ON THE GLOBAL SOLUTION OF THE N-BODY PROBLEM

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Abstract. In connection with the publication (Wang Qiu-Dong, 1991) the Poincaré type methods of obtaining the maximal solution of differential equations are discussed. In particular, it is shown that the Wang Qiu-Dong's *global solution of the N-body problem* has been obtained in Babadzanjanz (1979). First the more general results on differential equations have been published in Babadzanjanz (1978).

Key words: N-body problem, Poincaré type method, analytic continuation.

1. Introduction

In his work, Poincaré has proposed the method of solving the differential equations in series expantion (Poincaré, 1882). Furthermore, he has noticed that this method may be applied to the 3-body problem. In this section a family of methods based on the scheme of the method mentioned is considered. Then the notions of regularization and blowing up transformation are introduced and the Wang Qiu-Dong's *global solution of the N-body problem* (Wang Qiu-Dong, 1991) briefly discussed. In the remaining sections some results from Babadzanjanz (1978, 1979), among which there is *the global solution of the N-body problem*, are considered. About the contents of these sections we will say more at the close of this Introduction.

Poincaré type methods. Consider the system of equations

$$\mathrm{d}x_j/\mathrm{d}t = X_j(x_1, \ldots, x_n) \tag{1}$$

and the initial conditions

$$x_j(t_0) = x_{jo} \in R\,, \quad j \in [1:n]\,, \tag{2}$$

where X_j are the polynomials (with respect to all variables $x_1, \ldots, x_n$) with real coefficients.

Let $x(t) = (x_1, \ldots, x_n)$ and (ω_-, ω_+) be the solution of the problem (1), (2) and its maximal existence interval respectively. It is known that, for every fixed $x^0 = (x_{10}, \ldots, x_{n0}) \in R^n$, the implications

$$
\begin{aligned}
&(\omega_+ < +\infty) \Rightarrow \Big(\lim_{t\to\omega_+ - 0} \|x(t)\| = +\infty\Big) \\
&(\omega_- > -\infty) \Rightarrow \Big(\lim_{t\to\omega_- + 0} \|x(t)\| = +\infty\Big)
\end{aligned} \tag{3}
$$

are true. If t is considered as a complex variable and $x(t)$ is the solution of the problem (1), (2) extended along the real axis of t, then $x(t)$ is regular in a domain $\mathcal{D}_t$ containing (ω_-, ω_+).

Let us introduce instead of t in (1), (2) the new variable $s(t)$, defined by the formula

$$
\mathrm{d}s = \varphi(x_1, \dots, x_n)\,\mathrm{d}t \tag{4}
$$

with analytic φ and $1/\varphi$ for $x \in M \subset R^n$. If, for $j \in [1:n]$, one denotes $Y_j = X_j/\varphi$ and $y_j(s) = x_j(t(s))$, then

$$
\mathrm{d}y_j/ds = Y_j(y_1, \dots, y_n) \tag{5}
$$

$$
y_j(s_0) = x_{j0}\,, \qquad s_0 = s(t_0)\,. \tag{6}
$$

The function φ is supposed to satisfy the condition: ($\mathcal{A}$) *For every $x^0 \in M \subset R^n$ the solution of the problem (5), (6) is regular for $s \in (-\infty, +\infty)$.* This condition implies that, for any $x^0 \in M$, the corresponding solution $y(s) = (y_1(s), \dots, y_n(s))$ of the problem (5), (6) is regular in a simply connected open s-region $\mathcal{D}_s$, containing the s-interval $(-\infty, +\infty)$. Notice, that $\mathcal{D}_s$ depends on the initial data x^0.

The Riemann's theorem implies that there exists the function $\lambda : \mathcal{D}_s \to \mathcal{C}$, which transforms conformally $\mathcal{D}_s$ onto the circle $\mathcal{O}_1 = \{\lambda \in \mathcal{C} \mid |\lambda| < 1\}$ and satisfies the conditions:

$$
\lambda(s_0) = 0\,, \qquad \lambda((-\infty, +\infty)) = (-1, +1)\,. \tag{7}
$$

Using the substitution $s(\lambda)$ in (5) one obtains the differential equations for the variables

$$
z(\lambda) = (z_1(\lambda), \dots, z_n(\lambda)) = x(t(s(\lambda)))\,.
$$

Since the functions $z_k(\lambda)$, $k \in [1:n]$ are regular in $\mathcal{O}_1$, then $z_k(\lambda) = \sum_{m=0}^{\infty} \alpha_m^{(k)} \lambda^m$. Using the differential system for $z(\lambda)$ mentioned earlier one can obtain $\alpha_m^{(k)}$ step by step by the method of the undetermined coefficients.

We shall, from now on, agree to call the methods of obtaining the maximal solution of differential equations based on the scheme considered above Poincaré type methods. In the original Poincaré's method the function φ is introduced by the formula

$$\varphi(x_1, \dots, x_n) = 1 + X_1^2 + \cdots + X_{n+1}^2 , \tag{8}$$

where a polynomial $X_{n+1}(x_1, \dots, x_n)$ is chosen in such a way that all the highest power terms in $X_1, \dots, X_{n+1}$ are equal to 0 simultaneously iff $x_1 = \cdots = x_n = 0$. In this case it appears that there is $\rho > 0$ such that the functions $Y_1, \dots, Y_n$ of variables $x_1, \dots, x_n$ are analytic in the strip

$$\mathcal{S}_\rho = \{x = (x_1, \dots, x_n) \in \mathcal{C}^n \mid |\Im x_j| < \rho,\ j \in [1:n]\}$$

and the inequalities $|Y_j| \leq 1$ hold for every $x \in \mathcal{S}_\rho$. This implies that, for any fixed real initial data, there is $h > 0$ such that the corresponding solution of Equation (5) is regular in the strip $\mathcal{M}_{2h} = \{s \in \mathcal{C} \mid |\Im s| < h\}$. The function λ introduced by Poincaré's formula

$$\lambda(s) = (e^{\alpha s} - 1)(e^{\alpha s} + 1)^{-1} , \qquad \alpha = \pi/2h \tag{9}$$

transforms conformally the strip $\mathcal{M}_{2h}$ onto the circle $\mathcal{O}_1$.

To apply Poincaré type methods, as well as the original Poincaré's method, to the N-body problem (as well as to other problems) one has to get over certain difficulties. In particular, one sees that in order to make use of these methods and to make it possible to calculate $x(t) = x(t(s(\lambda))) = z(\lambda)$ the tools for calculating $\lambda(s(t))$ for $t \in (\omega_-, \omega_+)$ (or, at least, $t(s(\lambda))$ for $\lambda \in (\lambda(s(\omega_-)), \lambda(s(\omega_+))) \subset (-1, 1)$) are to be found. The straightforward way to do this is by constructing the Cauchy problem in such a way that there is, in addition to $z(\lambda)$, the function $T(\lambda) = t(s(\lambda))$ among its variables and, then, by making use of the method of undetermined coefficients for obtaining the power series expansions for $z(\lambda)$, $T(\lambda)$.

Regularization and blowing up transformation. As one can see, the numbers $\omega_\pm$ are functions of the initial data $x^0 = (x_{10}, \dots, x_{n0})$ in (2). Thus, using the functions $s(t)$, $\lambda(s)$ defined above one can introduce the functions $s_\pm(x^0) = s(\omega_\pm)$, $\lambda_\pm(x^0) = \lambda(s_\pm)$.

It is important to notice that the implications

$$\omega_\pm = \pm\infty \Rightarrow (\forall x^0)(s_\pm(x^0) = \pm\infty) , \tag{10}$$

$$\begin{aligned} &\omega_+ < +\infty \Rightarrow \text{either } s_+(x^0) < +\infty , \text{ or } s_+(x^0) = +\infty \\ &\omega_- > -\infty \Rightarrow \text{either } s_-(x^0) > -\infty , \text{ or } s_-(x^0) = -\infty \end{aligned} \tag{11}$$

hold for Poincaré type methods (as well as for the original Poincaré's method).

