EXISTENCE OF THE CONTINUATIONS IN THE N-BODY PROBLEM

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Abstract. The method of obtaining the estimates of the maximal t-interval (ω_-, ω_+) on which the solution of the N-body problem exists and which is such that some fixed mutual distance (e.g. Δ_{12}) exceeds some fixed non-negative lower bound, for all t contained in (ω_-, ω_+), is considered. For given masses and initial data, the increasing sequences of the numbers γ_k, each of which provides the estimate $\omega_+ > \gamma_k$, are constructed. It appears that if $\omega_+ = +\infty$, then $\gamma_k \xrightarrow[k \to +\infty]{} +\infty$.

1. Notation.

To introduce the designations, the symbol $\stackrel{\text{def}}{=}$ is used. The designations used here do not differ essentially from the traditional ones; for the convenience of the reader some of them are listed here.

The sets of the real, complex and whole numbers are denoted by R, C, Z respectively;

$\{a, b, \ldots, f\} \stackrel{\text{def}}{=}$ the set consisting of elements $a, b, \ldots, f$;

$\{x \in E \mid P(x)\} \stackrel{\text{def}}{=}$ the set consisting precisely of those $x \in E$, which satisfy the condition $P(x)$;

$A^2 \stackrel{\text{def}}{=} A \times A, \ldots, A^n \stackrel{\text{def}}{=} A^{n-1} \times A$;

$[m:n] \stackrel{\text{def}}{=} \{j \in Z \mid m \leqslant j \leqslant n\}$;

$\mathcal{O}_\rho \stackrel{\text{def}}{=} \{t \in C \mid |t| < \rho\}, \quad \rho > 0$;

$\mathcal{M}_r(\mathcal{I}) \stackrel{\text{def}}{=} \{t \in C \mid |\text{Im}\, t| < r;\ \text{Re}\, t \in \mathcal{I}\},\ \mathcal{I} \subset R$;

$f\colon A \to B \stackrel{\text{def}}{=}$ the mapping from A into B;

$t \to f(t) \stackrel{\text{def}}{=}$ the mapping which assigns the value $f(t)$ to each t;

$\{\alpha_j\}$ – a sequence.

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2. Introduction

Consider the point masses $m_o, \ldots, m_{N-1}$ moving according to Newton's law of gravitation with reference to the inertial coordinate system $Og_1g_2g_3$. Consider the system of equations

$$\frac{d^2 g_{ij}}{dt^2} = \varepsilon^2 \kappa^2 \sum_{r \in T(i)} m_r (g_{rj} - g_{ij}) \Delta_{ri}^{-3} \tag{1}$$

and the initial conditions

$$g_{ij}(1) = g_{ij}^0, \qquad \dot{g}_{ij}(1) = \varepsilon \dot{g}_{ij}^0, \tag{2}$$

where

$$i \in [0: N-1],\ j \in [1:3],\ T(i) \stackrel{\text{def}}{=} [0: N-1]/\{i\};\ g_{i1}, g_{i2}, g_{i3}$$

are the coordinates of m_i;

$$\Delta_{ri} \stackrel{\text{def}}{=} \left(\sum_{j=1}^{3} (g_{rj} - g_{ij})^2 \right)^{1/2};$$

κ is the Gaussian constant; ε is a positive parameter.

REMARK 1. The parameter ε has been introduced for the convenience of the applications; the case $\varepsilon \neq 1$ is reduced to the case $\varepsilon = 1$ by letting $\tau = \varepsilon t + 1 - \varepsilon$.

For given real non-negative $\Delta_{12}^* < \Delta_{12}(1)$, let us consider the function Δ such that

$$\Delta(t) \stackrel{\text{def}}{=} (\Delta_{12}(t) - \Delta_{12}^*)^{-1};$$

this satisfies the equation

$$\frac{d\Delta}{dt} = -\Delta^2 \Delta_{12}^{-1} \sum_{j=1}^{3} (g_{1j} - g_{2j})(\dot{g}_{1j} - \dot{g}_{2j}) \tag{3}$$

and the initial condition

$$\Delta(1) = \left(\sqrt{\sum_{j=1}^{3} (g_{1j}^0 - g_{2j}^0)^2} - \Delta_{12}^* \right)^{-1} \tag{4}$$

If (Ω_-, Ω_+), (ω_-, ω_+) are the maximal existence intervals of the Cauchy problems (1)–(2) and (1)–(4) respectively, then the inequalities

$$\Omega_- \leqslant \omega_- < 1 < \omega_+ \leqslant \Omega_+ \tag{5}$$

and, for all $t \in (\omega_-, \omega_+)$,

$$\Delta_{12}(t) > \Delta_{12}^* \tag{6}$$

are true. Consequently, solving the problem of obtaining the estimates of (ω_-, ω_+), implies that one should construct those estimates of $\mathscr{I}$ such that, for all $t \in \mathscr{I}$, $\Delta_{12}(t) > \Delta_{12}^*$. Analogously (introducting the other additional variables instead of Δ) we can investigate the other properties of the solution of the problem (1)–(2) such as some of the bodies being in the fixed spherical zone or the non-existence of a period less than a fixed number, and so on. We restrict ourselves to the case of the problem (1)–(4).

