Robust Schur Stability of a Polynomial Matrix Family*

Elizaveta Kalinina[1][0000−0003−0288−3938], Yuri Smol'kin[1], and Alexei Uteshev[1][0000−0002−8344−3266]

Faculty of Applied Mathematics, St. Petersburg State University,
7–9 Universitetskaya nab., St. Petersburg, 199034 Russia
{e.kalinina@spbu.ru,st040343@student.spbu.ru,a.uteshev@spbu.ru}
http://www.apmath.spbu.ru/ru/

Abstract. The problem of robust Schur stability of a polynomial matrix family is considered as that of discovering the structure of the stability domain in parameter space. The algorithms are proposed for establishing whether or not any given box in the parameter space belongs to this domain, and for finding the distance to instability from any internal point of the domain to its boundary. The treatment is performed in the ideology of analytical algorithm for elimination of variables and localization of zeros of algebraic systems. Some examples are given.

Keywords: Matrix polynomials · Robust Schur stability · Parameters · Discriminant.

1 Introduction

For a polynomial $f(z) \in \mathbb{C}[z]$, its Schur stability (or D-stability) property is defined as that of location of all its zeros inside the unit disc of the complex plane:

$$D = \{ z \in \mathbb{C} \mid |z| < 1 \} . \tag{1}$$

The same definition relates to the matrix $M \in \mathbb{C}^{n \times n}$ with numerical entries if the whole spectrum lies inside D. The D-stability property is of an importance for estimating the behavior of solutions to difference equation systems [11]. There exist several criteria for establishing the Schur stability of a polynomial in terms of its coefficients, for instance the Schur – Cohn or Jury's criteria [17, 20, 25].

The counterpart of the problem for matrices with entries depending on parameters varying within some set $\mathfrak{B}$ is sometimes referred to as the **robust Schur stability** problem with the meaning that all the matrices of this family should be D-stable. This property is of vital importance for the parameter synthesis in Control Theory. In digital signal processing applications such as sampling rate conversion, echo cancellation, phased-array antenna systems, time delay estimation, timing adjustment in all-digital receivers, modelling of music

* Supported by RFBR, project No. **17-29-04288**.

instruments, and speech coding and synthesis, there is a need to design a digital filter with predicted characteristics [4, 28, 34]. Therefore, the tolerances are to be estimated for the permissible parameter variations.

In the present paper, we will tackle the D-stability problem for the matrix family

$$\left\{M(\mu) = [m_{jk}(\mu)]_{j,k=1}^{n} \ \mid \ \mu = (\mu_1, \mu_2, \dots, \mu_k) \in \mathfrak{B}\right\} . \tag{2}$$

Here $\{m_{jk}(\mu)\}_{j,k=1}^{n}$ are real polynomials in μ while $\mathfrak{B}$ is a box:

$$\mathfrak{B} = \left\{\mu_1^- \le \mu_1 \le \mu_1^+, \mu_2^- \le \mu_2 \le \mu_2^+, \dots, \mu_k^- \le \mu_k \le \mu_k^+\right\} \subset \mathbb{R}^k . \tag{3}$$

For the case of symmetric matrices, the robust Schur stability problem is treated in recent book [26]. In [7, 15], necessary and sufficient conditions for the zeros of arbitrary polynomial matrix to belong to a given region D of the complex plane as a linear matrix inequality (LMI) feasibility problem are formulated. To solve this problem, interior-point methods are used. In [8], analysis and synthesis techniques for quadratic stability in LMI regions that embrace most practically useful stability regions are discussed.

In [6], robust Schur stability of a polynomial matrix family is reduced to the positivity of multivariate polynomials, and the Bernstein expansion method [19] is applied to test positivity of the obtained polynomials.

In Section 3, we first detail the structure of the boundary of the set of Schur stable matrices $M(\mu)$ in the parameter space. Then we propose an algebraical approach to the problem of testing the stability of family (2). The algorithm is based on the Le Verrier method for the computation of the characteristic polynomial of a matrix [33] and algebraic procedures for checking the positivity property of multivariate polynomials in the given domain [18]. Another problem dealt with in Section 3 is that of finding **the distance to instability** in the parameter space, i.e., the Euclidean distance $d_*(\mu^{(0)})$ from a given point $\mu^{(0)} \in \mathbb{R}^k$ corresponding to a stable matrix $M(\mu^{(0)})$ to the nearest point $\mu_* \in \mathbb{R}^k$ at the boundary of domain of stable matrices [26]. This notion should be distinguished from the one related to the *distance to instability in the matrix space* or **stability radius**. The latter is defined for a stable matrix $A \in \mathbb{R}^{n\times n}$ as the Frobenius norm of the matrix $E \in \mathbb{R}^{n\times n}$ such that the matrix $A+E$ is the nearest to A unstable matrix [1]. This definition can be treated as a particular case of the first one for the matrix set $\left\{A + [\mu_{j\ell}]_{j,\ell=1}^{n}\right\}$.

In Section 4, some numerical examples are presented illuminating the efficiency of the suggested algorithms.

Hereinafter the word *stability* should be understood in the meaning *Schur stability* (*D-stability*).

2 Algebraic Preliminaries

Here we give some auxiliary results regarding the properties of the zero sets of polynomials.

2.1 Newton sums of a polynomial

Consider a polynomial

$$f(z) = a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \ldots + a_n, \quad \{a_0 \neq 0, a_1, \ldots, a_n\} \subset \mathbb{R}, \qquad (4)$$

and denote $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset \mathbb{C}$ its zeros counted with their multiplicities.

The **Newton sums** of the polynomial $f(z)$ are formally defined as

$$s_0 = n; \; s_k = \sum_{j=1}^{n} \alpha_j^k \text{ for } \; k \in \mathbb{N},$$

while the **Newton identities** [14]

$$s_0 = n; s_1 = -a_1/a_0;$$
$$s_k = \begin{cases} -(a_1 s_{k-1} + a_2 s_{k-2} + \ldots + a_{k-1} s_1 + k a_k)/a_0 & \text{if } k \in \{2, \ldots, n\}, \\ -(a_1 s_{k-1} + a_2 s_{k-2} + \ldots + a_n s_{k-n})/a_0 & \text{if } k > n. \end{cases} \qquad (5)$$

allow one to compute recursively these sums as rational functions (polynomials if $a_0 = 1$) of the coefficients of $f(z)$.

Conversely, if for some reason, the canonical representation (4) of a normalized polynomial is not granted but we are able to compute somehow its Newton sums, the following inversions of the Newton identities

$$\begin{aligned} a_1 &= -s_1; a_2 = -(s_2 + a_1 s_1)/2; \\ a_k &= -(s_k + a_1 s_{k-1} + a_2 s_{k-2} + \ldots + a_{k-1} s_1)/k, \text{ if } k \in \{3, \ldots, n\} \end{aligned} \qquad (6)$$

allow one to restore the coefficients of $f(z)$. This opportunity gives rise to the Le Verrier method [12, 14] for computation of the characteristic polynomial of a matrix $A \in \mathbb{C}^{n \times n}$. Indeed, the Newton sums of this polynomial can be evaluated via computation of the traces of powers of the matrix A:

$$s_k = \operatorname{Tr}(A^k) \quad \text{for} \;\; k \in \mathbb{N}\,. \qquad (7)$$

It turns out that with the aid of the sequence of Newton sums of a polynomial, one can express some symmetric function of the pairs of zeros of this polynomial. We will utilize further two such functions.