In addition to $(\mathcal{A})$, let us consider the two conditions:

$(\mathcal{B})$ *For every* $x^0 \in M$, *the implications*

$$\omega_+ < +\infty \Rightarrow s_+(x^0) < +\infty\,, \quad \omega_- > -\infty \Rightarrow s_-(x^0) > -\infty$$

are true;

$(\mathcal{C})\,(\forall x^0 \in M)\,(s_\pm(x^0) = \pm\infty)$.

In connection with (10), (11) and conditions ($\mathcal{A}$), ($\mathcal{B}$), ($\mathcal{C}$) let us introduce the following definitions. The transformation (4) is said to be the *regularization transformation on* $M \subset \mathcal{R}^n$ if the function φ satisfies the conditions ($\mathcal{A}$) and ($\mathcal{B}$). The transformation (4) is said to be the *blowing-up transformation on* $M \subset \mathcal{R}^n$ if the function φ satisfies the conditions ($\mathcal{A}$) and ($\mathcal{C}$). Notice, that the term 'blowing-up transformation' is taken from Wang Qiu-Dong (1991). There, the transformation (12), (see later) was called blowing-up one.

If the transformation (4) is neither regularization nor blowing-up one, then, for fixed $x^0 \in M$, one does not know the λ-interval $(\lambda_-, \lambda_+) \subset (-1, 1)$. In particular, this means that in that case one cannot calculate the function $x(t) = x(t(s(\lambda)))$ for given $t \in (\omega_-, \omega_+)$. Unlike this, in the case of the blowing-up transformations one sees that $(\forall x^0 \in M)((\lambda_-, \lambda_+) = (-1, 1))$.

Regularization transformations (Sundman, 1913; Stiefel and Scheifele, 1971) as well as the fact that such a transformation for the N-body problem ($N > 3$) has not been discovered till now are well known. Moreover, it is possible that the transformation mentioned before does not exist in an appropriate form. Thus, if among the Poincaré type methods one seeks for the appropriate method for obtaining the maximal solution of the N-body problem ($N > 3$), then it is natural to use a method with a blowing-up transformation (4).

The Wang Qiu-Dong's method (Wang Qiu-Dong, 1991). Consider the point masses $m_0, \ldots, m_{n-1}$ moving according to Newton's law of gravitation with reference to the barycentric coordinate system $\mathcal{O}g_1g_2g_3$. Let:

$$g = (g_0, \ldots, g_{n-1})\,, \quad p = (p_0, \ldots, p_{n-1})\,, \quad g_i = (g_{i1}, g_{i2}, g_{i3})\,,$$

$$p_i = (p_{i1}, p_{i2}, p_{i3})\,, \quad p_{ij} = \dot{g}_{ij}\,,$$

$$M = \{(g, p) \in \mathcal{R}^{6n} \mid g \in \mathcal{R}^{3n} \backslash \mathcal{Q},\ p \in \mathcal{R}^{3n}\}\,, \quad \mathcal{Q} = \bigcup_{i,j} \{g \in \mathcal{R}^{3n} \mid g_i = g_j\}\,,$$

where g_i is the position of the i-th particle. Let $U(g), h$ be the potential function and the energy constant, respectively, and let t represent the time.

Omitting details, one can describe the scheme of this method as follows. Considering the initial data problem for the system of differential equations with respect to g, p and letting there

$$\mathrm{d}\tau = u^{-3/2}\mathrm{d}t\,, \tag{12}$$

$$u = \begin{cases} (2U(g)+h)^{-1}\,, & \text{if } h > 0\,, \\ (2U(g))^{-1}\,, & \text{if } h \leq 0\,, \end{cases} \tag{13}$$

one can obtain the corresponding differential system for the functions u, F, G, t of the new argument τ, where $F = (F_0, \dots, F_{n-1})$, $G = (G_0, \dots, G_{n-1})$, $F_i = u^{-1}g_i$, $G_i = m_i p_i u^{1/2}$. If the equalities

$$g(t_0) = g^0 = (g_0^0, \dots, g_{n-1}^0)\,, \qquad p(t_0) = p^0 = (p_0^0, \dots, p_{n-1}^0) \tag{14}$$

represent the initial condition in the original problem, then the corresponding equalities for the last system are:

$$u(\tau_0) = u^0\,, \qquad F(\tau_0) = F^0\,, \qquad G(\tau_0) = G^0\,, \qquad t(\tau_0) = t_0\,, \tag{15}$$

where

$$u^0 = \begin{cases} (2U(g^0)+h)^{-1}\,, & \text{if } h > 0\,, \\ (2U(g^0))^{-1}\,, & \text{if } h \leq 0\,, \end{cases} \tag{16}$$
$$G^0 = (G_0^0, \dots, G_{n-1}^0)\,, \qquad F^0 = (u^0)^{-1}g^0\,, \qquad G_i^0 = m_i p_i^0 (u^0)^{1/2}\,.$$

Using the energy integral one sees that F, G, u satisfy the following algebraic equations:

$$\begin{aligned} &\sum_{i=0}^{n-1} m_i^{-1}(G_{i1}^2 + G_{i2}^2 + G_{i3}^2) = 1\,, \;\; 1/2 - U(F) = uh && \text{if } h > 0\,, \\ &\sum_{i=0}^{n-1} m_i^{-1}(G_{i1}^2 + G_{i2}^2 + G_{i3}^2) = 1 + 2uh\,, \;\; 1/2 - U(F) = 0 && \text{if } h \leq 0\,. \end{aligned} \tag{17}$$

The solution u, F, G, t of the differential system mentioned earlier which satisfies (17) is called a proper solution. It appears, that there exist two real numbers A, B depending merely on $m_0, \dots, m_{n-1}$ and $(g^0, p^0) \in M$ such that the proper solution is regular in the region $\mathcal{H} = \{\tau \in \mathcal{C}\,|\,|\Im(\tau)| < Ae^{-B|\Re(\tau)|}\}$. Then, Wang Qiu-dong managed to construct the region $\mathcal{H}_3$ and conformal mapping $\varphi : \mathcal{H}_3 \to \mathcal{C}$ such that $(-\infty, \infty) \subset \mathcal{H}_3 \subset \mathcal{H}$, $\varphi(\mathcal{H}_3) = \mathcal{O}_1$. Thus, the Wang Qiu-Dong's method, as well as any other Poincaré type method with blowing-up transformation, may be used for obtaining *the global solution of the N-body problem.*

About the contents of Sections 2 and 3. Section 2 deals with the Cauchy problem (1), (2) for a polynomial system, i.e. an autonomous system of ordinary differential equations of the first order with polynomial (with respect to all dependent variables) right-hand sides. The main differential equations of celestial mechanics can be reduced to a system of that kind (see Section 2.1). Section 2.2 contains auxiliary results on such differential equations. In Section 2.3 we describe the Poincaré type method with blowing-up transformation which is valid for any polynomial system and, consequently, for the equations of the N-body problem. In Section 2.4 we derive the estimates of the existence interval for the Cauchy problem (1), (2) and the nesessary and sufficient conditions for a solution to be continuable along $[1, +\infty)$. Namely, in order to obtain the estimates of the existence interval of the solutions a function is constructed (from the original equations and initial data), which is regular in the circle on the complex plane. From the estimates of modulus of the function one can find the estimates of the existence interval. The sequence of such estimates of the right end of the existence interval is constructed. The sequence is divergent to $+\infty$, if this interval is semi-axis the whole time. All these results are obtained in the framework of the Poincaré type method.

In Section 3 we describe the Poincaré type method with blowing-up transformation for solving the N-body problem and obtaining the estimates and conditions mentioned before. The special form of the N-body problem essentially allow us to improve the method.

