 2a$) the values of stress concentration factor are close to the corresponding values for bi-material system that is modeled as two semi-infinite regions with joint curvilinear interface boundary [14]. Also it should be noted that when the stiffness ratio $\mu_1/\mu_2 \to 0$ the values of stress concentration factor pass to the corresponding values for solid with undulated surface [15].

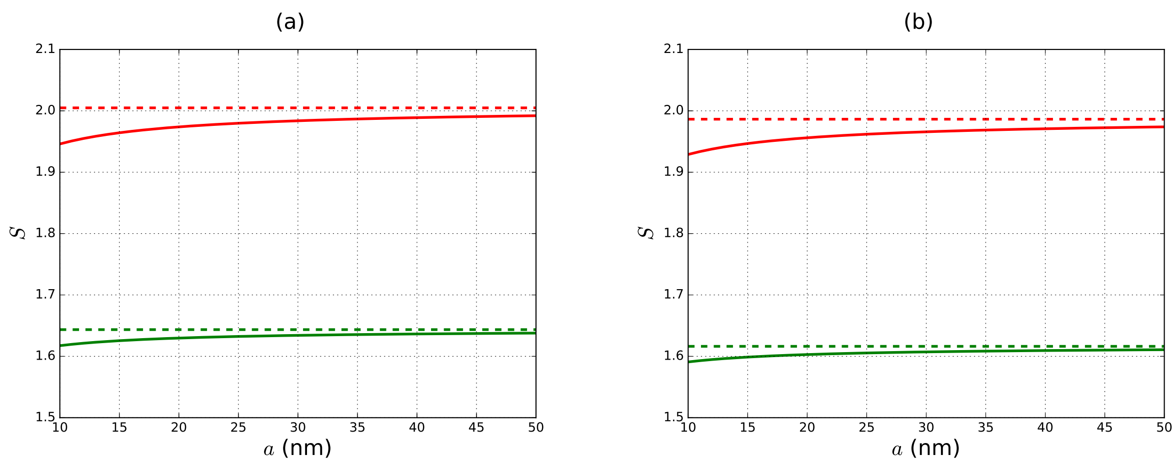

FIGURE 1. Stress concentration factor $S = \max \sigma^-_{tt}/T_2$ as a function of interface perturbation wavelength a.

As a conclusion, this research extends our previous model developed to study the stress distribution in thin film coating with slightly curved interface [6] to the case when the film thickness and the interface roughness are in nanometer range. The boundary value problem was formulated for a four-phase system involving two-dimensional constitutive equations for bulk materials and one-dimensional equations for membrane-type surface and interface under the assumption of plane strain conditions. The mixed boundary conditions were written in the terms of generalized Young-Laplace equations and relations describing the continuous of displacements across the surface and interphase regions. Using the perturbation technique combined with the superposition principle, the integral dependences of Goursat-Kolosov's complex potentials on the surface and interface stresses were derived in the first-order approximation. Then, based on the constitutive equations of Gurtin-Murdoch model, the inseparability conditions at surface and interface were satisfied providing the system of one singular and three hypersingular integral equations. The solution was obtained in terms of first-degree trigonometric polynomials. Numerical results have been presented and the size effect has been analysed. It should be mentioned, that the obtained solution is in a good agreement with our previous studies [6, 14, 15].

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