2 Relative Importance of Two Criteria

This chapter presents the initial concepts and results of the theory of relative importance of criteria. First of all, the statement “one criterion is more important than another” is defined and the main properties of corresponding definition are studied. A central result of the chapter shows how the Pareto set may be reduced by using that one criterion is more important than another.

Measuring of criterion values as well as a number of scales for measuring are also discussed.

2.1 Main Notions

Multicriteria Choice Problem and Initial Requirements

Consider a multicriteria choice model

\( \langle X, f, \succ_X \rangle \)

where

- \( X \) is a set of alternatives
- \( f = (f_1, f_2, \ldots, f_m) \) is a vector criterion
- \( \succ_X \) is a preference relation.

Recall that

\[ Y = f(X) = \{ y \in R^m \mid y = f(x) \text{ for some } x \in X \} \]

\( Y \subset \tilde{Y} = f_1(X) \times f_2(X) \times \ldots \times f_m(X) \subset R^m. \)

Moreover,

\[ y' \succ_Y y'' \iff x' \succ_X x'' \]

for all \( x', x'' \in X \), where \( y' = f(x') \in Y \), \( y'' = f(x'') \in Y \).

Further we shall assume that \( \succ \) is defined on the all space \( R^m \).

\[ y' \succ_Y y'' \iff y' \succ y'' \text{ for all } y', y'' \in Y \]

and Axioms 1-3 are satisfied. This relation is irreflexive and transitive as well as asymmetric.

A multicriteria choice model can be represented in terms of outcomes as the following pair

\( \langle Y, \succ \rangle \)

where
$Y \subset R^n$ is a set of outcomes [vectors]

$\succ$ is an irreflexive and transitive binary relation defined on $R^n$.

As it was mentioned in the previous chapter, all the results obtained in terms of alternatives could be reformulated in terms of outcomes and vice versa. That is why most commonly we shall deal with a multicriteria choice problem in terms of outcomes.

Under above conditions Pareto Axiom is available. By this axiom,

$$y' \succeq y^m \Rightarrow y' \succ y^m \quad \text{for all} \quad y', y^m \in R^n. \tag{2.1}$$

The relation $\succ$ is asymmetric. Hence, due to (2.1), the so-called **negative Pareto axiom**

$$y' \succeq y^m \Rightarrow [y^m \succ y' \text{ is false}] \tag{2.2}$$

is satisfied.

For every two vectors $y', y^m \in R^n$, one and only one of the following three cases may occur:

- $y' \succ y^m$ [i.e. $y'$ is preferred to $y^m$]
- $y^m \succ y'$ [i.e. $y^m$ is preferred to $y'$]
- neither $y' \succ y^m$ nor $y^m \succ y'$ [i.e. $y'$ and $y^m$ are incomparable].

**Preliminary Considerations**

Denote by $I$ a set of indices of the vector criteria $f$, i.e.

$I = \{1,2,\ldots,m\}$.

Consider the simplest choice between two vectors $y', y^m \in R^n$ that have a minimal number of distinct components.

If $y'$ and $y^m$ have all the same components except the $i$th, i.e.

$$y'_i = y^m_i; \quad y^m_j = y^m_j \quad \text{for all} \quad s \in I \setminus \{i\}$$

then either $y' \succeq y^m$ or $y^m \succeq y'$. By Axiom 3, either $y' \succ y^m$ or $y^m \succ y'$. Hence, in this case, Axiom 3 provides the choice.

Consider the outcomes $y'$ and $y^m$ that have exactly two distinct components, i.e.

$$y'_i \neq y^m_i, \quad y'_j \neq y^m_j; \quad y^m_i = y^m_i \quad \text{for all} \quad s \in I \setminus \{i, j\}$$

where at least one of the equalities $y'_i = y'_j$, $y^m_i = y^m_j$ is false. Then one and only one of the following four cases may occur:

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1 Recall that $y' \succeq y^m$ is equivalent to $y' \succeq y^m$, $y' \neq y^m$. 


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1) \( y_i' > y_i^* \), \( y_j' > y_j^* \)  
2) \( y_i^* > y_i' \), \( y_j^* > y_j' \)  
3) \( y_i' > y_i^* \), \( y_j' > y_j^* \)  
4) \( y_i^* > y_i' \), \( y_j^* > y_j' \).

Suppose that the DM has made his choice. This means that either \( y' > y^* \) or \( y^* > y' \). Without loss of generality assume that \( y' > y^* \). In other words, \( \text{Sel} \{y', y^*\} = \{y'\} \). The following question arises: how can explain such a choice of the DM?

In the first case [i.e. if \( y_i' > y_i^* \), \( y_j' > y_j^* \)], the choice \( \text{Sel} \{y', y^*\} = \{y'\} \) is a direct consequence of Pareto Axiom (2.1). In the second case, \( y^* > y' \) is impossible because of (2.2).

Let us consider the rest two cases. Due to symmetry, it is sufficient to consider one of them, say, the third one. The inequality \( y_i' > y_i^* \) means that the vector \( y' \) is preferred to \( y^* \) with respect to the \( i \)th criterion. On the other hand, with respect to the \( j \)th criterion, \( y^* \) is preferred to \( y' \), since \( y_j^* > y_j' \). One statement contradicts the other, and we ask: why has the choice \( \text{Sel} \{y', y^*\} = \{y'\} \) been made? What was a reason for doing this?

Selecting \( y' \) [instead of \( y^* \)] from the pair \( \{y', y^*\} \) the DM loses \( y_j^* - y_j' \) in the \( j \)th criterion in order to gain \( y_i^* - y_i' \) in the \( i \)th criterion. The most reasonable explanation of this is the following: for the DM the criterion \( i \) is more important than \( j \). Indeed, deleting \( y^* \) from the pair \( \{y', y^*\} \), the DM prefers to increase the value \( y_i^* \) in the more important criterion \( i \) despite decreasing of \( y_j^* \) in the less important criterion \( j \).

In the fourth case [\( y_i' > y_i^*, y_j' > y_j^* \)], we can similarly conclude that \( \text{Sel} \{y', y^*\} = \{y'\} \) means that for the DM the criterion \( j \) is more important than \( i \).

Main Definitions

The arguments given above are related to the simplest choice between two vectors. They lead to the following definition.

Definition 2.1. Let \( i, j \in I \), \( i \neq j \). We shall say that the criterion \( i \) [more precisely, \( f_i \)] is [relatively] more important than the criterion \( j \) [i.e. \( f_j \)] with two positive parameters \( w_i, w_j \) if for any two vector \( y', y^* \in \mathbb{R}^n \) such that

\[
\begin{align*}
y_i' &> y_i^*, & y_j^* > y_j', & y_i^* = y_i' & \text{for all } s \in I \setminus \{i, j\} \\
y_i^* - y_i' &= w_i, & y_j^* - y_j' &= w_j
\end{align*}
\]

(2.3)

the relationship \( y' > y^* \) is valid.
In other words, $f_i$ is more important than $f_j$ if the DM agrees to lose the amount $w_j^*$ in $f_j$ in order to gain the additional amount $w_i^*$ in the more important criterion $f_i$.