2. The General Poincaré Type Method with Blowing-Up Transformation (Babadzanjanz, 1978)

2.1. Reducing a System of Differential Equations to the Polynomial One

Let us consider a polynomial system and initial conditions (1), (2). In Section 2.3 we shall consider the Poincaré type method for the polynomial system of differential equations. Here we show how to reduce other defferential equations to such a system.

There exists a large class of differential systems all of which can be written in the form

$$\mathrm{d}y_j/\mathrm{d}t = P_j(y_1, \ldots, y_n\,;\; \varphi_1(y_1, \ldots, y_n, t), \ldots, \varphi_m(y_1, \ldots, y_n, t)\,;\; t)\,, \tag{18}$$

where $j \in [1 : n]$, and all the functions P_j, $f_{rk} = \partial\varphi_r/\partial y_k$, $f_r = \partial\varphi_r/\partial t$ are polynomials with respect to all variables $y_1, \ldots, y_n$, $\varphi_1, \ldots, \varphi_m$, t. On letting

$$x_1 = y_1\,, \ldots,\; x_n = y_n\,, \qquad x_{n+1} = \varphi_1\,, \ldots,\; x_{n+m} = \varphi_m\,, \qquad x_{n+m+1} = t\,,$$

one obtains the system

$$\mathrm{d}x_j/\mathrm{d}t = P_j(x_1,\ldots,x_{n+m+1})\,, \qquad j \in [1:n]\,,$$

$$\begin{aligned}\mathrm{d}x_j/\mathrm{d}t = \sum_{k=1}^{n} f_{jk}(x_1,\ldots,x_{n+m+1})\,P_k(x_1,\ldots,x_{n+m+1}) + \\ + f_j(x_1,\ldots,x_{n+m+1})\,, \qquad j \in [n+1:n+m]\,,\end{aligned} \tag{19}$$

$$\mathrm{d}x_{n+m+1}/\mathrm{d}t = 1\,,$$

which is of the form (1).

In particular, the main differential equations of celestial mechanics can be written in the form (18). As an example, let us consider the point masses $m_0,\ldots,m_{n-1}$ moving with reference to the coordinate system $Og_1g_2g_3$. Consider the system of equations

$$\mathrm{d}^2 g_{ij}/\mathrm{d}t^2 = k^2 \sum_{r\in T(i)} m_r(g_{rj} - g_{ij})\,\Delta_{ri}^{-3}\,, \tag{20}$$

where

$$i \in [0:n-1]\,, \qquad j \in [1:3]\,, \qquad T(i) = [0:n-1]\backslash\{i\}\,,$$

$$\Delta_{ri} = \Big(\sum_{j=1}^{3}(g_{rj} - g_{ij})^2\Big)^{1/2}\,,$$

the quantity k is the Gaussian constant and g_{i1}, g_{i2}, g_{i3} are the coordinates of m_i. By letting

$$y_1 = g_{01}\,, \qquad y_2 = g_{02}\,,$$

$$y_3 = g_{03},\ldots,y_{3n} = g_{n-1,3}\,, \qquad y_{3n+1} = \dot{y}_1,\ldots,y_{6n} = \dot{y}_{3n}$$

one obtains the system which is of the form (18) with

$$\varphi_1 = \Delta_{12}^{-1},\ldots,\varphi_m = \Delta_{n-2,n-1}^{-1}\,,$$

$$\partial\varphi_1/\partial y_1 = -\varphi_1^3(\,(y_1 - y_4)(y_{3n+1} - y_{3n+4}) + (y_1 - y_7)(y_{3n+1} - y_{3n+7}) +$$

$$+ (y_1 - y_{10})(y_{3n+1} - y_{3n+10})),\ldots\,; \qquad m = (n-1)n/2\,.$$

Then, on letting

$$x_1 = y_1,\ldots,x_{6n} = y_{6n}\,, \qquad x_{6n+1} = \varphi_1,\ldots,x_{6n+m} = \varphi_m\,,$$

one obtains the system which is of the form (1).

Remark 1. Without loss of generality one can assume that constant terms $a_j = X_j(0)$ in (1) are zero for all $j \in [1 : n]$. Indeed, on letting $x_{n+1} = 1$, one can write $a_j = a_j x_{n+1}$, $\mathrm{d}x_{n+1}/\mathrm{d}t = 0$, and, then, it appears that the system (1) is reduced to an analogous one which has none of constant terms.

2.2. Estimates of the solution and of its radius of convergence for polynomial differential system

Let us consider a polynomial differential system

$$\mathrm{d}x_j/\mathrm{d}t = X_j = \sum_{m=1}^{L+1} \sum_{i \in I(m)} a_j[i]\, x^i \tag{21}$$

and initial conditions

$$x_j(t_0) = x_{j0} \tag{22}$$

where

$$\begin{aligned} & i = (i_1, \dots, i_n) \in Z^n\,, \quad x = (x_1, \dots, x_n) \in C^n\,, \quad x^i = x_1^{i_1} \cdot \dots \cdot x_n^{i_n}\,, \\ & I(m) = \{ i \in Z^n \mid i_1 \geq 0, \dots, i_n \geq 0;\, |i| = m \}\,, \quad |i| = i_1 + \dots + i_n\,, \\ & j \in [1 : n]\,, \quad L \in [0 : +\infty)\,, \quad a_j[i] \in C\,, \quad x_{j0} \in C\,. \end{aligned} \tag{23}$$

PROPOSITION 1. *Let, for every $j \in [1 : n]$,*

$$|\, x_{j0} \,| \leq \gamma\,. \tag{24}$$

Then: (a) The solution $x = (x_1, \dots, x_n)$ of the problem (21), (22) is regular in circle $O_\rho = \{ t \in C \mid |t - t_0| < \rho \}$, where

$$\rho = (L s_a(\gamma))^{-1}\,, \qquad s_a(\gamma) = \max_{j \in [1:n]} \sum_{m=1}^{L+1} \gamma^{m-1} \sum_{i \in I(m)} |a_j[i]|\,. \tag{25}$$

(b) For all $t \in O_\rho$, $j \in [1 : n]$ it satisfies the inequalities

$$|x_j(t)| \leq \gamma\, (1 - |t - t_0|/\rho)^{-1/L}\,. \tag{26}$$

Proof. See Proposition 4 in (Babadzanjanz, 1979).

2.3. The Method

Consider the problem (21), (22) with $t_0 = 1, a_j[i] \in \mathcal{R}, x^0 = (x_{10}, \dots, x_{n0}) \in \mathcal{R}^n$.

FIRST STEP (*Blowing up transformation*).

Let us introduce instead of t the new variable s defined by the formulae:

$$s = t + \int_1^t \sum_{j=1}^{n+1} X_j^2(x(t))\,\mathrm{d}t\,, \tag{27}$$

$$X_{n+1} = 1/2 + 2\sum_{k=1}^{n} x_k X_k = 1/2 + \frac{\mathrm{d}}{\mathrm{d}t}\|x\|^2\,. \tag{28}$$

On letting

$$y_j(s) = x_j(t(s))\,,\ j \in [1:n]; \qquad y_{n+1}(s) = dt/ds\,,\ y_{n+2}(s) = t(s)\,, \tag{29}$$

one obtains the problem

$$\begin{aligned} &\mathrm{d}y_j/\mathrm{d}s = y_{n+1}X_j(y)\,, \qquad j \in [1:n]\,,\\ &\mathrm{d}y_{n+1}/\mathrm{d}s = -y_{n+1}^3\tilde{X}_{n+1}(y)\,,\\ &\mathrm{d}y_{n+2}/\mathrm{d}s = y_{n+1}\,, \end{aligned} \tag{30}$$

$$y_j(1) = x_{j0}\,, \qquad j \in [1:n+2]\,, \tag{31}$$

where

$$\begin{aligned} &y = (y_1, \dots, y_n)\,,\ x_{n+2,0} = 1\,,\ x_{n+1,0} = \Big(1 + \sum_{k=1}^{n+1} X_k^2(x^0)\Big)^{-1}\,,\\ &\tilde{X}_{n+1}(y) = 2\sum_{j=1}^{n+1}\sum_{k=1}^{n} X_j(y)X_k(y)\,\partial X_j(y)/\partial y_k\,. \end{aligned} \tag{32}$$

Notice that the degree of the polynomial $\tilde{X}_{n+1}$ is $3L + 4$ at most.