We briefly sketch now an outline of the method: For a given Cauchy problem (1)–(4) (i.e. for coefficients of the right-hand sides and initial data) the sequence of Taylor coefficients of the complex valued regular in the circle function w of the complex argument λ is constructed (see the Proposition 8). It appears that the function w is bounded if and only if the solution of the original Cauchy problem is not continuable along $[1, +\infty)$; this makes it possible to obtain the monotonically increasing sequence of the estimates from below the right end ω_+ of the maximal existence interval (ω_-, ω_+). The n-th estimate depends merely on the Taylor coefficients with the numbers $j \leqslant n$. If the solution is continuable along $[1, +\infty)$, then the sequence of the estimates mentioned is divergent to $+\infty$.

Analysing the above transition from the original Cauchy problem to the estimates, one can state that, for every natural n, the estimate of the existence interval can be found merely by use of the Taylor coefficients of the solution with numbers $j \leqslant n$. This seems to be a mistake since, for every fixed $r \in [0, +\infty]$, every polynomial can be added to the power series with radius of convergence equal to r. On the other hand, it is the Taylor coefficients of the function which satisfies a given Cauchy problem with which the above method deals, and this is an additional condition which allows us to cross from an unsolved problem of estimation of the existence interval of the solution merely by use of the Taylor coefficients with numbers $j \leqslant n$ to a solved problem of estimation of a regular function in circle $\mathcal{O}_1$ by use of analogous coefficients.

REMARK 2. The above procedure is analogous to the more general method described in [1], §8; the special form of the N-body problem allow us essentially to improve it.

For the convenience of the reader, some auxiliary results of [1] are given in Section 3.

3. The Preliminary Function – theoretical results ([1], §§ 6, 7)

As was stated above, one can reduce the problem of estimation of the existence interval to that of a regular function in a circle. Some results for solving the last problem by means of the Taylor coefficients are stated in the following proposition.

PROPOSITION 1. Let $w\colon \mathcal{O}_1 \to C$ be a regular function and $\sum_{j=0}^{\infty} \alpha_j \lambda^j$ its Taylor series expansion in $\mathcal{O}_1$. Let ρ_0 be a fixed number of [0, 1) and let

$$M(w) \overset{\text{def}}{=} \sup_{\lambda \in \mathcal{O}_1} |w(\lambda)|, \tag{7}$$

$$\gamma_n^{(1)} \stackrel{\text{def}}{=} \max_{\rho \in [\rho_0, 1]} \left(\frac{1}{n+1} \sum_{\gamma=0}^{n} |\alpha_0 + \cdots + \rho^{n-\gamma} \alpha_{n-\gamma}|^2 \right)^{1/2}, \tag{8}$$

$$\gamma_n^{(2)} \stackrel{\text{def}}{=} \left(\sum_{m=1}^{n} m|\alpha_m|^2 \right)^{1/2}. \tag{9}$$

Then, for each natural n, the inequalities

$$M(w) \geqslant \gamma_n^{(j)}, \qquad j = 1, 2 \tag{10}$$

are true.

In addition, if

$$\lim_{\substack{\lambda \to 1-0 \\ \lambda \in (0, +1)}} |w(\lambda)| = +\infty, \tag{11}$$

then

$$\lim_{n \to +\infty} \gamma_n^{(1)} = +\infty. \tag{12}$$

REMARK 3. Proof see in [1], § 7, Propositions 3, 4; it is based on the classic results (see [3], § 9.6).

Let us consider a polynominal system

$$\frac{\mathrm{d}x_j}{\mathrm{d}t} = \sum_{m=1}^{L+1} \sum_{i \in I(m)} a_j[i] X^i \tag{13}$$

and initial conditions

$$x_j(0) = x_{j0}, \tag{13'}$$

where

$$i \stackrel{\text{def}}{=} (i_1, \ldots, i_n) \in Z^n, \qquad X \stackrel{\text{def}}{=} (x_1, \ldots, x_n) \in C^n,$$

$$X^i \stackrel{\text{def}}{=} x_1^{i_1} \cdot \ldots \cdot x_n^{i_n}, \qquad j \in [1:n],\ L \in [0:+\infty),$$

$$I(m) \stackrel{\text{def}}{=} \{i \in Z^n \mid i_1 \geqslant 0; \ldots; i_n \geqslant 0; |i| = m\}, \tag{13''}$$

$$|i| \stackrel{\text{def}}{=} i_1 + \cdots + i_n, \qquad a_j[i] \in C, \qquad x_{j0} \in C.$$

In the Proposition 2, the system just written is reduced to an infinite linear autonomous system of ordinary differential equations of the first order.

For every $i \in \bigcup_{m=1}^{+\infty} I(m)$, let us introduce the new variables

$$x[i] \stackrel{\text{def}}{=} X^i \tag{14}$$

To order the set of the variables introduced, let us arrange the sets $\chi_m \stackrel{\text{def}}{=} \{x[i] \mid i \in I(m)\}$ from left to right in order $\chi_1, \chi_2, \ldots$ and, for every m, let us arrange the elements of χ_m from left to right so that $i_1 = k_1, \ldots, i_j = k_j, i_{j+1} > k_{j+1}$ imply

$x[i_1, \ldots, i_n]$ precedes $x[k_1, \ldots, k_n]$. Let us denote all the elements written from left to rignt by symbols $x_1, x_2, \ldots$, and let us introduce the designation $x = (x_1, x_2, \ldots)$. Notice that the sets χ_m and $\bigcup_{m=1}^{r} \chi_m$ consist of $s_m(n) =$

$$\frac{(m+n-1)!}{(n-1)!\,m!} \quad \text{and} \quad \sigma_r(n) = \frac{(r+n)!}{r!n!} - 1$$

elements respectively.