A ratio of the values $w_j^*$ and $w_i^*$ allows us to evaluate numerically a "degree" of the relative importance. But this ratio is unbounded above; by this reason it seems that the following definition is more convenient.

**Definition 2.2.** Let the criterion $f_i$ be more important than $f_j$ with the positive parameters $w_i^*, w_j^*$. The positive number $\theta_i$ defined by

$$\theta_i = \frac{w_j^*}{w_i^* + w_j^*}$$

we shall call a relative importance coefficient.

Obviously, $0 < \theta_i < 1$. If $\theta_i$ is close to 1 then $w_j^* > w_i^*$. This means that the DM pays much in the less important $j$th criterion for a relatively small gain in the more important $i$th criterion, i.e. the criterion $f_i$ has a high degree of importance in comparison with $f_j$.

If $\theta_i$ is close to zero then $w_j^* > w_i^*$. Consequently, the DM agrees to lose in the criterion $j$ only if he/she can obtain much more in the criterion $i$, i.e. a degree of importance of $f_i$ in comparison with $f_j$ is relatively low.

If $\theta_i = \frac{1}{2}$ then $w_i^* = w_j^*$, i.e. the value of the DM’s loss in criterion $f_i$ is equal to the value of addition in criterion $f_j$.

We should remark that the degree of relative importance of criteria, indicated above and expressed by the coefficient $\theta_i$, directly depends on the type of scales in which the criteria are measured (see Section 2.4).

**Further Considerations**

Let us establish some property of the relative importance of criteria.

**Theorem 2.1.** Let Axioms 2 and 3 be satisfied. If the $i$th criterion is more important than the $j$th one with the positive parameters $w_i^*, w_j^*$ then the $i$th criterion is more important than the $j$th with any pair of positive parameters $w_i', w_j'$ such that $w_i' > w_i^*$, $w_j' < w_j^*$. In other words if $f_i$ is more important than $f_j$ with the relative importance coefficient $\theta_i$ then $f_i$ is more important than $f_j$ with any relative importance coefficient which is less than $\theta_i$. 
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Proof. Let us fix two arbitrary positive numbers $w^*, w_j^*$ and two arbitrary vectors $y^i, y^s \in \mathbb{R}^n$ such that
\[ y^i_j - y^s_j = w^*_j > w^*_i, \quad 0 < y^i_j - y^s_j = w^*_j < w^*_i, \quad y^s_s = y^s_s \text{ for all } s \in I \setminus \{i, j\}. \]

Introduce the positive numbers $z_i, z_j$ by the following
\[ z_i - y^s_i = w^*_i, \quad y^s_j - z_j = w^*_j. \]

Since $w^*_i > w^*_j$, we have $y^i_j > z_i$ and $y^i_j > z_j$. Moreover, $y^s_j > z_j$.

Introduce $z^i$ whose components are
\[ z^i = z_i, \quad z^j = z_j; \quad z^s_s = y^s_s \text{ for all } s \in I \setminus \{i, j\}. \]

Obviously, $z^i > y^s$, since the $i$ th criterion is more important than the $j$ th one with the parameters $w^*_i, w^*_j$.

Further, we have
\[ y^i_j > z^i_j, \quad y^i_j > z^i_j; \quad y^s_s = z^i_s \text{ for all } s \in I \setminus \{i, j\}. \]

By Pareto Axiom, $y^i > y^s$. It together with $z^i > y^s$, by transitivity of $>$, yields $y^i > y^s$. The first part of Theorem 2.1 has proven.

The second part, dealing with the relative importance coefficients, follows from the first part in the following way.

Let $0 < \theta^i_j < \theta^j$. Introduce the following four parameters
\[ w^i_i = 1 - \theta^i_j, \quad w^i_j = \theta^i_j; \quad w^j_i = 1 - \theta^j_i, \quad w^j_j = \theta^j_i. \]

Obviously,
\[ \frac{w^i_i}{w^i_i + w^j_j} = \theta^i_j, \quad \frac{w^j_j}{w^i_i + w^j_j} = \theta^j_i \]
and $w^i_j > w^j_i$, $w^j_i < w^i_j$.

Then, by the first part of this theorem, the $i$ th criterion is more important than the $j$ th one with the positive parameters $w^i_i, w^j_j$. Hence, $f_i$ is more important than $f_j$ with the relative importance coefficient $\theta^j_i$.

Theorem 2.1 shows that if the DM agrees to lose the value $w^*_j$ in the less important criterion $f_j$ in order to gain the additional amount $w^*_i$ in the more important criterion $f_i$, then he/she also agrees to lose $w^*_j$ in order to gain $w^*_i$.

On the basis of Theorem 2.1 let us analyze all opportunities for arbitrary pair of criteria $f_i, f_j$. One and only one of the following three cases may occur:
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1) the interval (0,1) includes at least one number that is a relative importance coefficient and at least one number that is not such a coefficient
2) no one number of the interval (0,1) is a relative importance coefficient; in this case, we shall say that \( f_i \) is not more important than \( f_j \) with any two parameters \( w_i, w_j \)
3) each positive number of the interval (0,1) is a relative importance coefficient; in this case, we shall say that \( f_i \) is incomparably more important than \( f_j \).

Let us study the first case in more details. If at least one number of the interval (0,1) is a relative importance coefficient then, by Theorem 2.1, any smaller number of (0,1) is also a relative importance coefficient. Let us introduce two disjoint sets \( A \) and \( B \). The first set includes all the numbers of (0,1) that are relative importance coefficients. In the first case, obviously, \( A \neq \emptyset \). The second nonempty set \( B \) includes all the numbers of (0,1) that are not relative importance coefficients for \( f_i, f_j \). It is clear that \( A \cup B = (0,1) \) and \( a < b \) holds for all \( a, b \in B \). This means that \( A \) and \( B \) form a cross-section of the interval (0,1). By the Dedekind principle there exists the only number \( \bar{\theta} \in (0,1) \) which realizes this cross-section. The number \( \bar{\theta} \) may be called as an ultimate relative importance coefficient.

It must be noted that \( \bar{\theta} \) itself may either be a relative importance coefficient or not. In other words, \( \bar{\theta} \in A \) or \( \bar{\theta} \notin A \) [but not both] might be true.

Characterization of Lexicographic Relation

A lexicographic\(^2\) relation can be characterized in terms of incomparably more important criteria.

Theorem 2.2. Let an arbitrary binary relation \( \succ \) be defined on \( R^n \) and Axioms 2-3 be satisfied. The relation \( \succ \) is a lexicographic if and only if \( f_1 \) is incomparably more important than \( f_2 \), that is incomparably more important than \( f_3 \), ..., \( f_{m-1} \) is incomparably more important than \( f_m \).

Proof. Necessity. Let the relation \( \succ \) be lexicographic. In this case, for arbitrary two vectors \( y', y^* \in R^n \) the following statements are true:

1) \( y_1' > y_1^* \Rightarrow y' \succ y^* \)
2) \( y_1' = y_1^* \), \( y_2' > y_2^* \Rightarrow y' \succ y^* \)
3) \( y_1' = y_1^* \), \( y_2' = y_2^* \), \( y_3' > y_3^* \Rightarrow y' \succ y^* \)

\(^2\) A lexicographic relation was defined in Section 1.2.
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\[ m) \quad y'_i = y'^*_i, \quad i = 1, 2, \ldots, m-1; \quad y'_m > y'^*_m \quad \Rightarrow \quad y' > y'^*. \]

From the statement 1) it follows that \( y' > y'^* \) holds for arbitrary two vectors \( y', y'^* \in \mathbb{R}^m \) such that \( y'_1 > y'^*_1, y'_2 < y'^*_2, \ldots, y'_m = y'^*_m \). This means that the first criterion is incomparably more important than the second one.