PROPOSITION 2. *Let (ω_-, ω_+) be the maximal existence interval of the solution of the Cauchy problem (21), (22); let the function s of t be defined by (27); let $y_1, \dots, y_{n+2}$ be the solution of the problem (30), (31).*

Then: (a) The equalities

$$\lim_{t \to \omega_+ - 0} s(t) = +\infty\,, \qquad \lim_{t \to \omega_- + 0} s(t) = -\infty \tag{33}$$

hold for every $x^0 \in \mathcal{R}^n$;
(b) For every $s \in \mathcal{R}$ the solution mentioned is regular and satisfies the inequalities

$$\begin{aligned} &0 < y_{n+1}(s) < 1\,, \quad \omega_- < y_{n+2}(s) < \omega_+\,, \\ &|\mathrm{d}y_j(s)/\mathrm{d}s| \le 1/2\,, \ |y_j(s)| \le |s-1|/2 + |x_{j0}|\,, \ j \in [1:n]\,. \end{aligned} \tag{34}$$

Proof. The equalities (33) follow from (3) and the following equality

$$s = t + \int_1^t \Big(\sum_{j=1}^n X_j^2 + 1/4\Big)\,dt + \int_1^t \Big(\frac{\mathrm{d}}{\mathrm{d}t}\|x\|^2\Big)^2 dt + \int_1^t \frac{\mathrm{d}}{\mathrm{d}t}\|x\|^2\,\mathrm{d}t\,. \tag{35}$$

The inequalities (34) are evident.

SECOND STEP (*Reducing to bounded solutions*).
Let us introduce the functions $z_1, \ldots, z_{n+3}$ of s such that

$$\begin{aligned} &z_j(s) = s^{-1/E} y_j(s^{1/E})\,, \quad j \in [1:n]\,, \\ &z_{n+k}(s) = y_{n+k}(s^{1/E})\,, \quad k \in [1:2]\,, \\ &z_{n+3}(s) = s^{-1/E}\,, \end{aligned} \tag{36}$$

where $E = 3L + 7$.
These functions satisfy the following Cauchy problem

$$\begin{aligned} &\mathrm{d}z_j/\mathrm{d}s = Zz_{n+3}(-z_j + z_{n+1}X_j(z/z_{n+3}))\,, \quad j \in [1:n]\,, \\ &\mathrm{d}z_{n+1}/\mathrm{d}s = -Zz_{n+1}^3 \tilde{X}_{n+1}(z/z_{n+3})\,, \\ &\mathrm{d}z_{n+2}/\mathrm{d}s = Zz_{n+1}\,, \\ &\mathrm{d}z_{n+3}/\mathrm{d}s = -Zz_{n+3}^2\,, \end{aligned} \tag{37}$$

$$z_j(1) = x_{j0}\,, \quad j \in [1:n+3]\,, \tag{38}$$

where $x_{n+3,0} = 1$, $z = (z_1, \ldots, z_n)$, $Z = z_{n+3}^{E-1}/E$.

PROPOSITION 3. *Let $z_1, \ldots, z_{n+3}$ be the solution of the problem (37), (38); let (ω_-, ω_+) be the maximal existence interval of the solution of the problem (21), (22). Then the inequalities*

$$\begin{aligned} &|z_j(s)| \le \max\{1/2, (2^{1/E} - 1)/2 + 2^{1/E}|x_{j0}|\}\,, \ j \in [1:n], \\ &0 < z_{n+1}(s) < 1\,, \ 0 < z_{n+2}(s) < \omega_+\,, \ 0 < z_{n+3}(s) < 2^{1/E} \end{aligned} \tag{39}$$

hold for every $s \in [1/2\,,\ +\infty)$.

Proof. These inequalities follow from (34), (36).

Let us introduce the function $\tilde{z} = (\tilde{z}_1, \dots, \tilde{z}_{n+2}) = (z_1, \dots, z_{n+1},\ z_{n+3})$ and let us rewrite the equations for $\tilde{z}$ from (37) and the corresponding initial conditions in the following form

$$\mathrm{d}\tilde{z}_j/\mathrm{d}s = \sum_{m=1}^{W+1} \sum_{i \in I(m)} b_j[i]\, \tilde{z}^i\,, \qquad j \in [1:n+2]\,, \tag{40}$$

$$\tilde{z}_j(1) = x_{j0}\,, \qquad j \in [1:n+1]\,, \qquad \tilde{z}_{n+2}(1) = 1\,, \tag{41}$$

where $W = E + 1 = 3L + 8$. Introduce μ_j, μ, r, s_b as follows

$$\begin{aligned} &\mu_j = \max\{1/2, (2^{1/E} - 1)/2 + 2^{1/E}\,|x_{j0}|\}\,, \qquad j \in [1:n] \\ &\mu = \max\{2^{1/E}, \mu_1, \dots, \mu_n\}\,, \qquad r = (1 - 2^{-W})(W s_b(\mu))^{-1}\,, \\ &s_b(\mu) = \max_{j \in [1:n+2]} \sum_{m=1}^{W+1} \mu^{m-1} \sum_{i \in I(m)} |b_j[i]|\,, \end{aligned} \tag{42}$$

PROPOSITION 4. *Let* $\tilde{z}$ *be the solution of the problem (40), (41) and let* z_{n+2} *be the function satisfying the problem (37), (38). Let* $\mathcal{M} = \mathcal{M}_r([1/2, +\infty)) = \{s \in C \mid |\Im s| < r;\ \Re s \in [1/2, +\infty)\}$.

Then: (a) The function $\tilde{z}$ *is regular on* $\mathcal{M}$ *and satisfies there the inequalities*

$$|\tilde{z}_j(s)| \le 2\mu \tag{43}$$

for $j \in [1:n+2]$*;*
(b) The function z_{n+2} *is regular on* $\mathcal{M}$ *and there exists such* $\delta > 0$ *that*

$$|z_{n+2}(s)| \le \delta < 4\mu r/E + \omega_+ \tag{44}$$

for all $s \in \mathcal{M}$.

Proof. The statement *(a)* follows from Proposition 2. The statement *(b)* follows from the third equation in (37) and *(a)*.

THIRD STEP (*The global solution*).

For $s \in (1/2, +\infty)$, $\tau \in \mathcal{R}$, let us introduce the new variable τ defined by the formulae

$$\tau = 2 + s - 2/(2s - 1)\,, \qquad s = (2\tau - 3 + \sqrt{4\tau^2 - 20\tau + 41})/4\,. \tag{45}$$

On letting

$$u_j(\tau) = z_j(s(\tau)), \qquad j \in [1 : n+3], \tag{46}$$

$$u_{n+4}(\tau) = \frac{1}{\sqrt{4\tau^2 - 20\tau + 41}}, \qquad u_{n+5}(\tau) = \frac{2\tau - 5}{\sqrt{4\tau^2 - 20\tau + 41}}, \tag{47}$$

one obtains the polynomial Cauchy problem for the functions $u_1, \dots, u_{n+5}$. What are the variables τ, u_j introduced for? The matter is one can prove for $u = (u_1, \dots, u_{n+5})$ the same Proposition as Proposition 4 but with $\mathcal{M} = \mathcal{M}_r(\mathcal{R})$ instead of $\mathcal{M} = \mathcal{M}_r((1/2, +\infty))$ (see proof of the Proposition 5 later).

