# Error estimates for numerical integration of the $N$-body problem

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Estimates are established for the radius of convergence and the remainder term in the Taylor solution of the $N$-body problem in rectangular coordinates.

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1. One way to solve the differential equations of the N-body problem in rectangular coordinates is to approximate the solution by a Taylor-series segment in some neighborhood of the starting time, provided the coordinates and velocities of the bodies are specified at that time.

The principal result of this letter will be the following:

Theorem. Let $m_1, \ldots, m_N$ denote the masses of mass-points $M_1, \ldots, M_N$ moving under their mutual attraction according to Newton's law, and let $\mathscr{K}$ be the Gaussian constant. Denote by $g1_{ij}$, $g2_{ij}$ ($j = 1, 2, 3$) the coordinates and velocities of the i-th point ($i = 1, \ldots, N$) in a rectangular coordinate system centered at $M_1$, with axes parallel to those of some inertial system (corresponding to the customary notation $x_i$, $y_i$, $z_i$; $\dot{x}_i$, $\dot{y}_i$, $\dot{z}_i$). Let $t_0$ be a fixed epoch. Adopt the notation $gp_{ij0} \overset{\text{def}}{=} gp_{ij}(t_0)$ ($p = 1, 2$) for the initial data (in particular, $gp_{1j0} = 0$) and the notation

$$\delta_M gp_{ij}(t) \overset{\text{def}}{=} \sum_{l=M+1}^{\infty} (d^l gp_{ij}/dt^l)_{t=t_0} (t - t_0)^l / l!, \quad (p = 1, 2)$$

for the remaining terms of the power series for the coordinates and velocities. Calculate the quantities R and $a_k$ ($k = 2, \ldots, N$) from the expressions ($r, i = 1, \ldots, N$)

$$d_{ri0} = \Big(\sum_{j=1}^{3} (g1_{rj0} - g1_{ij0})^2\Big)^{1/2}; \quad e_{ri} = d_{ri0}^{-2} \text{ for } r \neq i, \text{ else } e_{ri} = 0;$$

$$q_{ri} = \mathscr{K}^2 \sum_{j=1}^{N} m_j (e_{jr} + e_{ji}); \quad h_{ri} = \max_{j=1,2,3} |g2_{rj0} - g2_{ij0}|;$$

$$f_{ri} = d_{ri0}^{-1} \Big| \sum_{j=1}^{3} (g1_{rj0} - g1_{ij0})(g2_{rj0} - g2_{ij0}) \Big|;$$

$$b_{ri} = \max\{h_{ri}, \sqrt{2/3} f_{ri}\}; \quad t_{ri} = (q_{ri} d_{rj0}/2)^{1/2};$$

$$c_{ri} = \sqrt{6}(2b_{ri}/d_{ri0} + q_{ri}/b_{ri}) \text{ for } b_{ri} \geqslant t_{ri},$$

otherwise

$$c_{ri} = 4(3q_{ri}/d_{ri0})^{1/2}; \quad s = \max_{r,i} c_{ri}; \quad R = 1/s;$$

$$a_k = \max\{b_{k1}, d_{k10} \cdot (s - (s^2 - 48q_{k1}/d_{k10})^{1/2}) / (2\sqrt{6})\}.$$

Then within a circle $\mathscr{O}_R(t_0)$ of radius R in the complex plane centered at $t_0$ the functions $gp_{ij}$ will be holomorphic with respect to t, and for $i = 2, \ldots, N$; $j = 1, 2, 3$ they will satisfy the inequalities

$$|\delta_M g2_{ij}(t)| \leqslant \Delta_i(M) \overset{\text{def}}{=} a_i (1 - |t - t_0|/R)^{-1} \times (|t - t_0|/R)^{M+1}, \quad (1)$$

$$|\delta_M g2_{ij}(t)| \leqslant \Delta_i(M) R/(M + 1). \quad (2)$$

The proof will be given in Sec. 3; it rests on an analogous result for a polynomial system of differential equations with constant coefficients, proved in Sec. 2 as Proposition 2. In Sec. 4 we apply the Theorem to the problem of the motion of the planets in the solar system, and we give some numerical results.