Theorem 1. *Set the Newton sums of the polynomial with zeros $\alpha_\ell \alpha_k$ of degree $n(n-1)/2$*

$$S_j := \sum_{1 \le \ell < k \le n} \alpha_\ell^j \alpha_k^j \quad \textit{for } j \in \mathbb{N}\,.$$

Then

$$S_j = (s_j^2 - s_{2j})/2\,. \qquad (8)$$

Proof. One has

$$s_j^2 = \left(\sum_{k=1}^{n} \alpha_k \right)^2 = \sum_{k=1}^{n} \alpha_k^2 + 2 \sum_{1 \le \ell < k \le n} \alpha_\ell^j \alpha_k^j = s_{2j} + 2 S_j\,.$$

□

The second symmetric function of the zeros of the polynomial $f(z)$ is formally defined as

$$\mathcal{D}(f(z)) := a_0^{2n-2} \prod_{1\le \ell < k \le n} (\alpha_\ell - \alpha_k)^2$$

and is known as the **discriminant** of the polynomial $f(z)$. It vanishes iff the polynomial $f(z)$ possesses a multiple zero (or, equivalently, iff $f(z)$ possesses a common zero with $f'(z)$). For the aim of expressing $\mathcal{D}(f(z))$ via the Newton sums of $f(z)$, we introduce the **Hankel determinant**

$$H_k := \det[s_{j+\ell}]_{j,\ell=0}^{k-1} = \left| \begin{array}{lllll} s_0 & s_1 & s_2 & \dots & s_{k-1} \\ s_1 & s_2 & s_3 & \dots & s_k \\ \vdots & \vdots & \vdots & & \vdots \\ s_{k-1} & s_k & s_{k+1} & \dots & s_{2k-2} \end{array} \right| . \tag{9}$$

Theorem 2. *The following equality is valid:*

$$\mathcal{D}(f(z)) = a_0^{2n-2} H_n \, . \tag{10}$$

It turns out that the sequence of Hankel determinants $\{H_k\}_{k=1}^n$ introduced by (9) permits one to establish the exact numbers of real zeros for the polynomial $f(z)$. Moreover, a slight modification of these determinants allows one to construct a sequence of polynomials that localize all its real zeros. For this aim, introduce the parameter dependent determinant

$$\mathcal{H}_k(z) := \left| \begin{array}{lllll} s_0 & s_1 & s_2 & \dots & s_k \\ s_1 & s_2 & s_3 & \dots & s_{k+1} \\ \vdots & \vdots & \vdots & & \vdots \\ s_{k-1} & s_k & s_{k+1} & \dots & s_{2k-1} \\ 1 & z & z^2 & \dots & z^k \end{array} \right| . \tag{11}$$

Its expansion by the last row yields the polynomial in z

$$\mathcal{H}_k(z) \equiv \sum_{j=0}^{k} h_{kj} z^{k-j} ,$$

which is sometimes called the kth **Hankel polynomial** generated by the sequence $\{s_j\}$. It is evident that $h_{k0} = H_k$.

Theorem 3 (Jacobi, Joachimsthal [16, 20, 27]). *Let* $H_n = 0, \dots, H_{\mathfrak{r}+1} = 0, H_{\mathfrak{r}} \ne 0, \dots, H_1 \ne 0$. *Then*

(i) *The number of distinct zeros of* $f(z)$ *equals* $\mathfrak{r}$*;*

(ii) *The number of distinct real zeros of* $f(z)$ *equals*

$$\mathcal{P}(1, H_1, \dots, H_{\mathfrak{r}}) - \mathcal{V}(1, H_1, \dots, H_{\mathfrak{r}}) \ ;$$

(iii) *The number of distinct real zeros of* $f(z)$ *lying in the interval* $[a, b], a < b$ *equals*

$$\mathcal{V}(1, \mathcal{H}_1(a), \dots, \mathcal{H}_{\mathfrak{r}}(a)) - \mathcal{V}(1, \mathcal{H}_1(b), \dots, \mathcal{H}_{\mathfrak{r}}(b)) \, .$$

Here $\mathcal{P}$ (or $\mathcal{V}$) stands for the number of permanences (variations) of sign for the given sequences[1].

Corollary 1. *The following identity is valid:*

$$\mathcal{H}_n(z) \equiv H_n f(z)/a_0 \,.$$

Remark. One may notice an evident relationship of the part (iii) of Theorem 3 to the algorithm of localization of zeros of the polynomial $f(z)$ based on the Sturm – Habicht sequence construction [2]. The principal distinction in the two procedures is that the Sturm – Habicht algorithm constructs the sequence of polynomials with decreasing degrees starting with the polynomial $f(z)$, whilst the Jacobi – Joachimsthal sequence is composed of polynomials with increasing degrees, with the final one coinciding with $f(z)$. This distinction is of importance for the class of problems where the polynomial $f(z)$ is not a priory represented in canonical form like the above mentioned problem related to characteristic polynomial of a matrix. With Newton sums evaluated via (7), computation of sequence $\mathcal{H}_1(z), \ldots, \mathcal{H}_n(z)$ results in this polynomial and furnishes, *free of an additional charge*, an opportunity to locate its real zeros.

Corollary 2. *[29]. If $H_n = 0, H_{n-1} \neq 0$, then the polynomial $f(z)$ possesses a single multiple zero with its multiplicity equal to 2. This zero can be expressed via the two coefficients of the polynomial $\mathcal{H}_{n-1}(z)$:*

$$\alpha = s_1 + h_{n-1,1}/H_{n-1} \,. \tag{12}$$

If $H_n = H_{n-1} = \ldots = H_{n-k+1} = 0, H_{n-k} \neq 0, k > 1$, then $\gcd(f(z), f'(z))$ is of the order k and can be expressed with the aid of the kth order minors of H_n [29].

Computation of the sequence of Hankel polynomials in part (iii) of Theorem 3 can be optimized with the following result.