Using Poincaré's transformation

$$\lambda = (e^{\nu(\tau-1)} - 1)(e^{\nu(\tau-1)} + 1)^{-1}, \qquad \nu = \pi/2r \tag{48}$$

and introducing the functions

$$v_j(\lambda) = u_j(\tau(\lambda)) \tag{49}$$

one obtains the following Cauchy problem

$$\begin{aligned}
&\mathrm{d}v_j/d\lambda = V v_{n+3}(-v_j + v_{n+1} X_j(v/v_{n+3})), \qquad j \in [1 : n], \\
&\mathrm{d}v_{n+1}/\mathrm{d}\lambda = -V v_{n+1}^3 \tilde{X}_{n+1}(v/v_{n+3}), \\
&\mathrm{d}v_{n+2}/\mathrm{d}\lambda = V v_{n+1}, \qquad \mathrm{d}v_{n+3}/d\lambda = -V v_{n+3}^2, \\
&\mathrm{d}v_{n+4}/\mathrm{d}\lambda = -4\Lambda v_{n+4}^2 v_{n+5}, \qquad \mathrm{d}v_{n+5}/\mathrm{d}\lambda = 4\Lambda(1 - v_{n+5}^2) v_{n+4},
\end{aligned} \tag{50}$$

$$v_j(0) = x_{j0}, \qquad j \in [1 : n+5], \tag{51}$$

where

$$x_{n+4,0} = 1/5, \qquad x_{n+5,0} = -3/5, \qquad v = (v_1, \dots, v_n),$$

$$\Lambda = \frac{2r}{\pi(1-\lambda^2)}, \qquad V = \frac{2r(1 + v_{n+5}) v_{n+3}^{E-1}}{\pi E(1-\lambda^2)}.$$

PROPOSITION 5. *Let the functions $u_1, \dots, u_{n+5}$ be defined by (46), (47) and let the functions $v_1, \dots, v_{n+5}$ be defined by (49). Let us use the designations (42) and let $r \le 0.5$, $\mathcal{M} = \mathcal{M}_r(\mathcal{R})$, $\mathcal{O}_1 = \{\lambda \in \mathcal{C} \mid |\lambda| < 1\}$. Let $\delta > 0$ be the number mentioned in Proposition 4.*

Then: (a) The functions $u_1, \dots, u_{n+5}$ are regular on $\mathcal{M}$ and u_{n+2} satisfies the inequality

$$|u_{n+2}(\tau)| \leq \delta < 4\mu r/E + \omega_+ \tag{52}$$

for all $\tau \in \mathcal{M}$*;*

(b) The functions $v_1, \ldots, v_{n+5}$ *are regular on* $\mathcal{O}_1$ *and* v_{n+2} *satisfy the inequality*

$$|v_{n+2}(\lambda)| \leq \delta < 4\mu r/E + \omega_+ \tag{53}$$

for all $\lambda \in \mathcal{O}_1$.

Proof. Consider the mapping $\tau : \mathcal{M}_r((1/2, +\infty)) \to \mathcal{C}$ defined by (45) and the mapping $\lambda : \mathcal{M}_r(\mathcal{R}) \to \mathcal{C}$ defined by (48). As one can show,

$$\tau(\mathcal{M}_r((1/2, +\infty))) \supset \mathcal{M}_r(\mathcal{R}) , \qquad \lambda(\mathcal{M}_r(\mathcal{R})) = \mathcal{O}_1 .$$

Then Proposition 4 implies *(a)* and *(a)* implies *(b)*.

Thus, using the method of the undetermined coefficients in the problem (50), (51) one may get the series expansion of the maximal solution of any original problem (21), (22) with $a_j[i] \in \mathcal{R}$, $x^0 \in \mathcal{R}^n$. In particular, every global solution of the N-body problem may be obtained in such a way.

2.4. Necessary and Sufficient Conditions for a Solution to be Continuable along $[1 : +\infty)$

The main result of this section is Proposition 7. Proposition 6 contains the necessary preliminary results (Babadzanjanz, 1978, 1979; Grenander and Szegö, 1958).

PROPOSITION 6. *Let* $w : \mathcal{O}_1 \to \mathcal{C}$ *be a regular function and let* $\sum_{j=0}^{\infty} \alpha_j \lambda^j$ *be its Taylor series expansion in* $\mathcal{O}_1$. *For* $m, r, s \geq 0$ *with* $r \leq s$, *let the numbers* h_{rs} *be defined by the formulae*

$$h_{rs} = \sum_{j=0}^{r} \alpha_{r-j} \bar{\alpha}_{s-j} , \qquad h_{sr} = \bar{h}_{rs} . \tag{54}$$

Let

$$\lambda_k = \sqrt{\mu_k} , \tag{55}$$

where μ_k *is the maximum eigen-value of the matrix*

$$H_k = \begin{pmatrix} h_{00}, & \ldots, & h_{0k} \\ \ldots & \ldots & \ldots \\ h_{k0}, & \ldots, & h_{kk} \end{pmatrix} . \tag{56}$$

Let ρ_0 *be a fixed number of* $[0, 1)$ *and let*

$$M(w) = \sup_{\lambda \in \mathcal{O}_1} |w(\lambda)| , \tag{57}$$

$$\gamma_k^{(1)} = \max_{\rho \in [\rho_0, 1]} \Big(\frac{1}{k+1} \sum_{\nu=0}^{k} |\alpha_0 + \ldots + \rho^{k-\nu} \alpha_{k-\nu}|^2 \Big)^{1/2} , \tag{58}$$

$$\gamma_k^{(2)} = \Big(\sum_{m=1}^{k} m\, |\alpha_m|^2 \Big)^{1/2} . \tag{59}$$

Then: (a) The sequence $\{\lambda_k\}$ *is non-decreasing and*

$$\lim_{k \to +\infty} \lambda_k = M(w) ; \tag{60}$$

(b) For each natural k, the inequalities

$$M(w) \geq \gamma_k^{(j)} , \qquad j = 1, 2 \tag{61}$$

are true. In addition, if

$$\lim_{(0,1) \ni \lambda \to 1-0} |w(\lambda)| = +\infty , \tag{62}$$

then

$$\lim_{k \to +\infty} \gamma_k^{(1)} = +\infty . \tag{63}$$

Propositions 5 and 6 imply the following result.

PROPOSITION 7. *Let* (ω_-, ω_+) *be the maximal existence interval of the solution of the Cauchy problem (21), (22). Let, for* $\lambda \in \mathcal{O}_1$, $w = v_{n+2}(\lambda) = \sum_{j=0}^{\infty} \alpha_j \lambda^j$ *be the function satisfying (50), (51) and let the numbers* h_{rs}, λ_k, $\gamma_k^{(j)}$ *be defined by (54), (59).*
Then: (a) Every of the equalities

$$\lim_{k \to +\infty} \lambda_k = +\infty , \tag{64}$$

$$\lim_{k \to +\infty} \gamma_k^{(1)} = +\infty \tag{65}$$

is the necessary and sufficient condition for the solution of the problem (21), (22) to be continuable along $[1, +\infty)$, *i.e.*

$$\lim_{k \to +\infty} \lambda_k = +\infty \Leftrightarrow \omega_+ = +\infty \Leftrightarrow \lim_{k \to +\infty} \gamma_k^{(1)} = +\infty . \tag{66}$$

(b) For each natural k, the inequalities

$$\omega_+ > \lambda_k - 4\mu r/E\ , \tag{67}$$

$$\omega_+ > \gamma_k^{(j)} - 4\mu r/E\ , \qquad j = 1,2\ , \tag{68}$$

with $E = 3L + 7$ *and* μ, r *defined by (42), hold true. In addition, if* $\omega_+ = +\infty$, *then*

$$\lim_{k\to+\infty} \gamma_k^{(1)} = +\infty\ . \tag{69}$$

3. The Poincaré Type Method with Blowing-up Transformation for Solving the N-body Problem

The special form of the N-body problem allows us essentially to improve the method of Sections 2.3, 2.4. The main result of Section 3.1 is obtaining the estimates of the maximal t-interval (ω_-, ω_+) on which the solution of the N-body problem exists and which is such that some fixed mutual distance (e.g. Δ_{12}) exceeds some fixed non-negative lower bound for $t \in (\omega_-, \omega_+)$. Namely, for given masses and initial data, the increasing sequences of the numbers γ_k, each of which provides the estimate $\omega_+ > \gamma_k$, are constructed. It appears that if $\omega_+ = +\infty$, then $\gamma_k \longrightarrow_{k\to+\infty} +\infty$. The rate of growth of the sequence is of particular interest in this case. As an illustrative application, the solution of relative equilibrium in the 3-body problem is considered in Section 3.2. In this case, γ_k tends to $+\infty$ as $\sqrt{\ln k}$.