Similarly, using 2), we can obtain that the second criterion is incomparably more important than the third one, and so on.

Sufficiency. If \( m = 2 \) then the proof is trivial.

Further we shall consider only the case \( m = 3 \) in order to avoid awkward considerations.

Take two arbitrary vectors \( y', y'^* \in \mathbb{R}^3 \) such that \( y'_1 > y'^*_1 \). To prove the theorem, we need to get \( y' > y'^* \).

If \( y'_2 \geq y'^*_2 \) and \( y'_3 \geq y'^*_3 \) then, by Pareto Axiom, these inequalities together with \( y'_1 > y'^*_1 \) imply \( y' > y'^* \).

Let \( y'_1 > y'^*_1 \) as well as \( y'_2 < y'^*_2, \quad y'_3 \geq y'^*_3 \). Consider the vector \( y^1 = (y'_1, y'_2, y'_3) \).

Comparing this vector with \( y'^* \), we obtain \( y^1 > y'^* \), since \( f_1 \) is incomparably more important than \( f_2 \). However, \( y^1 \geq y'^* \); consequently, either \( y' = y^1 \) [in this case, \( y' > y'^* \) is valid] or \( y' \geq y'^* \). If \( y' \geq y'^* \) then, by Pareto Axiom, we have \( y' > y^1 \) that together with \( y^1 > y'^* \) imply \( y' > y'^* \), since \( > \) is transitive.

Let us consider the case \( y'_1 > y'^*_1, \quad y'_2 = y'^*_2, \quad y'_3 < y'^*_3 \). Introduce the vector \( y^2 = (y'_1, y'^*_2, y'_3) \). The relationship \( y^2 > y'^* \) is true, since the first criterion is incomparably more important than the second one. On the other hand the second criterion is incomparably more important than the third one; hence, \( y' > y^2 \). The relationships \( y' > y^2 \), \( y^2 > y'^* \) imply \( y' > y'^* \).

Let \( y'_1 > y'^*_1, \quad y'_2 > y'^*_2, \quad y'_3 < y'^*_3 \). Introduce the vector \( y^3 = (y'^*_1, y'_2, y'_3) \). Since the second criterion is incomparably more important than the third one, \( y^3 > y'^* \) is valid. On the other hand, \( y' \geq y^3 \). By Pareto Axiom, we obtain \( y' > y^3 \) that together with \( y^3 > y'^* \) imply \( y' > y'^* \).

Consider the last case \( y'_1 > y'^*_1, \quad y'_2 < y'^*_2, \quad y'_3 < y'^*_3 \). Introduce the vector \( y^4 = (y'_1, y'^*_2, y'^*_3) \). Because of the first criterion is incomparably more important than the second one, \( y^4 > y'^* \). Due to the second criterion is incomparably more important than the third one, we have \( y' > y^4 \) that together with \( y^4 > y'^* \) imply \( y' > y'^* \).

In this way, 1) has proven. Using similar considerations, one can prove 2). By Axiom 3, the last statement 3) is true ■
2.2 Invariance and Cone Relations

**Invariance Axiom**

Recall that a binary relation $\mathcal{R}$ defined on $\mathbb{R}^n$ is said to be *invariant* with respect to positive linear transformation if for any vectors $y', y^*, c \in \mathbb{R}^n$ and each positive number $\alpha$ the relationship $y'\mathcal{R}y^*$ implies $(\alpha y' + c)\mathcal{R}(\alpha y^* + c)$. In other words $\mathcal{R}$ is invariant if for any $y', y^* \in \mathbb{R}^n$ both the following properties hold:

- [additivity] $y'\mathcal{R}y^*$, $c \in \mathbb{R}^n$ \implies $(y' + c)\mathcal{R}(y^* + c)$
- [homogeneity] $y'\mathcal{R}y^*$, $\alpha > 0$ \implies $(\alpha y')\mathcal{R}(\alpha y^*)$

are true.

The relations $=, >, \geq, \leq$ defined on $\mathbb{R}^n$ are invariant with respect to positive linear transformation. It is easy to understand that the lexicographic relation (see Section 1.2) is invariant too.

In order to develop a substantive theory of the relative importance of criteria we need to accept the following axiom.

**Axiom 4** [invariance of the preference relation]. The preference relation $\succ$ is invariant with respect to positive linear transformation.

It was mentioned above that invariance is equivalent to additivity together with homogeneity. In other words $\succ$ is invariant iff $y' \succ y^*$ implies $(y' + c) \succ (y^* + c)$ as well as $\alpha y' \succ \alpha y^*$ for all $c \in \mathbb{R}^n$ and all positive numbers $\alpha$.

**Cone relations**

Cone relations play a basic role in further presentation. However, before introducing this concept it is necessary to recall some auxiliary notions of convex analysis.

A set $A \subset \mathbb{R}^n$ is said to be *convex* if for any pair of its points $A$ includes the whole segment connecting these points. In other words $A$ is convex iff for all pairs of points $y', y^* \in A$ and any number $\lambda \in [0,1]$ the relationship $\lambda y' + (1 - \lambda)y^* \in A$ is valid.

A set $K \subset \mathbb{R}^n$ is said to be a *cone* if $\alpha y \in K$ for each point $y \in K$ and any positive number $\alpha$. A cone that is convex is called a *convex cone*. In other words a convex set $K$ is a convex cone if for any its point $K$ includes the whole ray which begins at the origin [generally, without the origin itself] and is passing through the point. It is can be verified that the sum of two [or more] arbitrary elements of a convex cone always belongs to this cone. Moreover, linear combinations of any set of cone points with nonzero nonnegative coefficients belong to this cone.

---

3 This means that the vector composed from the coefficients is not equal to zero.
A cone \( K \) is pointed if \( y \in K \), \(-y \in K \) imply \( y = 0 \). A cone that is not pointed must include at least one straight line passing through the origin [the origin itself may not belong to the straight line].

Let \( c \in \mathbb{R}^n \). The set \( L \) of all solutions [vectors] \( x \in \mathbb{R}^m \) of the linear inequality \( \langle c, x \rangle = c_1x_1 + c_2x_2 + \ldots + c_mx_m \geq 0 \) is a convex cone [this set is called a closed half-space]. To verify this, consider an arbitrary \( x \in L \), i.e. such that \( \langle c, x \rangle \geq 0 \). For any positive number \( \alpha \) we have \( \langle c, x \rangle = \langle c, \alpha x \rangle \geq 0 \). This means that \( L \) is a cone. Let us prove that \( L \) is a convex set. To this end, take two arbitrary points \( x', x'' \in L \). We have \( \langle c, x' \rangle \geq 0 \) and \( \langle c, x'' \rangle \geq 0 \). Multiplying the first inequality by \( \lambda \in [0,1] \) and the second one by \( 1 - \lambda \) and then adding the obtained inequalities term by term, we obtain

\[
\lambda \langle c, x' \rangle + (1 - \lambda) \langle c, x'' \rangle = \langle c, \lambda x' + (1 - \lambda) x'' \rangle \geq 0
\]

i.e. \( \lambda x' + (1 - \lambda) x'' \in L \).