PROPOSITION 2. Let

$$a_j[i] \overset{\text{def}}{=} 0 \tag{15}$$

if $|i| > L + 1$ or if i has a negative component, and let

$$\mathscr{I}(m) \overset{\text{def}}{=} \{i \in Z^n \mid i_1 \geqslant -1; \ldots; i_n \geqslant -1; |i| = m\}. \tag{16}$$

Then the functions $x[k]$ introduced by (14) satisfy the equations

$$\frac{dx[k]}{dt} = \sum_{m=0}^{L} \sum_{i \in I(m)} \alpha[k; i] x[k+i], \tag{17}$$

where

$$\alpha[k; i] \overset{\text{def}}{=} \sum_{j=1}^{n} k_j a_j[i_1, \ldots, i_j + 1, \ldots, i_n]. \tag{18}$$

REMARK 4. Proof see in [1], § 4, Proposition 1.

Let us write (17) in the form of the equation

$$\frac{dx}{dt} = \mathscr{A}x, \tag{19}$$

where

$$\frac{dx}{dt} \overset{\text{def}}{=} \left(\frac{dx_1}{dt}, \frac{dx_2}{dt}, \ldots\right) \tag{20}$$

and $\mathscr{A}$ is a complex infinite matrix such that its every row and column has only a finite number of non-zero elements. Let us write also the initial conditions

$$x(0) = x^0 \tag{19'}$$

with

$$x^0 \overset{\text{def}}{=} (x_{10}, \ldots, x_{n0}, x_{10}^2, \chi_{10}\, x_{20}, \ldots, x_{n0}^2, \ldots) \tag{21}$$

corresponding to (13').

In the Proposition 4, we shall obtain a new estimate of the radius of convergence of the solutions of the problem (13)–(13'); to prove it, we shall use the Proposition 3.

PROPOSITION 3. Let $p \in (\sigma_{q-1}(n) : \sigma_q(n)]$ ($\sigma_q(n)$ is defined just before the Proposition 2), and let there exist such γ, $M \in (0, +\infty)$ that

$$|x_p| \leqslant M\gamma^q, \tag{22}$$

for every natural q. Let

$$a_{mr} \overset{\text{def}}{=} \sum_{i \in \mathscr{I}(m)} |a_r[i_1, \ldots, i_r + 1, \ldots, i_n]| = \sum_{i \in I(m+1)} |a_r[i]|, \tag{23}$$

where $a_r[i]$ are the coefficients of the system (13), $\mathscr{I}(m)$, $I(m)$ are defined by (16), (13″), and let

$$s_a(\gamma) \overset{\text{def}}{=} \max_{r \in [1:n]} \sum_{m=0}^{L} a_{mr}\gamma^m. \tag{24}$$

Let $\mathscr{A}^j$ be the j-th power of $\mathscr{A}$, and let $(\mathscr{A}^j x)_p$ denotes the p-th component of the vector $\mathscr{A}^j x$.

Then

$$|(\mathscr{A}^j x)_p| \leqslant M s_a^j(\gamma)\gamma^q \prod_{m=0}^{j-1} (q + mL), \tag{25}$$

for every natural j, q.

REMARK 5. The above Proposition differs from the analogous one of [1] (§ 6.3, Proposition 4) by the formulation of $s_a(\gamma)$; besides, the coefficients $a_j[i]$ in [1] were the real numbers.

Proof. Let $k_1 + \cdots + k_n = q$, and let x_p is $x[k]$, then by (17), (18) one obtains

$$\begin{aligned} |\mathscr{A}x)_p| &\leqslant \\ &\leqslant \sum_{m=0}^{L} \sum_{i \in \mathscr{I}(m)} \sum_{j=1}^{n} k_j |a_j[i_1, \ldots, i_j + 1, \ldots, i_n]| \, |x[k + i]| \leqslant \\ &\leqslant M\gamma^q \sum_{m=0}^{L} \gamma^m \sum_{j=1}^{n} k_j \sum_{i \in \mathscr{I}(m)} |a_j[i_1, \ldots, i_j + 1, \ldots, i_n]| \leqslant \\ &\leqslant M s_a(\gamma)\gamma^q q. \end{aligned} \tag{26}$$

Thus, the inequality (25) has proved for $j = 1$. Let it be true for $j = r$, and let us introduce the quantities y_p, y, $y[k]$ such that

$$y[k] \overset{\text{def}}{=} \frac{\mathrm{d}^r x[k]}{\mathrm{d}t^r}, \qquad y_p \overset{\text{def}}{=} \frac{\mathrm{d}^r x_p}{\mathrm{d}t^r}, \qquad y \overset{\text{def}}{=} (y_1, y_2, \ldots).$$