Obtaining *the global solution of the N-body problem* is an auxiliary result in Section 3.1 (see 'First Step' there).

3.1. The Method (Babadzanjanz, 1979)

Consider the system (20) and the initial conditions

$$g_{ij}(1) = g_{ij}^0\ . \tag{70}$$

For given real non-negative $\Delta_{12}^* < \Delta_{12}(1)$, let us consider the function Δ such that $\Delta(t) = (\Delta_{12}(t) - \Delta_{12}^*)^{-1}$; this satisfies the equation

$$d\Delta/dt = -\Delta^2\Delta_{12}^{-1} \sum_{j=1}^{3} (g_{1j} - g_{2j})(\dot{g}_{1j} - \dot{g}_{2j}) \tag{71}$$

and the initial condition

$$\Delta(1) = \left(\left(\sum_{j=1}^{3} (g_{1j}^0 - g_{2j}^0)^2\right)^{1/2} - \Delta_{12}^*\right)^{-1}\ . \tag{72}$$

If (Ω_-, Ω_+), (ω_-, ω_+) are the maximal existence intervals of the Cauchy problems (20), (70) and (20), (70), (71), (72) respectively, then the inequalities

$$\Omega_- \leq \omega_- < 1 < \omega_+ \leq \Omega_+ \tag{73}$$

and, for all $t \in (\omega_-, \omega_+)$,

$$\Delta_{12}(t) > \Delta_{12}^* \tag{74}$$

are true. Consequently, solving the problem of obtaining the estimates of (ω_-, ω_+), implies that one should construct those estimates of $\mathcal{J}$ such that, for all $t \in \mathcal{J}$, $\Delta_{12}(t) > \Delta_{12}^*$. Analogously, considering the other additional variables instead of Δ, we can investigate the other properties of the solution of the problem (20), (70) such as some of bodies being in the fixed spherical zone or the non-existence of a period less than a fixed number, and so on. We restrict ourselves to the case of the problem (20), (70), (71), (72).

FIRST STEP (*The global solution of the N-body problem*).

Let us introduce instead of t in (20), (70), (71), (72) the new variable s defined by the formulae

$$s = t + \int_1^t X \, \mathrm{d}t \,, \tag{75}$$

$$\begin{aligned} X &= \sum_{T_1} \alpha_{ri}^{-3} \Delta_{ri}^{-3} + \alpha_{12}^{-3} \Delta^3 \,, \\ T_1 &= \{(r,i) \,|\, 0 \leq r < i \leq n-1 \,;\, (r,i) \neq (1,2)\} \,, \end{aligned} \tag{76}$$

where α_{ri} are positive parameters (one may attempt to improve the estimates by choosing of α_{ri}). Let us introduce the functions x_{ij}, y_{ij}, q of s such that

$$x_{ij}(s) = g_{ij}(t(s)) \,, \qquad y_{ij}(s) = p_{ij}(t(s)) \,, \qquad q(s) = \Delta(t(s)) \,, \tag{77}$$

where $p_{ij}(t) = \frac{d}{dt}\, g_{ij}(t) = \dot{g}_{ij}(t)$. These functions satisfy the equations

$$\mathrm{d}x_{ij}/\mathrm{d}s = X_{ij} \,, \qquad \mathrm{d}y_{ij}/\mathrm{d}s = Y_{ij} \,, \qquad \mathrm{d}q/\mathrm{d}s = Q \tag{78}$$

and the initial conditions

$$x_{ij}(1) = g_{ij}^0 \,,\ y_{ij}(1) = \dot{g}_{ij}^0 \,,\ q(1) = \left(\left(\sum_{j=1}^{3} (g_{1j}^0 - g_{2j}^0)^2\right)^{1/2} - \Delta_{12}^*\right)^{-1} , \tag{79}$$

where

$$Y_{ij} = k^2 z \sum_{r \in T(i)} m_r (x_{rj} - x_{ij}) u_{ri}^3 , \qquad u_{ri}(s) = \Delta_{ri}^{-1}(t(s)) ,$$

$$X_{ij} = z y_{ij} , \qquad z = \Big(1 + \sum_{T_1} \alpha_{ri}^{-3} u_{ri}^3 + \alpha_{12}^{-3} q^3\Big)^{-1} , \tag{80}$$

$$Q = -z u_{12} q^2 \sum_{j=1}^{3} (x_{1j} - x_{2j})(y_{1j} - y_{2j}) .$$

Let us introduce the functions $\tilde{z}$, u_{rij}, $\tilde{y}_{ij}$, $\tilde{u}_{ri}$, $\tilde{q}$, of s such that

$$u_{rij}(s) = (x_{rj}(s) - x_{ij}(s)) u_{ri}(s) , \qquad \tilde{u}_{ri}(s) = z^{1/3}(s) u_{ri}(s) ,$$
$$\tilde{y}_{ij}(s) = z^{1/3}(s) y_{ij}(s) , \qquad 0 \le r < i \le n-1 , \qquad j \in [1:3] ; \tag{81}$$
$$\tilde{z}(s) = z^{1/3}(s) , \qquad \tilde{q}(s) = z^{1/3}(s) q(s) ,$$

and, for $r > i$, let us use the designations

$$\tilde{u}_{ri} = \tilde{u}_{ir} , \qquad u_{rij} = -u_{irj} . \tag{82}$$

These functions satisfy the problem

$$\mathrm{d}\tilde{z}/\mathrm{d}s = \tilde{z}^2 \{\} ,$$

$$\mathrm{d}u_{rij}/\mathrm{d}s = \tilde{z} u_{ri} \Big(\tilde{y}_{rj} - \tilde{y}_{ij} - u_{rij} \sum_{k=1}^{3} u_{rik}(\tilde{y}_{rk} - \tilde{y}_{ik})\Big) ,$$
$$\mathrm{d}\tilde{y}_{ij}/\mathrm{d}s = \tilde{z}\tilde{y}_{ij}\{\} + k^2 \tilde{z}^2 \sum_{r \in T(i)} m_r u_{rij} \tilde{u}_{ri}^2 , \tag{83}$$
$$\mathrm{d}\tilde{u}_{ri}/\mathrm{d}s = \tilde{z}\tilde{u}_{ri}\{\} - \tilde{z}\tilde{u}_{ri}^2 \sum_{j=1}^{3} u_{rij}(\tilde{y}_{rj} - \tilde{y}_{ij}) ,$$
$$\mathrm{d}\tilde{q}/\mathrm{d}s = \tilde{z}\tilde{q}\{\} - \tilde{z}\tilde{q}^2 \sum_{j=1}^{3} u_{12j}(\tilde{y}_{1j} - \tilde{y}_{2j}) ,$$

$$\tilde{z}(1) = \tilde{z}^0 , \ u_{rij}(1) = u_{rij}^0 , \ \tilde{u}_{ri}(1) = \tilde{u}_{ri}^0 , \ \tilde{q}(1) = \tilde{q}^0 , \ \tilde{y}_{ij}(1) = \tilde{y}_{ij}^0 , \tag{84}$$

where

$$\{\} = \alpha_{12}^{-3} \tilde{q}^4 \sum_{j=1}^{3} u_{12j}(\tilde{y}_{1j} - \tilde{y}_{2j}) + \sum_{T_1} \alpha_{ri}^{-3} \tilde{u}_{ri}^4 \sum_{j=1}^{3} u_{rij}(\tilde{y}_{rj} - \tilde{y}_{ij})$$

with T_1 defined by (76) and $\tilde{z}^0$, u_{rij}^0, $\tilde{u}_{ri}^0$, $\tilde{q}^0$, $\tilde{y}_{ij}^0$ corresponding to (79)–(81).