A closed half-space is a cone but not a pointed cone, since \( \langle c, x \rangle = 0 \) imply \( \langle c, -x \rangle = 0 \) for \( x \neq 0 \). The set of all solutions of a system of the linear inequalities

\[
\langle c^i, x \rangle = c^i_1x_1 + c^i_2x_2 + \ldots + c^i_mx_m \geq 0, \quad i = 1,2,\ldots, p
\]

(2.5)
is also a convex cone, which is an intersection of a finite number of closed half-spaces. This cone is said to be polyhedral. Generally, a polyhedral cone is not pointed.

Let \( a^1, a^2, \ldots, a^p \in \mathbb{R}^n \). It is easy to verify that the set of all nonnegative linear combinations of these vectors [i.e. all vectors \( \lambda_1a^1 + \lambda_2a^2 + \ldots + \lambda_pa^p \), where \( \lambda_1, \lambda_2, \ldots, \lambda_p \) are nonnegative] is a convex cone \( K \) in \( \mathbb{R}^m \). In this case, we shall say that the vectors \( a^1, a^2, \ldots, a^p \) generates \( K \) and write \( K = \text{cone}\{a^1, a^2, \ldots, a^p\} \). By duality theory [28], [31], the cone \( K \) is polyhedral, i.e. \( K \) is an intersection of a finite number of closed half-spaces.

Let \( K \) be a convex cone. A subset \( F \) of \( K \) is a facet if and only if there exists a supporting hyperplane \( H \) of \( K \) such that \( K \cap H = F \). If a convex cone contains the origin then this point is a single its facet of dimensionality 0. One-dimensional facets are said to be edges of a convex cone. Since the edge is defined by any its fixed nonzero vector [point], this vector is also called an edge. Convex analysis provides the following: any pointed convex closed cone, which does not coincide with the origin, is generated by all its edges.

Let a convex cone be a set of solutions of the system (2.5). Principally (see [4]), all its edges can be computed by analyzing solutions of certain subsystems of the system of linear equations

\[
\langle c^i, x \rangle = c^i_1x_1 + c^i_2x_2 + \ldots + c^i_mx_m = 0, \quad i = 1,2,\ldots, p
\]

The nonnegative orthant
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\( R_+^n = \{ y \in R^n \mid y \geq 0 \} = \{ y \in R^n \mid y \geq 0 \} \setminus \{0_n\} \)

is a pointed convex cone [without the origin] generated by all unit vectors of the space \( R^n \).

The orthant \( R_+^2 \) coincides with the first quadrant of the plane (Figure 2.1). This one is generated by two unit vectors \( e^1 = (1,0) \), \( e^2 = (0,1) \) and it is an intersection of the right and the upper closed half-planes [without the origin].

![Fig. 2.1](image)

Some pointed cones \( K_1 \) and \( K_2 \) are shown in Figure 2.2.

![Fig. 2.2](image)

The upper half-plane is a convex but no pointed cone.

The reader may find more details related to convex sets and cones, e.g., in [4], [28], [31].

**Definition 2.3.** A binary relation \( \mathcal{R} \) defined on \( R^n \) [i.e. \( \mathcal{R} \subset R^n \times R^n \)] is called a **cone relation** if there exists a cone \( K \subset R^n \) such that

\( y' \mathcal{R} y'' \iff y' - y'' \in K \quad \text{for all} \quad y', y'' \in R^n. \)
The right side of the above equivalence can be rewritten as $y' \in y^* + K$ (see Figure 2.3), where

$$y^* + K = \{ y \in R^n \mid y = y^* + z \text{ for some } z \in K \} .$$

The relations $>,$ $\geq$ defined on $R^n$ are cone ones whose cones are $R^*_m = \{ y \in R^n \mid y > 0_m \}$ and $R^m$ respectively.

**Theorem 2.3.** Any irreflexive, transitive, and invariant with respect to positive linear transformation binary relation $\mathcal{R}$ defined on $R^n$ is a cone relation whose cone is a pointed convex cone without the origin. Conversely, any cone relation, whose cone is a pointed convex cone without the origin, is irreflexive, transitive, and invariant with respect to positive linear transformation.

**Proof.** Necessity. Let a relation $\mathcal{R}$ be irreflexive, transitive, and invariant with respect to positive linear transformation. Consider the set

$$K = \{ y \in R^n \mid y \mathcal{R} 0_m \} . \quad (2.6)$$

By homogeneity of $\mathcal{R}$, $K$ is a cone. The additivity property of $\mathcal{R}$ yields

$$y^* \mathcal{R} y^* \iff (y' - y^*) \mathcal{R} 0_m \iff y' - y^* \in K \text{ for all } y', y^* \in R^n .$$

Hence, $\mathcal{R}$ is a cone relation whose cone is $K$. It remains to check that $K$ is a pointed convex cone without the origin.

On the contrary, if $0_m \in K$ then, by (2.6), we have $0_m \mathcal{R} 0_m$. It contradicts the irreflexivity property of $\mathcal{R}$. Consequently, $0_m \not\in K$.

To prove convexity of $K$, take arbitrary $y', y^* \in K$ and $\alpha \in (0,1)$. Due to the homogeneity property of $\mathcal{R}$, the relationships $y' \mathcal{R} 0_m$, $y^* \mathcal{R} 0_m$ imply $\alpha y' \mathcal{R} 0_m$ and $(1-\alpha) y^* \mathcal{R} 0_m$. Therefore, $0_m \mathcal{R} 0_m$, which contradicts the irreflexivity property of $\mathcal{R}$. Consequently, $K$ is a pointed convex cone without the origin.
and \((1-\alpha)y^*\mathcal{R}0_m\). By additivity of \(\mathcal{R}\), the relationship \(\alpha y^\prime \mathcal{R} y^m\) implies \((\alpha y^\prime + (1-\alpha)y^m)\mathcal{R}(1-\alpha)y^m\). Since \(\mathcal{R}\) is transitive, the obtained relationship together with \((1-\alpha)y^\prime \mathcal{R}0_m\) yield \((\alpha y^\prime + (1-\alpha)y^m)\mathcal{R}0_m\), i.e. \((\alpha y^\prime + (1-\alpha)y^m)\in K\). This proves that \(K\) is a convex cone.

To verify that \(K\) is a pointed cone, assume the opposite, i.e. there exists a non-zero vector \(y^\prime \in K\) such that \(-y^\prime \in K\). Since the sum of two arbitrary cone elements belongs to this cone, we have \(y^\prime - y^\prime \in K\), or, equivalently, \(0_m \mathcal{R} y^m\). It contradicts irreflexivity of \(\mathcal{R}\).

Sufficiency. Let \(\mathcal{R}\) be an arbitrary cone relation whose cone \(K\) is pointed, convex, and also \(0_m \notin K\).

The relation \(\mathcal{R}\) is irreflexive because of \(0_m \notin K\).