Theorem 4 (Jacobi, Joachimsthal [16, 30]). *Let $k \in \{3, \ldots, n\}$. If $H_{k-1} \neq 0, H_k \neq 0$, then the following identity is valid:*

$$\frac{H_k}{H_{k-1}}\mathcal{H}_{k-2}(z) - \left(z - \frac{h_{k-1,1}}{H_{k-1}} + \frac{h_{k1}}{H_k}\right)\mathcal{H}_{k-1}(z) + \frac{H_{k-1}}{H_k}\mathcal{H}_k(z) \equiv 0. \tag{13}$$

Formula (13) permits one to compute the sequence of polynomials $\{\mathcal{H}_k(z)\}_{k=1}^n$ recursively with every polynomial computed as linear combination of two preceding ones and with the two constants involved also evaluated via the coefficients of these polynomials (v. [30]):

$$\left\{ \begin{array}{lcr} H_k & = & s_{k-1}h_{k-1,k-1} + s_k h_{k-1,k-2} + \ldots + s_{2k-2}h_{k-1,0}, \\ h_{k1} & = & -(s_k h_{k-1,k-1} + s_{k+1}h_{k-1,k-2} + \ldots + s_{2k-1}h_{k-1,0}). \end{array} \right. \tag{14}$$

[1] $\mathcal{P}(A_1, \ldots, A_K) := \sum_{j=1}^{K-1} \mathcal{P}(A_j, A_{j+1})$ where $\mathcal{P}(A_j, A_{j+1}) := 1$ if $A_j A_{j+1} > 0$ and $\mathcal{P}(A_j, A_{j+1}) := 0$ if $A_j A_{j+1} < 0$. $\mathcal{V}(A_1, \ldots, A_K)$ is defined similarly with $\mathcal{V}(A_j, A_{j+1}) := 1 - \mathcal{P}(A_j, A_{j+1})$.

Given polynomial (4) and a polynomial

$$g(z) = b_0 z^m + b_1 z^{m-1} + \cdots + b_{m-1} z + b_m \in \mathbb{C}[z],\ b_0 \neq 0$$

with zeros $\beta_1, \dots, \beta_m$, the **resultant** of these polynomials is formally defined by the formula

$$\mathcal{R}(f,g) = a_0^m b_0^n \prod_{\ell=1}^{n} \prod_{j=1}^{m} (\alpha_\ell - \beta_j), \tag{15}$$

while practically can be expressed as a polynomial in the coefficients of $f(z)$ and $g(z)$ using several determinantal representations. The mostly known is the Sylvester representation

$$\mathcal{R}(f,g) = \left| \begin{array}{cccccccccc} a_0 & a_1 & a_2 & \dots & & a_n & 0 & 0 & \dots & 0 \\ 0 & a_0 & a_1 & \dots & & & a_n & 0 & \dots & 0 \\ \dots & & & & & & & & & \\ 0 & 0 & 0 & \dots & a_0 & a_1 & a_2 & \dots & & a_n \\ b_0 & b_1 & \dots & & b_m & 0 & \dots & & & 0 \\ 0 & b_0 & b_1 & \dots & & b_m & 0 & 0 & \dots & 0 \\ \dots & & & & & & & & & \\ 0 & 0 & 0 & \dots & b_0 & b_1 & b_2 & \dots & & b_m \end{array} \right| \begin{array}{l} \left.\vphantom{\begin{array}{c}a\\a\\a\\a\end{array}}\right\} m \text{ rows} \\ \left.\vphantom{\begin{array}{c}a\\a\\a\\a\end{array}}\right\} n \text{ rows} \end{array} .$$

Obviously, polynomials $f(z)$ and $g(z)$ have a common zero iff $\mathcal{R}(f,g) = 0$.

Remark. In some further formulas involving discriminant or resultant, we will occasionally specify the variable of the considered polynomials with subscripts like $\mathcal{D}_z$ or $\mathcal{R}_z$.

2.2 Positivity of a multivariate polynomial

We present here a multidimensional counterpart for Theorem 3. We wish to establish the relative position of the manifold given implicitly in $\mathbb{R}^k, k \geq 2$ by an algebraic equation to a specified box

$$\mathfrak{B} = \left\{ x_1^- \leq x_1 \leq x_1^+, x_2^- \leq x_2 \leq x_2^+, \dots, x_k^- \leq x_k \leq x_k^+ \right\} \tag{16}$$

in this space. For the proof of the following result, we refer to [18].

Theorem 5. *Consider a multivariate polynomial*

$$G(X) \in \mathbb{R}[X], X := (x_1, \dots, x_k) \in \mathbb{C}^k, k \geq 2, \deg G = N \geq 1$$

and expand it in powers of one of the variables, say x_1:

$$G(X) \equiv a_0(x_2, \dots, x_k) x_1^N + \ldots + a_N(x_2, \dots, x_k) .$$

Let $a_0(x_2, \dots, x_k) \neq 0$ in the box

$$\mathfrak{B}_1 = \left\{ x_2^- \leq x_2 \leq x_2^+, \dots, x_k^- \leq x_k \leq x_k^+ \right\} .$$

Polynomial $G(X)$ does not vanish in the box (16) iff the following conditions hold:

(i) *For any $j \in \{2,3,\dots,k\}$, the polynomials*

$$G_j^- := G(x_1,\dots,x_{j-1},x_j^-,x_{j+1},\dots,x_k)$$

and

$$G_j^+ := G(x_1,\dots,x_{j-1},x_j^+,x_{j+1},\dots,x_k)$$

do not vanish in the box

$$\mathfrak{B}_j = \big\{\mu_1^- \le \mu_1 \le \mu_1^+,\dots, \quad \mu_{j-1}^- \le \mu_{j-1} \le \mu_{j-1}^+, \quad \mu_{j+1}^- \le \mu_{j+1} \le \mu_{j+1}^+, \quad \dots, \mu_k^- \le \mu_k \le \mu_k^+\big\}.$$

(ii) *The system of equations*

$$G(X)=0,\ \partial G(X)/\partial x_2 = 0,\dots,\ \partial G(X)/\partial x_k = 0 \tag{17}$$

does not have real zeros in the box (16).

For the case of a bivariate polynomial $G(x,y)$, the condition of the part (ii) of the above theorem can be immediately verified via application of Theorem 3. As for the condition (ii), system (17) takes then the form

$$G(x,y)=0, \quad \partial G(x,y)/\partial y = 0\,. \tag{18}$$

One can eliminate the variable y from this system using the discriminant technique introduced in Section 2.1. Assuming that this discriminant $F_1(x) := \mathcal{D}_y(G(x,y))$ is not identically zero (i.e., the set of zeros of (18) is zero-dimensional), it is a univariate polynomial. Under additional assumption of the absence of multiple zeros for $F_1(x)$, its real zeros yield the x-component of the real zeros of system (18). To establish the (non)existence of the latter in the box

$$\mathfrak{B} = \big\{a^- \le x \le a^+, b^- \le y \le b^+\big\}$$

one can first verify, via Theorem 3, if the polynomial $F_1(x)$ does not have real zeros in $[a^-,a^+]$. If it does, then any such a zero can be localized with the aid of the mentioned theorem within an arbitrary prescribed accuracy. Application of the result of Corollary 2 permits one to evaluate y-component of the corresponding zero of system (18) and to verify if it does not lie in $[b^-,b^+]$.

Remark. The suggested approach has an evident relationship to a real quantifier elimination (QE) problem [3, 24].

The just outlined scheme for checking the conditions of Theorem 5 for the bivariate case can be extended for the general case of a multivariate polynomial. It is connected with one notion that is introduced at the end of the next subsection.

2.3 Distance from a Point to an Algebraic Manifold

We treat here the problem of Euclidean distance evaluation from a point X_0 to the algebraic manifold defined implicitly by the equation

$$G(X) = 0 \tag{19}$$

in $\mathbb{R}^k, k \in \{2,3\}, \deg G > 1$. For this aim, we utilize the construction of the so-called **distance equation**, i.e., the algebraic univariate equation whose zero set coincides with that of the critical values of the squared distance function from the point to the manifold [31].