On differentiating the Equation (19), one obtains $\mathscr{A}^r x = y$,

$$(\mathscr{A}_x^{r+1})_p = (\mathscr{A}y)_p = \frac{\mathrm{d}y_p}{\mathrm{d}t}.$$

As one sees now, the formulae (17), (18) imply the inequalities

$$|(\mathscr{A}_x^{r+1})_p| \leqslant$$

$$\sum_{m=0}^{L} \sum_{i \in \mathscr{I}(m)} \sum_{j=1}^{n} k_j |a_j[i_1, \ldots, i_j + 1, \ldots, i_n]| \, |y[k + i]| \leqslant$$

$$\leqslant \sum_{m=0}^{L} \sum_{i \in \mathscr{I}(m)} \sum_{j=1}^{n} k_j |a_j[i_1, \ldots, i_j + 1, \ldots, i_n]| \times$$

$$\times \; M s_a^r(\gamma) \gamma^{|k+i|} \prod_{\nu=0}^{r-1} (|k + i| + \nu L) \leqslant$$

$$\leqslant M s_a^{r+1}(\gamma) \gamma^q q \prod_{\nu=0}^{r-1} (q + L + \nu L) = M s_a^{r+1}(\gamma) \gamma^q \prod_{\nu=0}^{r} (q + \tfrac{1}{2}L). \qquad (27)$$

Thus, the inequality (25) is true for $j = r + 1$.

COROLLARY 1. If $x_1, \ldots, x_n$ is the solution of the problem (13)–(13′) and, for every $p \in [1:n]$,

$$|x_{p0}| \leqslant \gamma, \qquad (28)$$

then

$$\left|\frac{d^j x_p(0)}{dt^j}\right| \leqslant \gamma s_a^j(\gamma) \prod_{m=0}^{j-1} (1 + mL). \qquad (29)$$

In fact, the first n of the components of the solution x of the problem (19)–(19′) form the solution of the problem (13)–(13′), and it satisfies the equalities

$$\frac{d^j x_p}{dt^j} = (\mathscr{A}^j x)_p. \qquad (30)$$

Thus, (29) follows from (25).

REMARK 5. The inequality (29) may be used to obtain estimates of the numerical integration of differential equations.

PROPOSITION 4. Let the inequalities (28) be true, and let $s_a(\gamma)$ be defined by (24). Then: (a) The solution $X = (x_1, \ldots, x_n)$ of the problem (13)–(13′) is regular in circle $\mathscr{O}_\rho$, where

$$\rho \overset{\text{def}}{=} \frac{1}{L s_a(\gamma)}. \qquad (31)$$

(b) For all $t \in \mathscr{O}_\rho$, $p \in [1:n]$ it satisfies the inequalities

$$|x_p(t)| \leqslant \gamma (1 - |t| \rho^{-1})^{-1/L}. \qquad (32)$$

Proof. Using the inequalities (29) one obtains

$$\left|\sum_{j=0}^{N} \frac{d^j x_p(0)}{dt^j} \cdot \frac{t^j}{j!}\right| \leqslant \gamma \left(1 + \sum_{j=1}^{N} \frac{s_a^j(\gamma) |t|^j}{j!} \prod_{m=0}^{j-1} (1 + mL)\right) <$$

$$< \gamma\left(1 + \sum_{j=1}^{N} \frac{|s_a(\gamma)Lt|^j}{j!} \prod_{m=0}^{j-1}\left(\frac{1}{L} + m\right)\right). \tag{33}$$

It remains to remember the binomial formula

$$(1 - \chi)^{-\alpha} = 1 + \sum_{j=1}^{\infty} \frac{\chi^j}{j!} \prod_{m=0}^{j-1} (\alpha + m). \tag{34}$$

REMARK 6. The above Proof is analogous to that of the Proposition 7 in § 6.4, [1].

COROLLARY 2. Let $\alpha_1, \ldots, \alpha_n$ be positive parameters, and let $\alpha^i = \alpha_1^{i_1} \cdot \ldots \cdot \alpha_n^{i_n}$. Then: (a) The solution $X = (x_1, \ldots, x_n)$ of the problem (13)–(13′) is regular in circle $\mathcal{O}_{\mathscr{R}}$, where

$$\mathscr{R} \stackrel{\text{def}}{=} \frac{1}{Ls_a(\mu)}, \tag{35}$$

$$S_a(\mu) \stackrel{\text{def}}{=} \max_{r \in [1:n]} \sum_{m=0}^{L} a_{mr}\mu^m, \tag{36}$$

$$a_{mr} \stackrel{\text{def}}{=} \alpha_r^{-1} \sum_{i \in I[m+1]} \alpha^i |a_r[i]|, \tag{37}$$

$$\mu \stackrel{\text{def}}{=} \max_{j \in [1:n]} \alpha_j^{-1} |x_{j0}|. \tag{38}$$

(b) For all $t \in \mathcal{O}_{\mathscr{R}}$, $p \in [1:n]$ it satisfies the inequalites

$$|x_p(t)| \leqslant \mu\alpha_p(1 - |t|\mathscr{R}^{-1})^{-1/L}. \tag{39}$$

In fact, the functions x_j and

$$y_j(t) \stackrel{\text{def}}{=} \alpha_j^{-1} x_j(t) \tag{40}$$

are regular in the same region. Using the substitution (40) in (13)–(13′) and then the Proposition 4, one obtains the results, which are sought for.