Prove the transitivity property of \(\mathcal{R}\). For three arbitrary vectors \(y^\prime, y^*, y^m \in \mathbb{R}^m\) such that \(y^\prime \mathcal{R} y^*, y^* \mathcal{R} y^m\) we have \(y^\prime - y^* \in K\) and \(y^* - y^m \in K\). The sum of two cone elements belongs to this cone. Hence, \(y^\prime - y^m \in K\), i.e. \(y^\prime \mathcal{R} y^m\). This proves transitivity.

The invariance property follows from the equivalences

\[
y^\prime \mathcal{R} y^* \iff y^\prime - y^* \in K \iff (y^\prime + c) - (y^* + c) \in K \iff (y^\prime + c) \mathcal{R} (y^* + c),
\]

\[
y^\prime \mathcal{R} y^* \iff y^\prime - y^* \in K \iff \alpha(y^\prime - y^*) \in K \iff \alpha y^\prime - \alpha y^* \in K \iff \alpha y^\prime \mathcal{R} \alpha y^* \]

for any \(c \in \mathbb{R}^m\) and \(\alpha > 0\).

**Corollary 2.1.** If a binary relation \(\succ\) satisfies Axioms 2-4 then this relation is a cone one whose cone is pointed, convex, and also it includes the nonnegative orthant \(\mathbb{R}^m_+\) and does not contain the origin. Conversely, a cone relation whose cone is a pointed, convex one which includes the nonnegative orthant \(\mathbb{R}^m_+\) and does not contain the origin, is a cone relation satisfying Axioms 2-4.

**Proof.** Necessity. If a binary relation \(\succ\) satisfies Axioms 2-4 then it is irreflexive, transitive, and invariant with respect to positive linear transformation. By Theorem 2.3, \(\succ\) is a cone relation. Denote by \(K\) the cone of \(\succ\). It remains to prove that \(K\) includes the nonnegative orthant. By Lemma 1.3 (Section 1.4), Pareto Axiom holds, i.e.

\(y^\prime \succeq y^m \Rightarrow y^\prime \succ y^m\)

or

\(y^\prime - y^m \in \mathbb{R}^m_+ \Rightarrow y^\prime - y^m \in K\).

Since the difference \(y^\prime - y^m\) could be arbitrary from \(\mathbb{R}^m_+\), the last implication can be rewritten as \(\mathbb{R}^m_+ \subseteq K\).

Sufficiency. If a cone relation \(\succ\) has a pointed convex cone \(K\) without the origin then, by Theorem 2.3, this relation is irreflexive, transitive, and invariant with
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respect to positive linear transformation. In other words \( \succ \) satisfies Axioms 2 and 4. Since \( \mathcal{K} \) includes the nonnegative orthant, the cone relation \( \succ \) satisfies Pareto Axiom. But Pareto Axiom implies Axiom 3. Therefore, \( \succ \) satisfies all Axioms 2-4.

By Corollary 2.1, any preference relation that satisfies Axioms 2-4 has a simple geometrical interpretation; namely, this is a cone relation whose cone is pointed, convex, and not narrower than \( R^n \).

2.3 Use of Information on Relative Importance

Simple Version of Main Definition

By Definition 2.1 (Section 2.1), the statement “one criterion is more important than another” has a quite definite meaning. In this definition there are two numerical parameters that form the relative importance coefficient in order to measure a degree of relative importance.

According to Definition 2.1, to verify that the \( i \)th criterion is more important than the \( j \)th one with two positive parameters \( w_i, w_j \) it is necessary to verify that

\[
y_i' > y_j' \quad \text{for all } s \in I \setminus \{i, j\}
\]

(2.3)

\[
y_i' - y_j' = w_i' \quad \text{and } y_j' - y_j' = w_j'
\]

Having \( w_i', w_j' \), we can evaluate the relative importance coefficient by

\[
\theta_i = \frac{w_i'}{w_i' + w_j'}.
\]

(2.4)

Obviously, the number of pairs of vectors such that (2.3) holds is infinite. By this reason practical meaning of Definition 2.1 is restricted, since nobody can check the relationship \( y', y'' \) for infinite number of vectors.

The invariance property of \( \succ \) allows us to simplify Definition 2.1 essentially.

Theorem 2.4. Let a relation \( \succ \) be invariant with respect to positive linear transformation on \( R^n \). The relationship \( y' \succ y'' \) is true for all \( y', y'' \in R^n \) satisfying (2.3) if and only if at least one of the following two relationships \( \bar{y} \succ 0_n \), \( \bar{y} \succ 0_n \) is valid, where

\[
\bar{y}_i = w_i' \quad \bar{y}_j = -w_j' \quad \bar{y}_s = 0 \quad \text{for all } s \in I \setminus \{i, j\}
\]

(2.7)

\[
\hat{y}_i = 1 - \theta_i \quad \hat{y}_j = -\theta_j \quad \hat{y}_s = 0 \quad \text{for all } s \in I \setminus \{i, j\}
\]

(2.8)

and \( \theta_i \) is the relative importance coefficient.
Proof. Consider two arbitrary vectors $y', y^*$ whose components are defined by (2.3). It is clear that

\[ y'_i > y^*_i \iff y'_i - y^*_i > 0; \quad y^*_j > y'_j \iff y^*_j - y'_j > 0 \]

Let us introduce the vector $\bar{y}$ whose components are $\bar{y}_i = y'_i - y^*_i = w^*_i$, $\bar{y}_j = y^*_j - y'^*_j = -w^*_j$, $\bar{y}_s = 0$ for all $s \in I \setminus \{i, j\}$. Using the additivity property of $\succ$, we have

\[ y' > y^* \iff (y' - y^*) > 0_m \iff \bar{y} > 0_m \]

where $\bar{y}$ is the same as in (2.7). The first part of the theorem has proven.

Continue the proof. By the homogeneity property of $\succ$, the relationship $\bar{y} > 0_m$ is equivalent to $\alpha \bar{y} > 0_m$ for any positive number $\alpha$. In particular, by putting $\alpha = -\theta_{ij} / \bar{y}_{ij}$ and denoting $\hat{y} = \alpha \bar{y}$ we have $\hat{y} > 0_m$, where

\[ \hat{y}_i = \alpha \bar{y}_i = -\frac{\theta_{ij}}{\bar{y}_{ij}} \bar{y}_i = \frac{\theta_{ij} w^*_i}{w^*_i} = \frac{w^*_i}{w^*_i + w^*_j} = 1 - \theta_{ij} \]

\[ \hat{y}_j = \alpha \bar{y}_j = -\frac{\theta_{ij} \bar{y}_j}{\bar{y}_j} = -\theta_{ij} \]

\[ \hat{y}_s = \alpha \bar{y}_s = 0 \text{ for all } s \in I \setminus \{i, j\}. \]

Thus the relationship $\bar{y} > 0_m$ is equivalent to $\hat{y} > 0_m$, where $\hat{y}$ is the same as in (2.8).

We assume that the preference relation $\succ$ satisfies Axiom 4. Then, by Theorem 2.4, we can reformulate Definition 2.1 in the following equivalent manner.