Theorem 6. *Let $G(0,0) \neq 0$, and $G(x,y)$ be an even polynomial in y. Expand G in powers of y^2 and denote $\widetilde{G}(x,y^2) \equiv G(x,y)$. Equation $G(x,y)=0$ does not define a real curve if*
(i) *equation $G(x,0)=0$ does not have real zeros and*
(ii) *distance equation*

$$\mathcal{F}(z) := \mathcal{D}_x(\widetilde{G}(x, z - x^2)) = 0 \tag{20}$$

does not possess positive zeros. If any of these conditions fails, then the distance from $X_0 = (0,0)$ to the curve $G(x,y)=0$ equals the minimal of the two values:
1. the minimal absolute value of real zeros of the equation $G(x,0)=0$,
2. the square root of the minimal positive zero of equation (20) provided that this zero is not a multiple one.

Remark. The condition of simplicity of the minimal positive zero of the distance equation appeared in theorems of the present subsection is essential since in some (fortunately exceptional) cases, this zero is generated by a pair of imaginary points in the manifold [32].

The generalization of this result to the case of an arbitrary polynomial $G(x,y)$, not necessarily even in any of its variables, can be performed by reduction to the just treated one via artificial *evenization* of the problem. Unfortunately, this causes the appearance of an extraneous factor in the distance equation.

Theorem 7. *Let $G(0,0) \neq 0$, and $G(x,y)$ be not an even polynomial in y. Split G into the sum of even and odd terms in this variable:*

$$G(x,y) \equiv G_1(x,y^2) + yG_2(x,y^2), \ \{G_1, G_2\} \subset \mathbb{R}[x,y^2]\,.$$

Let

$$\widetilde{G}(x,y^2) := G(x,y)G(x,-y) \equiv G_1^2(x,y^2) - y^2 G_2^2(x,y^2)$$

and compute the polynomial $\mathcal{F}(z)$ via (20). The latter is reducible over $\mathbb{R}$:

$$\mathcal{F}(z) \equiv \mathcal{F}_1(z)\mathcal{F}_2^2(z) \ \textit{with} \ \mathcal{F}_2(z) := \mathcal{R}_x(G_1(x, z-x^2), G_2(x, z-x^2))\,.$$

Equation $G(x,y)=0$ does not define a real curve if

(i) *equation* $G(x,0)=0$ *does not possess real zeros and*
(ii) *distance equation*

$$\mathcal{F}_1(z)=0 \tag{21}$$

does not possess positive zeros.

If any of these conditions fails, then the distance from $X_0=(0,0)$ *to the curve* $G(x,y)=0$ *equals the minimal of the two values:*

1. the minimal absolute value of real zeros of the equation $G(x,0)=0$*,*

2. the square root of the minimal positive zero of the equation (21) provided that this zero is not a multiple one.

Conditions **(i)** and **(ii)** of Theorems 6 and 7 can be verified using symbolic algebraic algorithms from Theorem 3. Equations (20) and (21) are the distance equations for the point $X_0=(0,0)$ and the curve $G(x,y)=0$. For arbitrary point $X_0=(x_0,y_0)$, this equation can be extracted from the corresponding theorem via shifting the origin: $\widehat{G}(x,y):=G(x+x_0,y+y_0)$. Generically, one gets $\deg \mathcal{F}_1(z)=(\deg G)^2$ for (21).

The treatment of the problem for the case of surfaces in $\mathbb{R}^3$ can be organized in a similar manner. Consider, for instance, a polynomial that is even in one of its variables, say $G(x_1,x_2,x_3^2)$. The distance equation for the point $(0,0,0)$ and the surface $G=0$ can be obtained as a result of elimination of the variables x_1,x_2 from the system of equations

$$\widetilde{G}=0,\ \partial\widetilde{G}/\partial x_1=0,\ \partial\widetilde{G}/\partial x_2=0 \quad \text{for } \widetilde{G}(x_1,x_2,z):=G(x_1,x_2,z-x_1^2-x_2^2)\,.$$

One may notice the similarity of the obtained system with system (17). The common underlying notion is known as the **multivariate discriminant**, i.e., an algebraic function of the coefficients of a multivariate polynomial $G(x_1,\dots,x_n)$ responsible for the existence of a multiple zero for this polynomial, i.e., zero for the system

$$G=0,\ \partial G/\partial x_1=0,\dots,\partial G/\partial x_n=0\,.$$

There are different approaches for constructive computation of this object with the universal one based on the Gröbner basis construction. For computations in Example 3 considered further, we utilize the procedure of the multivariate polynomial resultant computation based on a certain determinantal representation via its coefficients [5].

3 Stability Domain in the Parameter Space

3.1 Structure of the Boundary

Theorem 8. *For matrix (2), consider the characteristic polynomial and its reciprocal:*

$$f(z;\mu):=\det(zI-M(\mu)) \quad \textit{and} \quad f^*(z;\mu)\equiv z^n f(1/z;\mu)\,.$$

Assume that matrix (2) is nonsingular for μ in $\mathfrak{B}$ with the probable exception of manifold of codimension 1. Family (2) is stable iff its arbitrary member is stable and the polynomial

$$\Phi(\mu) := \mathcal{R}_z(f(z;\mu), f^*(z;\mu)) \tag{22}$$

is positive for $\mu \in \mathfrak{B}$.

Proof. If we denote by $\alpha_1(\mu), \dots, \alpha_n(\mu)$ the zeros of

$$f(z;\mu) = z^n + a_1(\mu)z^{n-1} + \dots + a_{n-1}(\mu)z + a_n(\mu),$$

then those of $f^*(z;\mu)$ are $1/\alpha_1(\mu), \dots, 1/\alpha_n(\mu)$. Due to the definition of resultant (15), one obtains

$$\Phi(\mu) = \mathcal{R}_z(f(z;\mu), f^*(z;\mu)) = a_n^n(\mu) \prod_{j,k=1}^{n} (\alpha_j(\mu) - 1/\alpha_k(\mu))$$

$$= \frac{(-1)^{n^2} a_n^n(\mu)}{\prod_{j=1}^{n} \alpha_j^n(\mu)} \prod_{j,k=1}^{n} (1 - \alpha_j(\mu)\alpha_k(\mu)) = \prod_{j,k=1}^{n} (1 - \alpha_j(\mu)\alpha_k(\mu))$$

$$= \prod_{j=1}^{n} (1 - \alpha_j(\mu)) \prod_{j=1}^{n} (1 + \alpha_j(\mu)) \prod_{1 \le j < k \le n} (1 - \alpha_j(\mu)\alpha_k(\mu))^2 \,. \tag{23}$$

If the matrix $M(\mu)$ is stable for some specialization of parameter $\mu = \mu_0 \in \mathfrak{B}$, then $\Phi(\mu_0) > 0$. When the parameter μ varies continuously within the (simply connected domain) $\mathfrak{B}$ starting from this value, the eigenvalues $\{\alpha_j(\mu)\}_{j=1}^n$ of the matrix drift continuously within disk (1). The inequality $\Phi(\mu) > 0$ keeps to be valid until either any real eigenvalue α_j reaches ± 1, or a pair of complex conjugate eigenvalues $\{\alpha_j(\mu), \alpha_k(\mu) = \overline{\alpha_j(\mu)}\}$ reaches the unit circle. Therefore, the condition $\Phi(\mu) > 0$ prevents the spectrum of the matrix to leave disc (1). □

Thus, the boundary of the set of stable matrices $M(\mu)$ in the parameter space $\mathbb{R}^k$ is given by the equation

$$\Phi(\mu) = 0 \,.$$

Next we determine the structure of this manifold.