PROPOSITION 8. *Let the functions $\tilde{z}$, u_{rij}, $\tilde{y}_{ij}$, $\tilde{u}_{ri}$, $\tilde{q}$ satisfy the problem (83), (84), and let (ω_-^*, ω_+^*) be the maximal existence interval of the solution of the problem (78), (79). Then the following inequalities hold for every $s \in (\omega_-^*, \omega_+^*)$:*

$$\begin{aligned} &|u_{rij}(s)| \le 1 , \qquad 0 < \tilde{u}_{ri}(s) < \alpha_{ri} , \\ &0 < \tilde{q}(s) < \alpha_{12} , \quad 0 < \tilde{z}(s) < 1 , \end{aligned} \tag{85}$$

$$|\tilde{y}_{ij}(s)| \le V_i = \sqrt{\frac{2}{m_i}(h + k^2 Bd)} \times (1 + Bd^3)^{-1/3} \tag{86}$$

where

$$B = \sum_{i=1}^{n-1} \sum_{r=0}^{i-1} (\alpha_{ri} m_r m_i)^{3/2} , \tag{87}$$

his the energy constant, i.e.

$$\frac{1}{2} \sum_{i=0}^{n-1} m_i \sum_{j=1}^{3} y_{ij}^2 - k^2 \sum_{i=1}^{n-1} \sum_{r=0}^{i-1} m_r m_i u_{ri} = h , \tag{88}$$

and dis the only positive root of the equation

$$\mathrm{d}^3 + \frac{2h}{k^2 B} \mathrm{d}^2 - \frac{1}{B} = 0 . \tag{89}$$

Proof. See Proposition 5 in (Babadzanjanz, 1979).

Remark 2. For fixed masses and α_{ri}, the less is energy constant h, the better is the estimate (86). To decrease h, one can choose the inertial coordinate system used in the problem (20), (70). One can show that the minimal value of h (obtained in the way just mentioned) is

$$\begin{aligned} h_0 = \frac{1}{2} \sum_{i=0}^{n-1} m_i \sum_{j=1}^{3} (\dot{g}_{ij}^0)^2 - \frac{1}{2m} \sum_{j=1}^{3} \Big(\sum_{i=0}^{n-1} m_i \dot{g}_{ij}^0 \Big)^2 - \\ - k^2 \sum_{i=1}^{n-1} \sum_{r=0}^{i-1} m_r m_i \Delta_{ri}^{-1}(1) , \end{aligned} \tag{90}$$

where $m = m_0 + \cdots + m_{n-1}$.

PROPOSITION 9. *Let (ω_-, ω_+) be the maximal existence interval of the solution of the Cauchy problem (20), (70), (71), (72). Let the function s of t be* *defined by (75) and let the initial data g_{1j}^0, g_{2j}^0 and Δ_{12}^* satisfy the condition*

$$0 \le \Delta_{12}^* < \Big(\sum_{j=1}^{3} (g_{1j}^0 - g_{2j}^0)^2\Big)^{1/2} . \tag{91}$$

Then:

$$\lim_{t \to \omega_+ - 0} s(t) = +\infty , \qquad \lim_{t \to \omega_- + 0} s(t) = -\infty . \tag{92}$$

Proof. See Proposition 6 in (Babadzanjanz, 1979).

PROPOSITION 10. *Let the functions $\tilde{z}$, u_{rij}, $\tilde{y}_{ij}$, $\tilde{u}_{ri}$, $\tilde{q}$ satisfy the problem (83), (84) and let the function t of s be defined by (75). Let $z(s) = dt(s)/ds$ and let μ_k, ν_r be positive parameters.*

Then the functions $\tilde{z}$, u_{rij}, $\tilde{y}_{ij}$, $\tilde{u}_{ri}$, $\tilde{q}$ and t, z are regular in the strip $\mathcal{M}_\rho(\mathcal{R})$ and, for every $s \in \mathcal{M}_\rho(\mathcal{R})$, satisfy the inequalities

$$\begin{aligned} &|\tilde{z}(s)| \le \mu_1 \Gamma , \quad |u_{rij}(s)| \le \mu_2 \Gamma , \quad |\tilde{u}_{ri}(s)| \le \mu_3 \Gamma , \\ &|\tilde{q}(s)| \le \mu_3 \Gamma , \quad |\tilde{y}_{rj}(s)| \le \nu_r \Gamma , \end{aligned} \tag{93}$$

$$|z(s)| \le \mu_1^3 \Gamma^3 , \qquad \Gamma = \mu (1 - 7\sigma |\Im s|)^{-1/7} \tag{94}$$

where

$$\rho = \frac{1}{7\sigma} , \qquad \sigma = \max\{\sigma_1, \sigma_2, \sigma_3\} \tag{95}$$

$$\begin{aligned} &\sigma_1 = A_2 \mu^2 + A_{41} \mu^4 , \quad \sigma_2 = A_{42} \mu^4 + A_7 \mu^7 , \quad \sigma_3 = A_{41} \mu^4 + A_7 \mu^7 , \\ &A_2 = \mu_1 \mu_2^{-1} \mu_3 \max_{r \ne i} (\nu_r + \nu_i) , \\ &A_7 = 3 \mu_1 \mu_2 \mu_3^4 \sum_{i=1}^{n-1} \sum_{r=0}^{i-1} (\nu_r + \nu_i) \alpha_{ri}^{-3} , \\ &A_{41} = 3 \mu_1 \mu_2 \mu_3 \max_{r \ne i} (\nu_r + \nu_i) , \\ &A_{42} = k^2 \mu_1^2 \mu_2 \mu_3^2 \max_i \Big(\nu_i^{-1} \sum_{r \in T(i)} m_r\Big) , \\ &\mu = \max\Big\{\mu_1^{-1}, \mu_2^{-1}, \mu_3^{-1} \max_{r<i} \alpha_{ri}, \max_i \nu_i^{-1} V_i\Big\} , \qquad r, i \in [0 : n-1] \end{aligned} \tag{96}$$

with V_i, $T(i)$ defined by (86), (20).

Proof. Using the substitutions

$$\tilde{z} \to \mu_1 \tilde{z}\,, \quad u_{rij} \to \mu_2 u_{rij}\,, \quad \tilde{u}_{ri}(s) \to \mu_3 \tilde{u}_{ri}\,,$$
$$\tilde{q} \to \mu_3 \tilde{q}\,, \quad \tilde{y}_{rj} \to \nu_r \tilde{y}_{rj}\,, \tag{97}$$

in (83), (84) and, then, applying Proposition 1 to the problem obtained one proves that $\tilde{z}$, u_{rij}, $\tilde{y}_{ij}$, $\tilde{u}_{ri}$, $\tilde{q}$ are regular and satisfy the inequalities (93) for $s \in \mathcal{M}_\rho(\mathcal{R})$. It remains to be noted that $z(s) = \tilde{z}^3(s)$.