**Definition 2.1'.** Let $i, j \in I, i \neq j$. The criterion $f_i$ is [relatively] more important than $f_j$ with two positive parameters $w^*_i, w^*_j$ [or with a coefficient of the relative importance $\theta_{ij} \in (0,1)$] if

- $\bar{y} > 0_m$ holds for the vector $\bar{y}$ satisfying (2.7) [respectively $\hat{y} > 0_m$ holds for the vector $\hat{y}$ satisfying (2.8)].

By Definition 2.1', in order to check whether the $i$ th criterion is relatively more important than the $j$ th one with the relative importance coefficient $\theta_{ij} \in (0,1)$ it is sufficient to verify that the vector $\hat{y}$ of the type (2.8) is preferred to the zero vector.

For instance, if $(0.7, -0.3, 0) \succ 0$, then the first criterion is more important than the second one with the relative importance coefficient $\theta_{z} = 0.3$.\[\]

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Proof. Consider two arbitrary vectors $y', y^*$ whose components are defined by (2.3). It is clear that

\[ y'_i > y^*_i \iff y'_i - y^*_i > 0; \quad y^*_j > y'_j \iff y^*_j - y'_j > 0 \]

Let us introduce the vector $\bar{y}$ whose components are $\bar{y}_i = y'_i - y^*_i = w^*_i$, $\bar{y}_j = y^*_j - y'^*_j = -w^*_j$, $\bar{y}_s = 0$ for all $s \in I \setminus \{i, j\}$. Using the additivity property of $\succ$, we have

\[ y' > y^* \iff (y' - y^*) > 0_m \iff \bar{y} > 0_m \]

where $\bar{y}$ is the same as in (2.7). The first part of the theorem has proven.

Continue the proof. By the homogeneity property of $\succ$, the relationship $\bar{y} > 0_m$ is equivalent to $\alpha \bar{y} > 0_m$ for any positive number $\alpha$. In particular, by putting $\alpha = -\theta_{ij} / \bar{y}_{ij}$ and denoting $\hat{y} = \alpha \bar{y}$ we have $\hat{y} > 0_m$, where

\[ \hat{y}_i = \alpha \bar{y}_i = -\frac{\theta_{ij}}{\bar{y}_{ij}} \bar{y}_i = \frac{\theta_{ij} w^*_i}{w^*_i} = \frac{w^*_i}{w^*_i + w^*_j} = 1 - \theta_{ij} \]

\[ \hat{y}_j = \alpha \bar{y}_j = -\frac{\theta_{ij} \bar{y}_j}{\bar{y}_j} = -\theta_{ij} \]

\[ \hat{y}_s = \alpha \bar{y}_s = 0 \text{ for all } s \in I \setminus \{i, j\}. \]

Thus the relationship $\bar{y} > 0_m$ is equivalent to $\hat{y} > 0_m$, where $\hat{y}$ is the same as in (2.8).

We assume that the preference relation $\succ$ satisfies Axiom 4. Then, by Theorem 2.4, we can reformulate Definition 2.1 in the following equivalent manner.

**Definition 2.1'.** Let $i, j \in I, i \neq j$. The criterion $f_i$ is [relatively] more important than $f_j$ with two positive parameters $w^*_i, w^*_j$ [or with a coefficient of the relative importance $\theta_{ij} \in (0,1)$] if

- $\bar{y} > 0_m$ holds for the vector $\bar{y}$ satisfying (2.7) [respectively $\hat{y} > 0_m$ holds for the vector $\hat{y}$ satisfying (2.8)].

By Definition 2.1', in order to check whether the $i$ th criterion is relatively more important than the $j$ th one with the relative importance coefficient $\theta_{ij} \in (0,1)$ it is sufficient to verify that the vector $\hat{y}$ of the type (2.8) is preferred to the zero vector.

For instance, if $(0.7, -0.3, 0) \succ 0$, then the first criterion is more important than the second one with the relative importance coefficient $\theta_{z} = 0.3$.\[\]
Main Result

The following theorem shows how information on the relative importance of criteria allows us to reduce the Pareto set.

**Theorem 2.5** [in terms of vectors]. Let Axioms 1-4 be satisfied and the criterion \( f_i \) be more important than \( f_j \) with the pair of positive parameters \( w'_i, w'_j \). Then for any nonempty set of selected vectors \( \text{Sel}Y \) it follows that

\[
\text{Sel}Y \subset \hat{P}(Y) \subset P(Y)
\]  

(2.9)

where

- \( P(Y) \) is a set of Pareto-optimal vectors with respect to \( f \)
- \( \hat{P}(Y) = f(P_j(X)) \)
- \( P_j(X) \) is a set of Pareto-optimal alternatives with respect to \( f \)
- \( \hat{f}_j = (\hat{f}_1, \hat{f}_2, \ldots, \hat{f}_m) \)

and also

\[
\hat{f}_j = w'_i f_i + w'_j f_j, \quad \hat{f}_s = f_s \text{ for all } s \in I \setminus \{j\}.
\]  

(2.10)

**Proof.** Part I. Let \( K \) be the pointed convex cone of \( \succ \). By Definition 2.1', the relationship \( \tilde{f} \succ 0_m \) holds for \( \tilde{f} \) that is given by (2.7). In other words, \( \tilde{f} \in K \).

By \( e^i, e^{i+1}, \ldots, e^m \) let us designate the unit vectors of \( R^m \); i.e. the \( s \)th component of \( e^i \) is equal to one and all the rest are equal to zero, \( s = 1, 2, \ldots, m \). Let \( M \) denotes a convex cone [without the origin] generated by the set of linearly independent vectors

\[
e^i, \ldots, e^{i+1}, \tilde{f}, e^{i+1}, \ldots, e^m.
\]  

(2.11)

The cone \( M \) coincides with the set of all linear combinations

\[
\lambda_1 e^i + \cdots + \lambda_{i-1} e^{i-1} + \lambda_i \tilde{f} + \lambda_{i+1} e^{i+1} + \cdots + \lambda_m e^m
\]

where \( \lambda_1, \lambda_2, \ldots, \lambda_m \) are nonnegative coefficients such that \( (\lambda_1, \lambda_2, \ldots, \lambda_m) \neq 0_m \).

Let us verify that the cone \( M \) is pointed. Assume the opposite; then there exists a nonzero vector \( y \in M \) such that \( -y \in M \). According to above we have

\[
y = \lambda_1 e^i + \cdots + \lambda_{i-1} e^{i-1} + \lambda_i \tilde{f} + \lambda_{i+1} e^{i+1} + \cdots + \lambda_m e^m
\]

\[
-y = \lambda_1 e^i + \cdots + \lambda_{i-1} e^{i-1} + \lambda_i \tilde{f} + \lambda_{i+1} e^{i+1} + \cdots + \lambda_m e^m
\]

where all the coefficients \( \lambda_1, \lambda_2, \ldots, \lambda_m \) are nonnegative and \( (\lambda_1, \lambda_2, \ldots, \lambda_m) \neq 0_m \), \( (\lambda_i^*, \lambda_i^*, \ldots, \lambda_i^*) \neq 0_m \). Since the sum of two cone elements belongs to this cone, by adding the two last equalities, we obtain

\[\lambda_i e^i + \cdots + \lambda_{i-1} e^{i-1} + \lambda_i \tilde{f} + \lambda_{i+1} e^{i+1} + \cdots + \lambda_m e^m = 0_m.\]

\[\therefore \lambda_i = 0_{i-1} \cdot e^i + \cdots + \lambda_m e^m.
\]  

\[\lambda_i^* e^i + \cdots + \lambda_m^* e^m = 0_m.
\]  

(2.10)
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\[ 0_m = (\lambda'_1 + \lambda'_m)e^1 + \ldots + (\lambda'_r + \lambda'_m)e^r + \ldots + (\lambda'_m + \lambda'_m)e^m \]

where at least one coefficient in parentheses must be nonzero. On the other hand, due to linear independence of (2.11), all these coefficients are equal to zero. This contradiction proves that the cone \( M \) is pointed.