Theorem 9. *Under conditions and in notation of Theorem 8, one has*

$$\Phi(\mu) \equiv \det f^*(M(\mu);\mu)$$

$$= \det(I + a_1(\mu)M(\mu) + \dots + a_{n-1}(\mu)M^{n-1}(\mu) + a_n(\mu)M^n(\mu)) \,. \tag{24}$$

Proof. It follows from the more general result [14]. For any polynomial $g(z) = b_0 z^m + \dots + b_m \in \mathbb{C}[z]$ and any matrix $A \in \mathbb{C}^{n \times n}$, the following equality is valid

$$\det g(A) = \mathcal{R}_z(\det(zI - A), g(z)) \,.$$

□

For low order matrices $M(\mu)$, computation of $\Phi(\mu)$ via formula (24) does not cause troubles. However, for higher orders, any simplification of computations is valuable. One of the opportunities for such a potential simplification is provided by representation (23).

Corollary 3. *Polynomial (22) is reducible over* $\mathbb{R}$*:*

$$\Phi(\mu) \equiv f(1;\mu) f(-1;\mu) \Phi_1^2(1;\mu)\,. \tag{25}$$

Here

$$\Phi_1(z;\mu) \equiv \prod_{1\le j<k\le n} (z - \alpha_j(\mu)\alpha_k(\mu))\,. \tag{26}$$

Being symmetric functions of the zeros of the polynomial $f(z;\mu)$, the coefficients of $\Phi_1(z;\mu)$ can be expressed as polynomials over $\mathbb{Z}$ in the coefficients of $f(z;\mu)$.

Corollary 4. *Family (2) is stable iff $M(\mu^{(0)})$ is stable for a particular specialization of the parameter $\mu = \mu^{(0)}$ in (3) and polynomials*

$$f(1;\mu),\ f(-1;\mu),\ \Phi_1(1;\mu) \tag{27}$$

are positive in (3). Equations

$$f(1;\mu)=0,\ f(-1;\mu)=0,\ \Phi_1(1;\mu)=0 \tag{28}$$

define implicit manifolds in $\mathbb{R}^k$ that form the boundary for the ***stability domain*** *in the parameter space, i.e., of parameter specializations responsible for stability of the matrix $M(\mu)$.*

Example 1. In terms of the coefficients of the characteristic polynomial $f(z) := z^n + \sum_{j=1}^{n} a_j z^{n-j}$, one has

$$\begin{aligned} \Phi_1(1;a_1,a_2,a_3) &\equiv -a_3^2 + a_3a_1 - a_2 + 1 \quad \text{for} \ \ n=3, \\ \Phi_1(1;a_1,a_2,a_3,a_4) &\equiv (a_4-1)^2(1-a_2+a_4) + (a_1a_4-a_3)(a_3-a_1) \\ &\quad \text{for} \ \ n=4\,. \end{aligned}$$

From (26) and Viète formulas, one can notice that the degree of $\Phi_1(1;a_1,\dots,a_n)$, treated as a polynomial in all the coefficients of $f(z)$, equals $n-1$, and it contains the term $(-1)^{n(n-1)/2}a_n^{n-1}$. □

To obtain the general expression for $\Phi_1(1;\mu)$ for any n via factorization of polynomial (24), looks like a nontrivial task even if we know two its factors from identity (25). There exist several approaches to factoring polynomials (see [22] and references therein). All the known algorithms run in polynomial time or are conjectured so (using randomization and for dense polynomials). Their running time bounds, however, seem to have high exponents [23] (Theorem 3.12). An alternative procedure for the construction of the polynomial $\Phi_1(1;\mu)$ can be suggested on the base of Theorem 1. We describe it in the next section.

3.2 The Algorithm

To check the robust stability of family (2), perform the following steps

0. Take arbitrary point $\mu^{(0)}$ in $\mathfrak{B}$. If $M(\mu^{(0)})$ is not stable, then the claim is wrong. Otherwise

1. Calculate the powers of matrix M and their traces: $s_k = \mathrm{Tr}(M^k)$ for $k \in \{1, 2, \dots, n(n-1)\}$.

2. By (6), calculate the coefficients of $f(z;\mu) := \det(zI - M(\mu))$.

3. By formulae (8), calculate the Newton sums S_k for $k \in \{1, 2, \dots, n(n-1)/2\}$.

4. By (6), calculate the coefficients of polynomial $\Phi_1(z;\mu)$ defined by (26).

5. By Theorem 5, verify that polynomials $f(1;\mu), f(-1;\mu)$ and $\Phi_1(1;\mu)$ do not have real zeros in the box $\mathfrak{B}$.

First, consider the computational complexity of the first 4 steps of the algorithm. Here, matrix multiplication is the most expensive operation. The square matrix multiplication has an asymptotic complexity of $O(n^3)$, if carried out naively, and the complexity of $O(n^{\log_2 7}) \approx O(n^{2.807})$ if utilized Strassen's algorithm. The exponent appearing in the complexity of matrix multiplication has been improved several times, and a final (up to date) complexity of $O(n^{2.3728639})$ has the Le Gall algorithm that generalizes the Coppersmith – Winograd algorithm [9].

To compute M^k for $k \in \{0, 1, 2, \dots, n^2 - n\}$, we have to perform $n^2 - n - 1$ matrix multiplications, so totally we have $O(n^5)$ operations. Then we find traces of matrices M^k for $k \in \{0, 1, 2, \dots, n^2 - n\}$ and coefficients of (27). This yields $\approx O(n^4)$ operations in total. Hence, there are $O(n^5)$ operations, if we do not take into account operations for testing positiveness of polynomials (27) in the box $\mathfrak{B}$.

For the same computations, the most expensive operation in the algorithm described in [6] is calculation of the determinant of matrix $I - M(\mu) \cdot M(\mu)$, where bialternate product is defined as

$$M \cdot M := [n_{ij,k\ell}] \quad \text{where } n_{ij,k\ell} := \begin{vmatrix} m_{ik} & m_{i\ell} \\ m_{jk} & m_{j\ell} \end{vmatrix}$$

for the indices ordered lexicographically. Hence, the matrix $M \cdot M$ is of the order $(n^2 - n)/2$, and one needs $O(n^6)$ operations to compute its determinant.

Therefore, the algorithm presented here is more efficient in its first part, i.e., for computation of polynomials (27).

Now consider the last step of the algorithm. To test the positivity of the obtained polynomials in [6] numerical procedures based on Bernstein expansion method [10, 13] are used. Even applied for the order 3 matrices with only two parameters (with a sample one considered in the next section), they require more than 100 iterations [6].