2. Consider the equation

$$\dot{X} = \sum_{m=1}^{L+1} \sum_{i \in I(m)} a[i] x_1^{i_1} \cdot \ldots \cdot x_n^{i_n}, \quad (3)$$

where $X \overset{\text{def}}{=} (x_1, \ldots, x_n)$ represents a vector function with argument $t \in C$ and with values in $C^n$; $a[i] \overset{\text{def}}{=} (a_1[i], \ldots, a_n[i])$ is a constant vector in $C^n$; n, L are fixed natural numbers; $i_1, \ldots, i_n$ are integers; and

$$I(m) \overset{\text{def}}{=} \{i = (i_1, \ldots, i_n) \mid i_1 \geqslant 0, \ldots, i_n \geqslant 0;\ i_1 + \ldots + i_n = m\}.$$

We shall assume that the solution $X = X(t)$ of Eq. (3) satisfies the initial condition

$$X(t_0) = X_0. \quad (4)$$

Introduce the notation

$$XT_M(t) \overset{\text{def}}{=} \sum_{l=0}^{M} X_l (t - t_0)^l / l!, \quad X_l \overset{\text{def}}{=} (d^l X/dt^l)_{t=t_0}. \quad (5)$$

Let $\rho < \mathscr{R}$, where $\mathscr{R}$ denotes the radius of convergence of the series $XT_\infty(t)$. In the Taylor-series method of numerical integration of the problem (3), (4), for all $t \in \mathscr{O}_\rho(t_0)$ the solution X will be replaced by $XT_M$.

We pose the problem: To estimate $\mathscr{R}$ and the quantity

$$\delta_M X(t) \overset{\text{def}}{=} X(t) - XT_M(t). \quad (6)$$

The solution is given by Proposition 2, whose proof rests on the following result:

Proposition 1 (Babadzhanyants,[1] p. 49). Define the quantities

$$s(\gamma) \overset{\text{def}}{=} \gamma^{-1} \max_{j=1,\ldots,n} \sum_{m=1}^{L+1} \gamma^m \sum_{i \in I(m)} |a_j[i]|, \ \gamma \in (0, +\infty),$$
$$|X| \overset{\text{def}}{=} \max_{j=1,\ldots,n} |x_j| \text{ for } X = (x_1, \ldots, x_n). \quad (7)$$

Then the quantities $X_l$ defined in Eq. (5) will satisfy the inequalities

$$|X_l| \leqslant |X_0| (s(|X_0|))^l \prod_{m=0}^{l-1} (1 + mL), \ l = 1, 2, \ldots \quad (8)$$

Proposition 2. Adopt the definitions (5)-(7) and define $\rho \overset{\text{def}}{=} 1/(Ls(|X_0|))$. Then the solution X of the problem (3), (4) will be holomorphic for $t \in \mathscr{O}_\rho(t_0)$ and for those t will satisfy the inequality

TABLE I. Sample Error Estimates for Outer Planets

| $t_0$(JD) | $R$(d) | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ |
|---|---|---|---|---|---|---|
| 2415000.5 | 104.9 | 6.0 | 5.3 | 3.6 | 3.1 | 2.2 |
| 2420000.5 | 98.7 | 7.1 | 5.6 | 3.1 | 2.8 | 2.2 |
| 2441200.5 | 103.3 | 7.0 | 5.1 | 3.7 | 2.8 | 3.1 |
| 2441600.5 | 99.4 | 7.3 | 5.7 | 3.6 | 2.8 | 3.1 |
| 2442000.5 | 133.6 | 5.4 | 5.9 | 3.5 | 2.9 | 3.1 |

$$|\delta_M X(t)| \leqslant |X_0| (1 - |t-t_0|/\rho)^{-1/L} (|t-t_0|/\rho)^{M+1} \quad (9)$$

Proof. Using the inequality (8), we obtain for any $k = 1, 2, \ldots$:

$$\left|\sum_{l=M+1}^{M+1+k} X_l (t-t_0)^l / l!\right| \leqslant |X_0| (|t-t_0|/\rho)^{M+1} \left(1 + \sum_{l=1}^{k} (|t-t_0|/\rho)^l \times \prod_{m=0}^{l-1} (1/L + m)/l!\right).$$

Passing to the limit as $k \to +\infty$ and recalling the binomial law

$$(1-a)^{-b} = 1 + \sum_{l=1}^{\infty} a^l \prod_{m=0}^{l-1} (b+m)/l!,$$

we complete the proof.