Part II. Let us prove that the cone \( M \) coincides with the set of all nonzero solutions of the following system of linear inequalities

\[ y_s \geq 0 \quad \text{for all } s \in I \setminus \{j\} \]

(2.12)

\[ w_j y_i + w_j y_j \geq 0 \]

Find a fundamental system of solutions of (2.12). To this end, consider the corresponding system of linear equations

\[ y_s = 0 \quad \text{for all } s \in I \setminus \{j\} \]

(2.13)

\[ w^*_j y_i + w^*_j y_j = 0 \]

It can be rewritten as

\[ \langle e^i, y \rangle = 0 \quad \text{for all } s \in I \setminus \{j\} \]

(2.14)

where \( \vec{y} = (\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_m) \) whose components are

\[ \vec{y}_i = w^*_i, \quad \vec{y}_j = w^*_j; \quad \vec{y}_s = 0 \quad \text{for all } s \in I \setminus \{i, j\} \]

In (2.14) the number of equations is equal to \( m \). It is easy to verify that any \( m-1 \) vectors among \( e^1, \ldots, e^{i-1}, \vec{y}, e^{i+1}, \ldots, e^m \) are linear independent. Hence, a fundamental system of solutions of (2.12) can be constructed by successive removing one equation from (2.14) and finding a nonzero solution [i.e. a vector] of the obtained subsystem such that this solution satisfies (2.12).

If we remove the last equation in (2.14) then, e.g., \( e^j \) is a nonzero solution of the obtained “shortened” system. Removing \( \langle e^i, y \rangle = 0 \) [for all \( s \neq i \)], we get \( e^i \) as a nonzero solution of the corresponding “shortened” system. If \( \langle e^i, y \rangle = 0 \) is deleted then we may take \( \vec{y} \). As a result, we obtain the solution vectors \( e^1, \ldots, e^{i-1}, \vec{y}, e^{i+1}, \ldots, e^m \) each of them satisfies (2.12). Consequently, these vectors compose a fundamental system of solutions of (2.12).

---

5 A general solution of a system of linear inequalities is a linear combination (with nonnegative coefficients) of some finite number of solutions which, by varying the coefficients, can represent any solution of this system (see [4], p. 243). A fundamental system of solutions is the minimal (with respect to their number) general solution.

6 Recall that \( \langle a, b \rangle \) denotes a scalar product of two vectors \( a \) and \( b \).
Since they are the same as in (2.11), the cone $M$ coincides with the set of non-zero solutions of the system of linear inequalities (2.12).

**Part III.** Recall that $\bar{y} \in K$. By Corollary 2.1, $R^n_+ \subset K$. The orthant $R^n_+$ is generated by the unit vectors $e^1, e^2, \ldots, e^n$. Hence, $K$ includes all the vectors $e^1, e^2, \ldots, e^n, \bar{y}$. Since $K$ is a convex cone, one includes the set of all nonzero non-negative linear combinations of $e^1, e^2, \ldots, e^n, \bar{y}$. In Part I this set of linear combinations was denoted by $M$. Consequently, we obtain

$$R^n_+ \subset M \subset K$$

that implies

$$\text{Ndom } Y \subset \hat{P}(Y) \subset P(Y)$$

(2.15)

where

$$\hat{P}(Y) = \{y^* \in Y \mid \text{there does not exist } y \in Y \text{ such that } y - y^* \in M\}$$

is a set of all nondominated vectors of $Y$ with respect to the cone relation whose cone is $M$.

Let $x, x^* \in X$, $y = f(x)$, $y^* = f(x^*)$, and also $f(x) \not= f(x^*)$. By Part II, $f(x) - f(x^*) \in M$ is valid iff the vector $y = f(x) - f(x^*)$ is a nonzero solution of (2.12), i.e.

$$\begin{align*}
&\begin{cases}
    f_i(x) - f_i(x^*) \\
    \ldots \\
    f_{j_1}(x) - f_{j_1}(x^*) \\
    w_i^j(f_i(x) - f_i(x^*)) + w_j^i(f_j(x) - f_j(x^*)) \\
    f_{j_2}(x) - f_{j_2}(x^*) \\
    \ldots \\
    f_m(x) - f_m(x^*)
  \end{cases} \\
\geq 0_n.
\end{align*}$$

Using the vector function $\hat{\mathbf{f}}$ whose components are given by (2.10), we can rewrite the last inequality as follows $\hat{\mathbf{f}}(x) - \hat{\mathbf{f}}(x^*) \in R^n_+$ or $\hat{\mathbf{f}}(x) \geq \hat{\mathbf{f}}(x^*)$. Therefore, for every two distinct vectors $y = f(x)$, $y^* = f(x^*)$ the relationship $y - y^* \in M$ is equivalent to $\hat{\mathbf{f}}(x) \geq \hat{\mathbf{f}}(x^*)$. This implies $\hat{P}(Y) = f(P_f(X))$.

To obtain (2.9), i.e. to finish the proof, it remains to add $\text{Sel } Y \subset \text{Ndom } Y$ to (2.15).
By the Edgeworth-Pareto principle (see Section 1.4), the Pareto set includes all selected vectors or, equivalently, only Pareto-optimal vectors should be selected. If it is known that one criterion is more important than another then the Pareto set may be reduced without the loss of selected vectors. In other words we may remove some Pareto-optimal vectors from further consideration, since they should not be selected a fortiori. The reduction of the Pareto set may essentially facilitate the decision process.

To be exact, it must be noted that in some cases [especially, if the relative importance coefficient is close to 0, i.e. if the criteria $j_f$ and $\hat{j_f}$ are almost the same] may be $\hat{P}(Y) = P(Y)$. In these cases, the information on the relative importance of criteria is useless.

Divide the both sides of the first equality in (2.10) by $**_{ji} w_{ij} +$ and remain the designation $\hat{j_f}$ for the obtained left side. Then we can state the following.

**Corollary 2.2.** Theorem 2.5 is true if the components of $\hat{f}$ are determined by $\hat{j_f} = \theta_{j_f} f_{j_f} + (1 - \theta_{j_f}) f_{\hat{j_f}}, \quad \hat{j_s} = f_{j_s}$ for all $s \in I \setminus \{j\}$ instead of (2.10).

In terms of alternatives Theorem 2.5 can be formulated in the following way.