The algebraic approach that we propose allows one to find all the required values with arbitrary precision.

4 Numerical Examples

Example 2. It is shown in [6] that the family

$$M(\mu_1,\mu_2)=\left[\begin{array}{rrr} -0.14 & 0.235 & 0.29 \\ -0.94 & -0.811 & 1.246 \\ -0.22 & -0.35 & 0.95 \end{array}\right]+\mu_1\left[\begin{array}{rrr} -0.3 & 0.15 & 0.275 \\ -0.275 & -0.3 & 0.55 \\ -0.35 & -0.25 & 0.625 \end{array}\right]$$

$$+\mu_2\left[\begin{array}{rrr} 0.4 & -0.1 & -0.4 \\ -0.6 & -0.325 & 0.225 \\ 0.725 & 0.225 & -0.45 \end{array}\right]$$

is stable for $(\mu_1,\mu_2)\in[-1,1]\times[-1,1]$. We will demonstrate that it is stable in the larger box $\mathfrak{B}:=[-2.8,1]\times[-1.03,1.1]$ and find the distance to instability from $\mu^{(0)}=(0,0)$.

Solution. It can be verified that the matrix $M(0,0)$ is stable. To check the other conditions of Corollary 4, we compute the traces of powers of the matrix[2] $M(\mu_1,\mu_2)$

$$\begin{array}{rcl} s_1 &=& \frac{1}{40}\mu_1-\frac{3}{8}\mu_2-\frac{1}{1000};\\ s_2 &=& \frac{297}{20000}\mu_1+\frac{9317}{20000}\mu_2+\frac{33}{1600}\mu_1^2+\frac{13}{160}\mu_1\mu_2+\frac{7}{64}\mu_2^2+\frac{138221}{1000000};\\ \dots & & \\ s_6 &=& -\frac{487405339}{2048000000}\mu_1^3\mu_2^3+\frac{519633}{204800000}\mu_1^5\mu_2+\frac{526764063}{4096000000}\mu_1^2\mu_2^4\\ &+& \frac{85443417}{2048000000}\mu_1\mu_2^5+\frac{81273}{4096000000}\mu_1^6-\frac{1399171}{2048000000}\mu_2^6+\dots\\ &+& \frac{167430804832241561}{1000000000000000000} \end{array}$$

and then restore by (6) the coefficients of its characteristic polynomial $f(z;\mu)$:

$$\begin{array}{rcl} a_1 &=& -s_1;\\ a_2 &=& -\frac{1}{100}\mu_1^2-\frac{1}{20}\mu_1\mu_2+\frac{1}{64}\mu_2^2-\frac{149}{20000}\mu_1-\frac{4651}{20000}\mu_2-\frac{6911}{100000};\\ a_3 &=& -\frac{31}{16000}\mu_1^3-\frac{2249}{12800}\mu_1^2\mu_2+\frac{13131}{64000}\mu_1\mu_2^2-\frac{139}{16000}\mu_2^3+\dots-\frac{117957}{500000}. \end{array}$$

Then compute the sums S_1,S_2,S_3 by (8):

$$\begin{array}{rcl} S_1 &=& a_2;\\ S_2 &=& \frac{1}{2560000}(124\mu_1^4-8340\mu_2^4+9385\mu_1^3\mu_2-181806\mu_1^2\mu_2^2+197521\mu_1\mu_2^3)+\dots\\ S_3 &=& \frac{54880506251}{256000000000}\mu_1^3\mu_2+\frac{19234505889}{2560000000000}\mu_1^2\mu_2^2-\frac{255226105707}{3200000000000}\mu_1^3\\ &+& \frac{171599531007}{1600000000000}\mu_2^3+\dots \end{array}$$

[2] We treat the matrix entries as rational fractions.

and coefficients A_1, A_2, A_3 of polynomial $\Phi_1(z;\mu)$:

$$\begin{aligned}
A_1 &= -S_1;\\
A_2 &= \frac{1}{2560000}(124\mu_1^4 - 8340\mu_2^4 + 9385\mu_1^3\mu_2 - 181806\mu_1^2\mu_2^2 + 197521\mu_1\mu_2^3) + \ldots\\
A_3 &= -\frac{961}{256000000}\mu_1^6 - \frac{19321}{256000000}\mu_2^6 - \frac{123193537}{4096000000}\mu_1^4\mu_2^2 + \frac{147589151}{2048000000}\mu_1^3\mu_2^3\\
&- \frac{69719}{102400000}\mu_1^5\mu_2 - \frac{184927601}{4096000000}\mu_1^2\mu_2^4 + \frac{1825209}{512000000}\mu_1\mu_2^5 + \ldots
\end{aligned}$$

Next we get polynomials (27)

$$\begin{aligned}
f(1;\mu) &= 1 + a_1 + a_2 + a_3\\
&= \frac{1}{2^9 5^6}\big(- 15500\mu_1^3 - 1405625\mu_1^2\mu_2 + 1641375\mu_1\mu_2^2 - 69500\mu_2^3\\
&- 1013085\mu_1^2 - 3658925\mu_1\mu_2 + 3113455\mu_2^2 - 2963542\mu_1 - 52708\mu_2\\
&+ 5567808\big);\\
f(-1;\mu) &= 1 - a_1 + a_2 - a_3 = \ldots\\
\Phi_1(1;\mu) &= -\frac{1}{15500^2}\big[(-15500\mu_1^3 - 1405625\mu_1^2\mu_2 + 1641375\mu_1\mu_2^2 - 69500\mu_2^3\\
&\quad -933085\mu_1^2 - 3258925\mu_1\mu_2 + 2988455\mu_2^2 - 2603942\mu_1\\
&\quad -2692308\mu_2 - 1891312)^2 - 16000000(40625\mu_1^2 + 181250\mu_1\mu_2\\
&\quad +78125\mu_2^2 + 29750\mu_1 + 930950\mu_2 + 4276441)\big]\,.
\end{aligned}$$

To verify that these polynomials do not possess real zeros in the box $\mathfrak{B}$, we utilize the algorithm from Theorem 5. For our particular example, system (17) takes the form

$$G(\mu_1,\mu_2) = 0, \quad \partial G(\mu_1,\mu_2)/\partial\mu_2 = 0\,.$$

Using discriminant (10), the first component of any zero to this system satisfies the univariate equation $\mathcal{D}_{\mu_2}(G) = 0$. For $G(\mu) := f(1;\mu)$, this equation, up to a numerical factor, is as follows

$$\begin{gathered}
191212466544777822265625\mu_1^6 + 2329084130401050056250000\mu_1^5\\
+11209700007594350089281250\mu_1^4 + 23296298499543138006825000\mu_1^3\\
+11685796422886314151153325\mu_1^2 - 24715629423274671343748560\mu_1\\
-27001282400664725495388016 = 0.
\end{gathered}$$

From part (iii) of Theorem 3, it follows that in $[-2.8, 1]$, it has a single real zero. We can improve its approximation like, for instance, $\mu_1^{(1)} = -2.121108 \pm 10^{-6}$. For this value, the polynomial $G(\mu_1^{(1)}, \mu_2)$ in μ_2 has a multiple zero. It can be evaluated via (12) as $\mu_2^{(1)} = -4.888746 \pm 10^{-6}$. The point $(\mu_1^{(1)}, \mu_2^{(1)})$ does not belong to the box $\mathfrak{B}$. Therefore, the conditions (ii) of Theorem 5 are satisfied.