3. Proof of Theorem. In order to make use of Proposition 2, we reduce the equations of the N-body problem in rectangular coordinates to the polynomial form (3). Let the $\alpha_{ri}$ be positive parameters, and adopt the notation $d_{ri} \overset{\text{def}}{=} \left(\sum_{j=1}^{3} (g1_{rj} - g1_{ij})^2\right)^{1/2}$. For j = 1, 2, 3 and r, i = 1,..., N (r ≠ i) introduce the variables

$$u_{ri} \overset{\text{def}}{=} d_{ri0}/d_{ri}; \quad v_{ri} \overset{\text{def}}{=} u_{ri}^2; \quad x_{rij} \overset{\text{def}}{=} u_{ri}(g1_{rj} - g1_{ij})/d_{ri};$$
$$y_{rij} \overset{\text{def}}{=} (g2_{rj} - g2_{ij})/\alpha_{ri}; \quad z_{rij} \overset{\text{def}}{=} y_{rij}u_{ri};$$
$$w_{rij} \overset{\text{def}}{=} x_{rij}u_{ri}; \quad \vartheta_{ri} \overset{\text{def}}{=} \sqrt{2/3} \sum_{j=1}^{3} x_{rij}y_{rij}.$$

Let $\xi 1_{ij}$, $\xi 2_{ij}$ denote the coordinates and velocities of the i-th body in an inertial coordinate system with axes parallel to those of the adopted relative coordinate system; then $gp_{rj} - gp_{ij} = \xi p_{rj} - \xi p_{ij}$. Expressing all the variables $u_{ri}, \ldots, \vartheta_{ri}$ that we have introduced in terms of $\xi p_{rj} - \xi p_{ij}$ and using the standard equations of the N-body problem in an inertial coordinate system, we obtain the equations

$$\dot u_{ri} = -\sqrt{3/2}\sigma_{ri}u_{ri}\vartheta_{ri}; \quad \dot v_{ri} = -\sqrt{6}\sigma_{ri}v_{ri}\vartheta_{ri};$$
$$\dot x_{rij} = \sigma_{ri}(u_{ri}z_{rij} - \sqrt{6}x_{rij}\vartheta_{ri}); \quad \dot y_{rij} = q_{rij};$$
$$\dot z_{rij} = -\sqrt{3/2}\sigma_{ri}\vartheta_{ri}z_{rij} + u_{ri}q_{rij};$$
$$\dot w_{rij} = \sigma_{ri}(z_{rij}v_{ri} - 3\sqrt{3/2}\vartheta_{ri}w_{rij});$$
$$\dot\vartheta_{ri} = -\sqrt{6}\sigma_{ri}\vartheta_{ri}^2 + \sqrt{2/3}\sum_{j=1}^{3}(\sigma_{ri}z_{rij}^2 + q_{rij}x_{rij}), \quad (10)$$

where

$$\sigma_{ri} \overset{\text{def}}{=} \alpha_{ri}/d_{ri0}; \quad q_{rij} \overset{\text{def}}{=} \alpha_{ri}^{-1}\mathscr{K}^2 \sum_{\nu=1}^{N} m_\nu (e_{\nu r}w_{\nu rj} - e_{\nu i}w_{\nu ij}).$$