REMARK 8. Notice that instead of the Proposition 4 one can use the Cauchy's theorem (see [2], §§ 4, 5), but it should be noted that ρ and $\mathscr{R}$ from the Proposition 4 and the Corollary 2 do not depend on the dimension n of the system (13) while the analogous quantity ρ' from the theorem is inversely proportional to $n + 1$; besides, the quantities ρ, $\mathscr{R}$ can be found by a straightforward counting while for counting ρ' one should evaluate the maximum in a complex polydisk of the modulus of the right-hand sides of the system. In the case of N-body problem, let us give the numerical results obtained by I. A. Chernov in her graduation thesis: if $t_0 = 2\,415\,000.5$ I.D, the problem of the Sun and the five outer planets are considered, then $\mathscr{R}_{\max} \geqslant 91 d$, $\mathscr{R}_s = 1.5 \times 10^{-25} d$, where

$$\mathscr{R}_{\max} \overset{\text{def}}{=} \max_{\alpha} \mathscr{R}$$

(see (35)) and $\mathscr{R}_s$ is the analogous estimate given by Siegel ([2], § 5).

4. Estimation of the Existence Interval in the N-body Problem

4.1. FIRST STEP

Let us introduce instead of t in (1)–(4) the new variable s defined by the formulae

$$s = t + \int_1^t X \, \mathrm{d}t, \tag{41}$$

$$\left.\begin{aligned} &X \overset{\text{def}}{=} \sum_{T_1} \alpha_{ri}^{-3} \Delta_{ri}^{-3} + \alpha_{12}^{-3} \Delta^3, \\ &T_1 \overset{\text{def}}{=} \{(r, i) | 0 \leqslant r < i \leqslant N - 1\} / \{(1, 2)\}, \end{aligned}\right\} \tag{42}$$

where α_{ri} are positive parameters (one may attempt to improve the estimates by choosing of α_{ri}). Let us introduce the functions x_{ij}, y_{ij}, q of s such that

$$x_{ij}(s) \overset{\text{def}}{=} g_{ij}(t(s)), \qquad Y_{ij}(s) \overset{\text{def}}{=} p_{ij}(t(s)), \qquad q(s) \overset{\text{def}}{=} \Delta(t(s)), \tag{43}$$

where

$$p_{ij}(t) \overset{\text{def}}{=} \varepsilon^{-1} \frac{\mathrm{d}g_{ij}}{\mathrm{d}t}; \tag{44}$$

these functions satisfy the equations

$$\frac{\mathrm{d}x_{ij}}{\mathrm{d}s} = X_{ij}, \qquad \frac{\mathrm{d}y_{ij}}{\mathrm{d}s} = y_{ij}, \qquad \frac{\mathrm{d}q}{\mathrm{d}s} = Q, \tag{45}$$

and the initial conditions

$$x_{ij}(1) = g_{ij}^0, \qquad y_{ij}(1) = \dot{g}_{ij}^0, \qquad q(1) = \left(\sqrt{\sum_{j=1}^{3} (g_{1j}^0 - g_{2j}^0)^2} - \Delta_{12}^* \right)^{-1}, \tag{46}$$

where

$$\left.\begin{aligned} &Y_{ij} \stackrel{\text{def}}{=} \varepsilon\kappa^2 z \sum_{r\in T[i]} m_r(x_{rj} - x_{ij})u_{ri}^3, \\ &u_{ri}(s) \stackrel{\text{def}}{=} \Delta_{ri}^{-1}(t(s)), \qquad X_{ij} \stackrel{\text{def}}{=} \varepsilon z y_{ij}, \\ &z \stackrel{\text{def}}{=} \left(1 + \sum_{T_1} \alpha_{ri}^{-3}u_{ri}^3 + \alpha_{12}^{-3}q^3\right)^{-1}, \end{aligned}\right\} \tag{47}$$

$$Q \stackrel{\text{def}}{=} -\varepsilon z u_{12} q^2 \sum_{j=1}^{3} (x_{1j} - x_{2j})(y_{1j} - y_{2j}).$$

The principal results of the first step are stated in the Propositions 6, 7; the proof of these propositions will be based on an auxiliary system of differential equations and on estimates (see (50) and (52), (53)).

Let us introduce the functions $\tilde{z}$, u_{rij}, $\tilde{y}_{ij}$, $\tilde{u}_{ri}$, $\tilde{q}$ of s such that

$$\begin{aligned} &\tilde{z}(s) \stackrel{\text{def}}{=} z^{1/3}(s), \qquad u_{rij}(s) \stackrel{\text{def}}{=} (x_{rj}(s) - x_{ij}(s))u_{ri}(s), \\ &\tilde{u}_{ri}(s) \stackrel{\text{def}}{=} z^{1/3}(s)u_{ri}(s), \qquad \tilde{q}(s) \stackrel{\text{def}}{=} z^{1/3}(s)q(s), \\ &\tilde{y}_{ij}(s) \stackrel{\text{def}}{=} z^{1/3}(s)y_{ij}(s), \qquad 0 \leqslant r < i \leqslant N-1, \qquad j \in [1:3], \end{aligned} \tag{48}$$

and, for $r > i$, let us use the designations

$$\tilde{u}_{ri} \stackrel{\text{def}}{=} \tilde{u}_{ir}, \qquad u_{rij} \stackrel{\text{def}}{=} -u_{irj}. \tag{49}$$