The absence of real zeros for boundary univariate polynomials $G(\mu_1, -1.03)$, $G(\mu_1, 1.1)$, $G(-2.8, \mu_2)$, and $G(1, \mu_2)$ in the corresponding intervals composing the box $\mathfrak{B}$ can be established via application of part (iii) of Theorem 3. Hence, the polynomial $f(1;\mu)$ does not possess zeros in $\mathfrak{B}$.

Analogously, it can be verified that polynomials $f(-1;\mu)$ and $\Phi_1(1;\mu)$ do not have real zeros in the box $\mathfrak{B}$.

For the curve $G(\mu_1,\mu_2) := f(1;\mu_1,\mu_2) = 0$ and the point $(\mu_1,\mu_2) = (0,0)$, distance equation (21) is as follows:

$$\begin{aligned}\mathcal{F}_1(z) \quad &:= \quad 16564379277892466763025576286707180020473301410675048828125 0\, z^9 + \dots \\ &- \quad 1418671204757960500059517541617921148525764211238280835065504137216 = 0\,.\end{aligned}$$

According to Theorem 3, it has 3 real (and positive) zeros with the minimal one equal to $z_* = 1.225741 \pm 10^{-6}$. Distance to this curve equals $d_* = \sqrt{z_*} = 1.107132 \pm 10^{-6}$ and is achieved at $(\mu_1^*, \mu_2^*) = (1.055645 \pm 10^{-6}, 0.333698 \pm 10^{-6})$. Minimal positive zero of the distance equation constructed for the curve $f(-1;\mu_1,\mu_2) = 0$ equals 1.524132 ± 10^{-6}. For the curve $\Phi_1(1;\mu_1,\mu_2) = 0$, distance equation is of the order 24, and it possesses 6 real (and positive) zeros with the minimal one equal to 1.509424 ± 10^{-6}. Therefore, the distance to instability from $(\mu_1,\mu_2) = (0,0)$ equals d_* (Fig. 1).

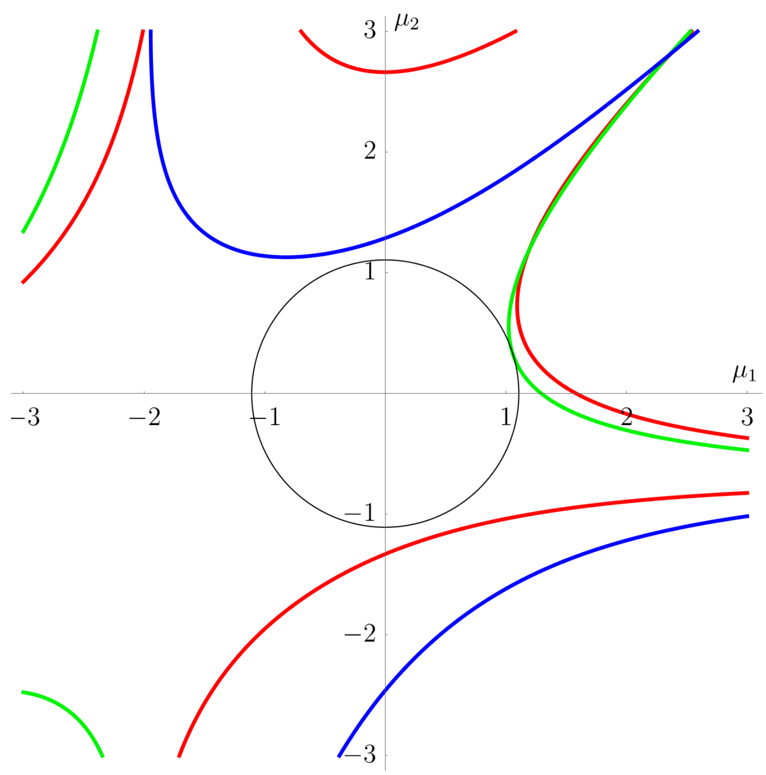

Fig. 1. Example 2. Plots of $f(1;\mu) = 0$ (green); $f(-1;\mu) = 0$ (blue); $\Phi_1(1;\mu) = 0$ (red)

One can notice that the asymptotes for the three drawn curves look identical. This is indeed the case. It is known [21] that the asymptotes for an algebraic curve $G(x,y)=0, \deg G := N > 1$ are determined by the coefficients of the two highest order forms in expansion of $G(x,y)$ in the decreasing powers of variables:

$$G(x,y) \equiv G_N(x,y) + G_{N-1}(x,y) + \dots$$

If for $(x_0,y_0) \in \mathbb{R}^2, (x_0,y_0) \neq (0,0)$, the following conditions hold

$$G_N(x_0,y_0)=0,\ (\partial G_N(x_0,y_0)/\partial x)^2 + (\partial G_N(x_0,y_0)/\partial y)^2 \neq 0,$$

then the equation

$$x \partial G_N(x_0,y_0)/\partial x + y \partial G_N(x_0,y_0)/\partial y + G_{N-1}(x_0,y_0) = 0$$

gives an asymptote for the curve. Generically, the degree of the polynomial $a_n(\mu_1,\mu_2) \equiv (-1)^n \det M(\mu_1,\mu_2)$ is much higher than those of the other coefficients of $f(z;\mu_1,\mu_2)$. Its forms of the two highest orders coincide with those of the polynomials $f(\pm 1;\mu_1,\mu_2)$. According to the remark in Example 1, the two highest order forms of $\Phi_1(1;\mu_1,\mu_2)$ coincide up to a sign with those of $a_n^{n-1}(\mu_1,\mu_2)$. □

The complexity of computations increases drastically with the number of parameters and degrees of the matrix entries.

Example 3. For the matrix family

$$M(\mu_1,\mu_2,\mu_3) = \left[\begin{array}{rrr} -0.3 & -0.1 & 0 \\ 0.2 & 0.3 & 0.3 \\ -0.1 & 0 & 0.3 \end{array}\right] + \mu_3 \left[\begin{array}{rrr} 0.1 & -0.2 & -0.3 \\ 0.3 & 0.1 & 0.2 \\ 0.1 & 0.3 & 0.2 \end{array}\right]$$

$$+\mu_2 \left[\begin{array}{rrr} -0.2 & 0 & 0.1 \\ -0.3 & 0.1 & -0.3 \\ -0.1 & 0.1 & -0.3 \end{array}\right] + \mu_2\mu_3 \left[\begin{array}{ccc} 0.2 & 0.3 & 0.1 \\ 0 & 0.1 & 0.2 \\ 0.1 & 0 & 0 \end{array}\right]$$

$$+\mu_1^2 \left[\begin{array}{rrr} 0.1 & -0.2 & 0 \\ 0.3 & 0.3 & 0.1 \\ 0 & 0 & -0.2 \end{array}\right],$$

find the distance to instability from $\mu^{(0)} = (0,0,0)$.