The solution of the system (10) satisfies the initial conditions

$$u_{ri}(t_0) = v_{ri}(t_0) = 1; \quad x_{rij}(t_0) = w_{rij}(t_0) = (g1_{rj0} - g1_{ij0})/d_{ri0};$$
$$y_{rij}(t_0) = z_{rij}(t_0) = (g2_{rj0} - g2_{ij0})/\alpha_{ri};$$
$$\vartheta_{ri}(t_0) = \sqrt{2/3}\sum_{j=1}^{3} x_{rij}(t_0)\,y_{rij}(t_0). \quad (11)$$

The parameters $\alpha_{ri}$ in the problem (10), (11) will obey the constraints $\alpha_{ri} \geq b_{ri}$; in this case all the initial values (11) will be no larger than unity in absolute value. Representing the quantities (10), (11) in the form (3), (4) and applying Proposition 2 to this problem, we find that its solution is holomorphic in the circle $\mathcal{O}_\rho(t_0)$ and that the inequality (9) is valid, with

$$\rho = 1/q, \quad q = \sqrt{6}\max_{r,i}(2\alpha_{ri}/d_{ri0} + q_{ri}/\alpha_{ri}).$$

If in addition the parameters $\alpha_{ri}$ are subject to the condition that q be a minimum, we obtain q = s, $\rho$ = R. If we select the smallest of all the $\alpha_{r1}$ that satisfy the conditions adopted, then we will have $\alpha_{r1} = a_r$.

Since the quantities $g2_{ij}$ are among the variables of the problem (10), (11), we have already proved they are holomorphic for $t \in \mathcal{O}_R(t_0)$, and we have demonstrated the inequality (1). Sinc $g1_{ij} = g2_{ij}$, the functions $g1_{ij}$ will also be holomorphic, for the same t. From the inequality (1) we may infer the inequality (2).

4. To illustrate the application of the theorem proved above, we now give numerical results for the problem of the motion of the sun, Jupiter, Saturn, Uranus, Neptune, and Pluto (numbering them from 1 to 6 in that order).

As initial data we have adopted from Lieske[2] and Oesterwinter and Cohen[3] values for the coordinates and velocities of the outer planets in a heliocentric equatorial coordinate system (equator and equinox of 1950.0) for five epochs: $t_0$ = 1899 Dec 12.0, 1913 Aug. 21.0, 1971 Sep 6.0, 1972 Oct 10.0, 1973 Nov 14.0. From the data for each of these epochs we have used the equations in the conditions of the Theorem to calculate the quantities R (days) and $a_i$ appearing in the estimates (1), (2).

Table I, which gives the values of R and $\alpha_i \equiv 1000a_i$ for the five epochs, illustrates the practical efficacy of our theorem for an a priori choice of step and for estimating the error involved in numerical integration of the equations of the N-body problem by the Taylor-series method. Suppose, for example, that the masses, rectangular heliocentric coordinates, and velocities of the outer planets are known for epoch JD 2442000.5 = 1973 Nov 14.0, and that it is required to calculate the coordinates and velocities of these planets at epoch $t_0$ + h by solving the corresponding differential equations by Taylor series. Looking at the last line of the table (which corresponds to this epoch $t_0$), we see that the Theorem assures the possibility of choosing a step h as long as $133^d.6$. If we set $h = 20^d$ (the customary value to adopt in this problem) and use a Taylor polynomial of order M = 10 to obtain an approximate solution at $t_0 + h$, then we may infer from the Theorem that the absolute error in approximating the coordinates of Jupiter is no more than $10^{-3} \times 5.4 \times (1-20/133.6)^{-1} \times (20/133.6)^{11} \times (133.6/11) \approx 6.6 \times 10^{-11}$ AU, as given by the inequality (2) with i = 2 (we have assigned to Jupiter the number i = 2).

[1]L. K. Babadzhanyants, "The existence of continuations in the N-body problem," Celest. Mech. 20, 43-57 (1979).
[2]H. Lieske, "Newtonian planetary ephemerides 1800-2000; development ephemeris No. 28," Jet Propulsion Lab. Tech. Rep. 32-1206 (NASA CR-90430) (Nov. 1967).
[3]C. Oesterwinter and C. J. Cohen, "New orbital elements for moon and planets," Celest. Mech. 5, 317-395 (1972).