The functions just defined satisfy the system

$$\left.\begin{aligned} &\frac{d\tilde{z}}{ds} = \varepsilon\tilde{z}^2\{\ \}, \\ &\frac{du_{rij}}{ds} = \varepsilon\tilde{z}\tilde{u}_{ri}\left(\tilde{y}_{rj} - \tilde{y}_{ij} - u_{rij}\sum_{k=1}^{3} u_{rik}(\tilde{y}_{rk} - \tilde{y}_{ik})\right), \\ &\frac{d\tilde{y}_{ij}}{ds} = \varepsilon\tilde{z}\tilde{y}_{ij}\{\ \} + \varepsilon\kappa^2\tilde{z}^2 \sum_{r\in T(i)} m_r u_{rij}\tilde{u}_{ri}^2, \\ &\frac{d\tilde{u}_{ri}}{ds} = \varepsilon\tilde{z}\tilde{u}_{ri}\{\ \} - \varepsilon\tilde{z}\tilde{u}_{ri}^2 \sum_{j=1}^{3} u_{rij}(\tilde{y}_{rj} - \tilde{y}_{ij}), \\ &\frac{d\tilde{q}}{ds} = \varepsilon\tilde{z}\tilde{q}\{\ \} - \varepsilon\tilde{z}\tilde{q}^2 \sum_{j=1}^{3} u_{12j}(\tilde{y}_{1j} - \tilde{y}_{2j}), \end{aligned}\right\} \tag{50}$$

where

$$\{\ \} \stackrel{\text{def}}{=} \alpha_{12}^{-3}\tilde{q}^4 \sum_{j=1}^{3} u_{12j}(\tilde{y}_{1j} - \tilde{y}_{2j}) + \sum_{T_1} \alpha_{ri}^{-3}\tilde{u}_{ri}^4 \sum_{j=1}^{3} u_{rij}(\tilde{y}_{rj} - \tilde{y}_{ij}), \tag{51}$$

and T_1 is defined by (42).

PROPOSITION 5. Let the functions u_{rij}, $\tilde{u}_{ri}$, $\tilde{q}$, $\tilde{z}$ satisfy the Equations (50) and the initial conditions (46), and let (ω_-^*, ω_+^*) be the maximal existence interval of the solution of the problem (45)–(46). Then the following inequalities hold for every $s \in (\omega_-^*, \omega_+^*)$:

$$\begin{aligned} &|u_{rij}(s)| \leqslant 1, \qquad 0 < \tilde{u}_{ri}(s) < \alpha_{ri}, \\ &0 < \tilde{q}(s) < \alpha_{12}, \qquad 0 < \tilde{z}(s) < 1, \end{aligned} \tag{52}$$

$$|\tilde{y}_{ij}(s)| \leqslant V_i \stackrel{\text{def}}{=} \sqrt{\frac{2}{m_i}(h + \kappa^2 Bd)}(1 + Bd^3)^{-1/3}, \tag{53}$$

where

$$B \stackrel{\text{def}}{=} \sum_{i=1}^{N-1} \sum_{r=0}^{i-1} (\alpha_{ri} m_r m_i)^{3/2}, \tag{54}$$

h is the energy constant, i.e.

$$\frac{1}{2} \sum_{i=0}^{N-1} m_i \sum_{j=1}^{3} y_{ij}^2 - \kappa^2 \sum_{i=1}^{N-1} \sum_{r=0}^{i-1} m_r m_i u_{ri} = h, \tag{55}$$

and d is the only positive root of the equation

$$d^3 + \frac{2h}{\kappa^2 B} d^2 - \frac{1}{B} = 0. \tag{56}$$

Proof. The inequalities (52) are evident. The inequalities (53) we shall prove in two steps.

Step 1. For every s of (ω_-^*, ω_+^*), using the energy integral (55) and the inequality $u_{12} \leqslant q$ one obtains the inequalities

$$|y_j^i(s)| \leqslant \sqrt{\frac{2}{m_i}\left(h + \kappa^2 \sum_{i=1}^{N-1} \sum_{r=0}^{i-1} m_r m_i u_{ri}\right)}, \tag{57}$$

$$|\tilde{y}_{ij}| = |z y_{ij}^3| \leqslant \left(\frac{2}{m_i}\right)^{3/2} \Theta, \tag{58}$$

where

$$\Theta \stackrel{\text{def}}{=} \left(h + \kappa^2 \sum_{i=1}^{N-1} \sum_{r=0}^{i-1} m_r m_i u_{ri}\right)^{3/2} \times$$

$$\times\left(1+\sum_{i=1}^{N-1}\sum_{r=0}^{i-1}\alpha_{ri}^{-3}u_{ri}^{3}\right)^{-1}. \tag{59}$$