Solution. Here the matrix entries are even polynomials in μ_1, and the distance equations can be constructed using the approach outlined in Section 2.3. These equations are polynomials over $\mathbb{Z}$ with the magnitude of some of their coefficients exceeding 10^{500}.

Surface	distance ($\pm 10^{-6}$)	achieved at ($\pm 10^{-6}$)	distance equation degree
$f(1,\mu)=0$	3.775852	$(\pm 2.062413, 0.572611, -3.110566)$	51
$f(-1;\mu)=0$	2.322106	$(\pm 2.322106, 0.6986351, -0.305716)$	51
$\Phi_1(1;\mu)=0$	1.749203	$(\pm 1.620479, -0.450343, 0.480575)$	162

The distance to instability from $\mu^{(0)} = (0,0,0)$ equals 1.749203 ± 10^{-6}. □

5 Conclusions

We have investigated Schur stability property for the matrices with the entries polynomially depending on parameters. The first task has been stated as that of the description of the domain of stability in the parameter space, i.e., finding its boundaries. The second task has been aimed at the estimation of possible tolerances for the parameter specializations that would not disturb the stability property of a particular matrix. In this paper, the purely algebraic procedures based on symbolic algorithms for the elimination of variables and localization of the real zeros for algebraic equation systems have been suggested for solving the stated problems. This provides one with precise information on the obtained solution, i.e., the results do not depend on the precision of calculations and round-off errors (which are known to make a tight bottleneck for any numerical algorithm relating the problems solved). For further investigation, it remains the optimization of computational efficiency of developed algorithms.

Acknowledgments. The authors are grateful to Prof Evgenii Vorozhtzov and to the anonimous referees for valuable suggestions that helped to improve the quality of the paper.

References

1. Ackermann, J.: Robust Control: The Parameter Space Approach. Springer-Verlag, London (2002)
2. Akritas, A.G.: Elements of Computer Algebra With Applications. Wiley, New York (1989)
3. Basu, S., Pollack, R., Roy, M.-F.: Algorithms in Real Algebraic Geometry. Springer-Verlag, Berlin, Heidelberg (2010)
4. Bhattacharya, S.P., Chapellat, H., Keel, L.: Robust Control: The Parametric Approach. Prentice-Hall, New Jersey (1995)
5. Bikker, P., Uteshev, A.Yu.: On the Bézout construction of the resultant, J. Symbolic Comput. **28**, 45–88 (1999)
6. Büyükköroğlu, T., Çelebi, G., Dzhafarov, V.: On the robust stability of polynomial matrix families, Electron. J. Linear Algebra **30**, 905–915 (2015)
7. Chesi, G.: Exact robust stability analysis of uncertain systems with a scalar parameter via LMIs. Automatica **49**, 1083–1086 (2013)
8. Chilali, M., Gahinet, P., Apkarian, P.: Robust pole placement in LMI regions. IEEE Trans. Automat. Control. **44**(12), 2257–2270 (1999)
9. Coppersmith, D., Winograd, Sh.: Matrix multiplication via arithmetic progressions. J. Symbolic Comput. **9**(3), 251–280 (1990)
10. Dzhafarov, V., Büyükköroğlu, T.: On nonsingularity of a polytope of matrices. Linear Algebra Appl. **429**, 1174–1183 (2008)
11. Elaydi, S.: An Introduction to Difference Equations. Springer, New York (2005)
12. Faddeev, D. K., Faddeeva, V. N.: Computational Methods of Linear Algebra. Freeman, San Francisco (1963)
13. Farouki, R.T.: The Bernstein polynomial basis: A centennial retrospective. Comput. Aided Geom. Des. **29**(6), 379–419 (2012)

14. Gantmakher, F. R.: The Theory of Matrices. V. I, II. Chelsea, New York (1959)
15. Henrion, O., Bachelier, O., Sebek, M.: *D*-stability of polynomial matrices. Int. J. Control **74**(8), 845–856 (2001)
16. Joachimsthal, F.: Bemerkungen über den Sturmschen Satz. J. Reine Angew. Math. **48**, 386–416 (1854)
17. Jury E.I.: Innors and Stability of Dynamic Systems. Wiley, New York (1974)
18. Kalinina, E.A.: Stability and D-stability of the family of real polynomials, Linear Algebra Appl. **438**, 2635–2650 (2013)
19. Keel, L.H., Bhattacharya, S.P.: Robust stability via sign-definite decomposition. IEEE Trans. on Automat. Control **56**(1), 140–145 (2011)
20. Krein, M. G., Naimark, M. A.: The method of symmetric and Hermitian forms in the theory of the separation of the roots of algebraic equations, Linear Multilinear A. **10**, 265–308 (1981)
21. Kunz, E.: Introduction to Plane Algebraic Curves. Birkhäuser, Boston (2005)
22. Lee, M. M.-D.: Factorization of Multivariate Polynomials. Technische Universität Kaiserslautern (2013)
23. Lenstra, A. K.: Factoring multivariate polynomials over finite fields, J. Comput. Syst. Sci. **30**(2), 235–248 (1985)
24. Mishra, B.: Algorithmic Algebra. Springer-Verlag, New York (1993)
25. Schur, J.: Über Potenzreihen, die im Innern des Einheitskreises beschränkt sind. J. Reine Angew. Math., **147**, 205–232 (1917), **148**, 122–145 (1918)
26. Skalna, I.: Parametric Interval Algebraic Systems. Springer, Cham (2018)
27. Uteshev, A.Yu., Shulyak, S.G.: Hermite's method of separation of solutions of systems of algebraic equations and its applications, Linear Algebra Appl. **177**, 49–88 (1992)
28. Smith III, J.O.: Introduction to Digital Filters: with Audio Applications. W3K Publishing, USA (2007)
29. Uteshev, A.Yu., Cherkasov, T.M.: The search for the maximum of a polynomial, J. Symbolic Comput. **25**(5), 87–618 (1998)
30. Uteshev, A.Yu., Borovoi, I.I.: Solution of a rational interpolation problem using Hankel polynomials. Vestn. St.-Peterbg. Univ. Ser. 10 Prikl. Mat. Inform. Protsessy Upr. **4**, 31–43 (2016) (in Russian)
31. Uteshev, A.Yu., Goncharova, M.V.: Metric problems for algebraic manifolds: analytical approach. In: Constructive Nonsmooth Analysis and Related Topics — CNSA 2017 Proceedings, 7974027 (2017)
32. Uteshev A.Yu., Yashina M.V. Metric problems for quadrics in multidimensional space, J. Symbolic Comput. **68**, 287–315 (2015)
33. Wayland, H.: Expansion of determinantal equations into polynomial form. Quart. Appl. Math. **2**(4), 277–305 (1945)
34. Yli-Kaakinen, J., Saramaki, T.: Efficient recursive digital filters with variable magnitude characteristics. In: Proc. 7th Nordic Signal Processing Symposium — NORSIG 2006, 30–33 (2006)