COROLLARY. *For every $s \in \mathcal{M}_\rho(\mathcal{R})$, the following inequalities hold:*

$$|\Im t(s)| \leq \chi\,, \tag{98}$$

$$|\Re t(s) - t(\Re s)| \leq \chi\,, \tag{99}$$

$$|t(s)| \leq |t(\Re s)| + \chi\,, \qquad \chi = \mu_1^3 \mu^3 |\Im s|\,(1 - 7\sigma\,|\Im s|)^{-3/7}\,. \tag{100}$$

One may now get the series expansion, which is valid for every global solution of the N-body problem (see the Section 'Poincaré type methods' in the Introduction). Namely, using the substitution (9) (with $h = \rho$) in the problem (20), (70), (71), (72) (with $\Delta_{12}^* = 0$) one obtains the differential equations for the variables $\tilde{z}(s(\lambda)), \ldots$ These functions are regular in $\mathcal{O}_1$, i.e. $\tilde{z} = \sum_{m=0}^{\infty} \alpha_m \lambda^m, \ldots$. Using the differential system mentioned for $\tilde{z}(s(\lambda)), \ldots$, one obtains $\alpha_m, \ldots$ step by step by the method of undetermined coefficients. Then, using (81) and the equation

$$\mathrm{d}t/\mathrm{d}\lambda = z(t(s(\lambda)))\,\mathrm{d}s/\mathrm{d}\lambda \tag{101}$$

one obtains $x_{ij}(s(\lambda)) = g_{ij}(t(s(\lambda)))$, $y_{ij}(s(\lambda)) = (\mathrm{d}/\mathrm{d}t) g_{ij}(t(s(\lambda)))$, and $t(s(\lambda))$.

SECOND STEP (*Obtaining the estimates*).

The purpose of this step is to obtain the sequences of estimates $\omega_+ > \gamma_k$. First notice that the function $\lambda : \mathcal{M}_r((0,\infty)) \to \mathcal{C}$ such that

$$s(\lambda) = \frac{2r}{\pi} \operatorname{arsh}\left(\frac{1+\lambda}{1-\lambda} \operatorname{sh}\left(\frac{\pi}{2r}\right)\right)\,, \tag{102}$$

transforms conformally the half-strip $\mathcal{M}_r((0,\infty))$ onto the circle $\mathcal{O}_1$.

For $r = r_0$, introduce the regular function $w : \mathcal{O}_1 \to \mathcal{C}$ such that

$$w(\lambda) = t(s(\lambda))\,. \tag{103}$$

For $\lambda \in \mathcal{O}_1$, from (103) and (100) with $r_0 \leq \rho$ it follows

$$|w(\lambda)| < \omega_+ + 1.2\mu_1^3 \mu^3 r_0\,. \tag{104}$$

By this inequality and Proposition 6 one can prove the following results.

PROPOSITION 11. *Under the assumptions of Proposition 6, let w be defined by (103). Let (ω_-, ω_+) be the maximal existence interval of the solution of the Cauchy problem (20), (70), (71), (72). Let the quantities μ, ρ be defined by (95), (96) and let r_0 be a fixed number satisfying the inequality $r_0 \leq \rho/3$.*

Then, for $j = 1, 2$ and for every natural k, the inequalities

$$\omega_+ > \gamma_k^{(j)} - 1.2\mu_1^3\mu^3 r_0 \tag{105}$$

hold true. In addition, if $\omega_+ = +\infty$, then

$$\lim_{k\to+\infty} \gamma_k^{(1)} = +\infty\ . \tag{106}$$

We shall consider an example in Section 3.2. If we should apply Proposition 11 there, then the development of the function w should take a very complicated form. In order to simplify the developments just mentioned we estimate there the quantity $\omega = \max\{\omega_+, |\omega_-|\}$ instead of ω_+, making use of the following:

PROPOSITION 12. *Under the assumptions of the Proposition 11, let, for $\lambda \in \mathcal{O}_1$, the function s of λ be defined by (Poincaré transformation - see (9))*

$$s(\lambda) = \frac{2r}{\pi}\ln\frac{1+\lambda}{1-\lambda} + 1 \tag{107}$$

instead of (102), and let $\omega = \max\{\omega_+, |\omega_-|\}$.
Then, for $j = 1, 2$ and for every natural k, the inequalities

$$\omega > \gamma_k^{(j)} - 1.2\mu_1^3\mu^3 r_0 \tag{108}$$

hold true. In addition, if $\omega = +\infty$, then

$$\lim_{k\to+\infty} \gamma_k^{(1)} = +\infty\ . \tag{109}$$

3.2. Example

For obtaining γ_k^j in the N-body problem we restrict ourselves to a problem such that one can obtain $w(\lambda)$ as an explicit function.

Let us return to the system (20), (71) and let $\Delta_{12}^* = 0$ there; then $\omega_\pm = \Omega_\pm$ (see (73), (74)) and $\Delta = \Delta_{12}^{-1} = u_{12}$ (see (80)). In order to obtain the function $w(\lambda)$ in a simple form one should consider such a problem that the function t of s defined by (75) be a simple one. If the mutual distances in N-body problem do not depend on t, then (see (75), (76))

$$s = (1 + X)t - X\ , \tag{110}$$

where

$$X = \sum_{i=1}^{n-1} \sum_{r=0}^{i-1} \alpha_{ri}^{-3} \Delta_{ri}^{-3}\ . \tag{111}$$

It is well known that the N-body problem has the solutions satisfying the condition just mentioned, namely the solutions of relative equilibrium. Consider the solution of relative equilibrium of the 3-body problem such that

$$m_0 = m_1 = m_2 = m_{\odot} = 1\ , \qquad \Delta_{01} = \Delta_{02} = \Delta_{12} = 1 \text{ AU}\ . \tag{112}$$

In this case, the equilateral triangle formed by three bodies rotates around the centre of mass G with the angular velocity $k\sqrt{3}$, while G has a uniform motion with reference to the inertial coordinate system.

In order to obtain the estimates of ω (see Proposition 12) let us consider the function w. Using (103), (107), (110) we have

$$w(\lambda) = c_1\Big(1 + \frac{2r}{\pi} \ln \frac{1+\lambda}{1-\lambda}\Big) + c_2\ , \tag{113}$$

where

$$c_1 = (1+X)^{-1}\ , \qquad c_2 = c_1 X\ , \qquad r = r_0 \le \rho/3\ , \tag{114}$$

and ρ is calculated by (95). For every $\lambda \in \mathcal{O}_1$ the expansion

$$w(\lambda) = 1 + \frac{4rc_1}{\pi} \sum_{m=1}^{\infty} \frac{1}{2m-1} \lambda^{2m-1} \tag{115}$$

holds true. By means of the formula

$$\sum_{m=1}^{k} (2m-1)^{-1} = \tfrac{1}{2} \ln k + \tfrac{1}{2}\,\mathrm{C} + \ln 2 + \frac{B_2}{8k^2} + \cdots\ , \tag{116}$$

where C is the Euler constant and $B_2, \ldots$ are the Bernoulli numbers, one obtains

$$\gamma_{2k-1}^{(2)} = \gamma_{2k}^{(2)} = \frac{4rc_1}{\pi} (\tfrac{1}{2} \ln k + \cdots)^{1/2}\ . \tag{117}$$

In order to estimate ρ one should choose the parameters α_{ri}, ν_i, μ_1, μ_2, μ_3 in (96) and estimate V_i defined by (86). Let us choose such an inertial coordinate system in the problem (20), (70), (71), (72) that $h = h_*$ (see (90)); in the example considering $h_* = -1.5k^2$. Putting $\alpha_{ri} = 1$, by (86), (87), (90) one obtains $V_i < 1.2k$. On choosing the parameters in formulae (95), (96) such that

$$\mu_1 = \mu_2 = \mu_3 = \nu_i / V_i = \mu^{-1} ,$$

one obtains

$$\rho > 0.28\,\text{days} , \qquad \mu_1 \mu = 1 . \tag{118}$$

Using the Proposition 12, the equalities (117) and $c_1 = 1/4$, and putting $r = \rho/3$ one obtains the inequality

$$\omega = \max\{\omega_+, |\omega_-|\} > \left(\frac{0.09}{\pi}(\ln k + \cdots)^{1/2} - 0.12\right)\,\text{days} \tag{119}$$

for every $k = 1, 2, \ldots$.

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