Step 2. For fixed masses and α_{ri}, consider Θ as a function of $u_{ri} > 0$. It is easily shown that the quantities u_{ri} satisfying the system

$$\frac{\partial\Theta}{\partial u_{ri}} = 0, \qquad \Theta > 0, \tag{60}$$

may be given by the formula

$$u_{ri} = \sqrt{m_r m_i \alpha_{ri}^3}\, d, \tag{61}$$

where d is a constant. On substituting (61) into (60) one obtains the Equation (56) which has the only positive root. This implies that, for positive u_{ri} and fixed positive α_{ri}, $m_r m_i$, the function Θ has only one local maximum

$$\Theta^* \stackrel{\text{def}}{=} (h + \kappa^2 Bd)^{3/2}(1 + Bd^3)^{-1}. \tag{62}$$

One can show that it is equal to $\sup_{\{u_{ri}>0\}} \Theta$. It remains to use (58).

Remark 9. For fixed masses and α_{ri}, the less is the energy constant h, the better is the estimate (53). To decrease h, one can choose the inertial coordinate system used in the problem (1)–(4). One can show that the minimal value of h (obtained in way just mentioned) is

$$h_* \stackrel{\text{def}}{=} \frac{1}{2}\sum_{i=0}^{N-1} m_i \sum_{j=1}^{3} (\dot{g}_{ij}^0)^2 - \frac{1}{2m}\sum_{j=1}^{3}\left(\sum_{i=0}^{N-1} m_i \dot{g}_{ij}^0\right)^2 -$$

$$- \kappa^2 \sum_{i=1}^{N-1}\sum_{r=0}^{i-1} m_r m_i \Delta_{ri}^{-1}(1), \tag{63}$$

where $m \stackrel{\text{def}}{=} m_0 + \cdots + m_{N-1}$.

The following Proposition shows that the solution of the problem (45)–(47) is continuable along R.

PROPOSITION 6. Let (ω_-, ω_+) be the maximal existence interval of the solution of the Cauchy problem (1)–(4); let the functions s of t be defined by (41); let the initial data g_{1j}^0, g_{2j}^0 and Δ_{12}^* satisfy the condition

$$0 \leqslant \Delta_{12}^* < \sqrt{\sum_{j=1}^{3} (g_{1j}^0 - g_{2j}^0)^2}. \tag{64}$$

Then

$$\lim_{t\to\omega_- + 0} s(t) = -\infty, \qquad \lim_{t\to\omega_+ - 0} s(t) = +\infty. \tag{65}$$

Proof. We shall prove the second of the equalities (65); in the same way one proves the first. From the inequalities

$$|u_{ri}(x_{rj} - x_{ij})| \leqslant 1 \tag{66}$$

it follows (see (43)–(47))

$$\left|\frac{\mathrm{d}q}{\mathrm{d}s}\right| \leqslant \varepsilon \sum_{j=1}^{3} |zq^2(y_{1j} - y_{2j})|, \qquad \left|\frac{\mathrm{d}u_{ri}}{\mathrm{d}s}\right| \leqslant \varepsilon \sum_{j=1}^{3} |zu_{ri}^2(y_{rj} - y_{ij})|. \tag{67}$$

For positive u_{ri}, by (57) one obtains the inequalities

$$|zu_{ri}^2(y_{rj} - y_{ij})| \leqslant \Theta_1, \tag{68}$$

$$|zq^2(y_{1j} - y_{2j})| \leqslant \Theta_1, \tag{69}$$

where Θ_1 is a positive constant. On letting

$$R(t) \stackrel{\text{def}}{=} \sum_{i=1}^{N-1} \sum_{r=0}^{i-1} \Delta_{ri}^{-1}(t) + \Delta(t), \tag{70}$$

one sees from the inequalities just obtained that

$$\left|\frac{\mathrm{d}R}{\mathrm{d}s}\right| \leqslant \kappa, \tag{71}$$

where κ is a positive constant. Since $\mathrm{d}s/\mathrm{d}t > 1$, on integrating the inequality

$$\frac{\mathrm{d}R}{\mathrm{d}t} \leqslant \left|\frac{\mathrm{d}R}{\mathrm{d}s}\right| \frac{\mathrm{d}s}{\mathrm{d}t} \leqslant \kappa \frac{\mathrm{d}s}{\mathrm{d}t} \tag{72}$$

between 1 and t, one obtains the inequality

$$s(t) \geqslant s(1) + \kappa^{-1}(R(t) - R(1)). \tag{73}$$

In order to complete the proof, notice that if $\omega_+ < +\infty$, then $\lim_{t \to \omega_+ - 0} R(t) = +\infty$ and if $\omega_+ = +\infty$, then (65) follows from (41).

PROPOSITION 7. Let the function t of s be defined by (41); let the function z be such that

$$z(s) \stackrel{\text{def}}{=} \frac{\mathrm{d}t(s)}{\mathrm{d}s},$$

and let μ_k, ν_r be positive parameters. Then t, z are regular in the strip $\mathscr{M}_\rho(R)$ and, for every $s \in \mathscr{M}_\rho(R)$, z satisfies the inequality

$$|z(s)| \leqslant \mu_1^3 \mu^3 (1 - 7\sigma|\mathrm{Im}\, s|)^{-3/7}, \tag{74}$$

where

$$\rho \stackrel{\text{def}}{=} \frac{1}{7\sigma}, \qquad \sigma \stackrel{\text{def}}{=} \varepsilon \max\{\sigma_1, \sigma_2, \sigma_3\}, \tag{75}$$

$$\left.\begin{aligned}
&\sigma_1 \stackrel{\text{def}}{=} A_2\mu^2 + A_{41}\mu^4,\\
&\sigma_2 \stackrel{\text{def}}{=} A_{42}\mu^4 + A_7\mu^7,\\
&\sigma_3 \stackrel{\text{def}}{=} A_{41}\mu^4 + A_7\mu^7,\\
&A_7 \stackrel{\text{def}}{=} 3\mu_1\mu_2\mu_3^4 \sum_{i=1}^{N-1}\sum_{r=0}^{i-1}(\nu_r + \nu_i)\alpha_{ri}^{-3},\\
&A_{41} \stackrel{\text{def}}{=} 3\mu_1\mu_2\mu_3 \max_{r\neq i}(\nu_r + \nu_i),\\
&A_{42} \stackrel{\text{def}}{=} \kappa^2\mu_1^2\mu_2\mu_3^2 \max_i\left(\nu_i^{-1}\sum_{r\in T(i)} m_r\right),\\
&A_2 \stackrel{\text{def}}{=} \mu_1\mu_2^{-1}\mu_3 \max_{r\neq i}(\nu_r + \nu_i),\\
&\mu \stackrel{\text{def}}{=} \max\{\mu_1^{-1}, \mu_2^{-1}, \mu_3^{-1}\max_{r<i}\alpha_{ri}, \max_i \nu_i^{-1}V_i\},\\
&r, i\in[0: N-1],
\end{aligned}\right\} \tag{76}$$

and V_i, $T(i)$ are defined by (53) and (1_1).

Proof. Applying the Corollary 2 (see the Proposition 4) to the Equations (50) and using instead of (40) the substitutions

$$\tilde{z} \mapsto \mu_1\tilde{z}, \quad u_{rij} \mapsto \mu_2 u_{rij}, \quad \tilde{u}_{ri} \mapsto \mu_3\tilde{u}_{ri}, \tag{77}$$

$$\tilde{q} \mapsto \mu_3\tilde{q}, \quad \tilde{y}_{rj} \mapsto \nu_r\tilde{y}_{rj},$$

one proves that $\tilde{z}$ is regular in $\mathcal{M}_\rho(R)$ and satisfies the inequality

$$|\tilde{z}(s)| \leqslant \mu_1\mu(1 - 7\sigma|\text{Im}\, s|)^{-1/7}. \tag{78}$$

It remains to notice that $z(s) = \tilde{z}^3(s)$.

COROLLARY 3. For every $s\in\mathcal{M}_\rho(R)$, the following inequalities hold:

$$|\text{Im}\, t(s)| \leqslant \chi_1, \tag{79}$$

$$|\text{Re}\, t(s) - t(\text{Re}\, s)| \leqslant \chi_1, \tag{80}$$

$$|t(s)| \leqslant |t(\text{Re}\, s)| + \chi_1, \tag{81}$$

where

$$\chi_1 \stackrel{\text{def}}{=} \mu_1^3 \mu^3 \,|\mathrm{Im}\, s|\,(1 - 7\sigma\,|\mathrm{Im}\, s|)^{-3/7}. \tag{82}$$

4.2. SECOND STEP

As was said above, the purpose of the method described briefly in Section 2 is to obtain the sequences of estimates $\omega_+ > \gamma_k$. First of all notice that the function $\lambda: \mathscr{M}_r((0, +\infty)) \to C$ such that

$$s(\lambda) = \frac{2r}{\pi} \operatorname{arsh}\left(\frac{1+\lambda}{1-\lambda} \operatorname{sh} \frac{\pi}{2r}\right), \tag{83}$$

transforms conformally the half-strip $\mathscr{M}_r((0, +\infty))$ onto the circle $\mathscr{O}_1$.

For $r \stackrel{\text{def}}{=} r_0$ introduce the regular function $w : \mathscr{O}_1 \to C$ such that

$$w(\lambda) \stackrel{\text{def}}{=} t(s(\lambda)). \tag{84}$$

For $\lambda \in \mathscr{O}_1$, from (84) and (81) it follows

$$|w(\lambda)| < \omega_+ + 1.2\mu_1^3\mu^3 r_0. \tag{85}$$

By the inequality and the Proposition 1, one proves the following results.

PROPOSITION 8. Under the assumptions of the Proposition 1, let w be defined by (84). Let (ω_-, ω_+) be the maximal existence interval of the solution of the Cauchy problem (1)-(4); let the quantities μ. ϱ be defined by (75), (76) and let r_0 be a fixed number satisfying the inequality $r_0 \leqslant \frac{1}{3}\rho$.

Then for $j = 1, 2$ and for every natural n, the inequalities

$$\omega_+ > \gamma_n^{(j)} - 1.2\mu_1^3\mu^3 r_0 \tag{86}$$

hold true.

In addition, if $\omega_+ = +\infty$, then

$$\lim_{n \to +\infty} \gamma_n^{(1)} = +\infty. \tag{87}$$

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