**Theorem 2.5 [in terms of alternatives].** Let Axioms 1-4 be satisfied and the criterion $f_i$ be more important than $f_j$ with the pair of positive parameters $w_{ij}, w_{ji}$. Then for any nonempty set of selected alternatives $\text{Sel} X$ it follows that $\text{Sel} X \subset P_X(X) \subset P(X)$ (2.17)

where $P(X)$ is a set of Pareto-optimal alternatives with respect to $f$

$P_X(X)$ is a set of Pareto-optimal alternatives with respect to $\hat{f}$

$\hat{f} = (\hat{f}_1, \hat{f}_2, ..., \hat{f}_m)$ is given by (2.10) or (2.16).

Figure 2.4 illustrates (2.17).
Let us discuss the hypotheses of Theorem 2.5. First of all we need to pay attention to the absence of any kind of requirements to the set of alternatives \( X \) and the vector criterion \( f \). This means that Theorem 2.5 might be applied to any multicriteria choice problem; the set \( X \) may include a finite as well as infinite number of elements; the functions \( f_1, f_2, \ldots, f_m \) may be linear or nonlinear, concave or no concave, continuous or discontinuous etc. The only requirement is that all Axioms 1-4 must be satisfied, i.e. the DM’s behavior should be “reasonable”.

Formulas (2.10) and (2.16), to compute the “new” vector criterion \( \hat{f} \), are very simple. Indeed, to obtain “new” \( \hat{f} \) from the “old” \( f \), in accordance with (2.16) we should to replace the less important criterion \( jf \) by a convex combination of the criteria \( if \) and \( jf \) with the relative importance coefficient \( \theta \in (0,1) \). All the rest “old” criteria stay the same. And also if \( if \) and \( jf \) are continuous, convex, concave or linear then the “new” criterion \( \hat{f} \) will possess the same characteristics.

For linear criteria, formulas (2.10), (2.16) have the simplest form.

**Corollary 2.3.** Let \( X \subset R^n \) and the functions \( if, jf \) be linear, i.e.

\[
f_i(x) = \langle c^i, x \rangle = \sum_{k=1}^{n} c^i_k x_k, \quad k = i, j
\]

where \( c^i = (c^i_1, c^i_2, \ldots, c^i_n) \). Then

\[
\hat{f}_j(x) = \langle \hat{c}, x \rangle
\]

where

\[
\hat{c} = w^*_i c^i + w^*_j c^j \quad \text{or} \quad \hat{c} = \theta_j c^j + (1-\theta_j) c^i.
\]  

(2.18)

Corollary 2.3 follows from (2.10), (2.16) due to the linearity property of a scalar product.

The second representation in (2.18) has a simple geometrical interpretation on the plane, i.e. for \( n = 2 \) (Figure 2.5).
The closer the relative importance coefficient $\theta_{ij}$ to 0, the closer the end point of the vector $\hat{c}$ to the end point of $c^j$. If $\theta_{ij} = 0.5$ then the end point of $\hat{c}$ is placed in the center of the interval between the end points of $c^i$ and $c^j$. If the relative importance coefficient is close to 1 then the end point of $\hat{c}$ is placed on this interval near the end point of $c^i$; in this case, the vector criterion $\hat{f}_j$ is almost the same as $f_j$, i.e. $\hat{f}_j$ becomes “nonessential”.

**Geometrical aspects**

By Theorem 2.3, the DM’s [in general, unknown] preference relation $\succ$ is a cone one whose cone is a pointed convex cone $K$ without the origin. By Corollary 2.1, $K$ includes the nonnegative orthant, i.e. $R^+_n \subset K$. This inclusion implies $\mathrm{Ndom}Y \subset P(Y)$. Using the last inclusion and Lemma 1.2, we have $\mathrm{Sel} \subset \mathrm{Ndom}Y \subset P(Y)$. Finally,

$$\mathrm{Sel} \subset P(Y)$$

i.e. the Pareto set is an upper estimate for the set of selected vectors.

Let $f_i$ be more important than $f_j$ with two positive parameters $w_i^*, w_j^*$. This means that there is the vector $\overline{y} \in K$ that is given by (2.7). Hence, $K$ includes the nonnegative orthant as well as $\overline{y}$.

Consider the cone $M$ that coincides with the set of all nonzero nonnegative linear combinations of $e_1, \ldots, e^{i-1}, \overline{y}, e^{i+1}, \ldots, e^n$; $M \neq R^+_n$. This cone was introduced in Part I of the proof of Theorem 2.5. There was also shown that $R^+_n \subset M \subset K$. This imply

$$\mathrm{Sel} \subset \mathrm{Ndom}Y \subset \mathrm{Ndom}_y \subset P(Y)$$
2 Relative importance of two criteria

where

\[ P(Y) = \{ y' \in Y \mid \text{there does not exist } y \in Y \text{ such that } y - y' \in R^n_+ \} \]

\[ \text{Ndom}_{\text{st}} Y = \hat{P}(Y) = \{ y' \in Y \mid \text{there does not exist } y \in Y \text{ such that } y - y' \in M \} \]

\[ \text{Ndom} Y = \{ y' \in Y \mid \text{there does not exist } y \in Y \text{ such that } y - y' \in K \}. \]

Thus, we obtain a more precise upper estimate for an unknown set of selected vectors than \( P(Y) \), i.e.

\[ \text{Sel} Y \subset \text{Ndom}_{\text{st}} Y. \]

It easy to understand that the wider \( M \) than the nonnegative orthant \( R^n_+ \) the narrower \( \text{Ndom}_{\text{st}} Y \) than \( P(Y) \).

In this way, information on the relative importance of criteria provides us to construct the cone \( M \) that is wider than \( R^n_+ \) (Figure 2.6) and obtain a more precise upper estimate than \( P(Y) \) for the set of selected vectors.

\[ y_2 \]
\[ y_1 \]
\[ M \]

Fig. 2.6.

**Example 2.1.** Let \( f = (f_1, f_2) \), \( Y = f(X) = \{ y^1, y^2, y^3 \} \), where

\[ y^1 = (4,1), \quad y^2 = (3,2), \quad y^3 = (1,3). \]

Here, all three vectors are Pareto-optimal, i.e. the Edgeworth-Pareto principle does not allow us to facilitate a choice.

Let the first criterion be more important than the second one with the relative importance coefficient 0.5. By Definition 2.1, this implies that \( \hat{y} = (0.5, -0.5) \in K \).

Figure 2.7 shows these three vectors [points] and the cone \( M \) translated into the end points of these vectors.
Relative importance of two criteria

Fig. 2.7.

Obviously, neither $y^1$ nor $y^2$ should be selected, since
$$y^1 \in y^2 + M, \quad y^2 \in y^3 + M.$$  
Therefore, the only $y^1$ may be selected. In other words if the set of selected vectors is nonempty then one contains only the first vector.

We can arrive to the same conclusion if Theorem 2.5 is applied. Indeed, owing to (2.16), the “new” second criteria is $0.5y_1 + 0.5y_2$. Now easily to calculate
$$\hat{Y} = \hat{f}(X) = \{(4, 2.5), (3, 2.5), (1, 2)\}; \quad P(\hat{Y}) = \{(4, 2.5)\}; \quad P(Y) = \{(4.1)\}.$$

2.4. Invariance of Measurements and Scales

Quantitative and Qualitative Scales

Dealing with the quantitative information on the relative importance of criteria, we mean that all criteria $f_1, f_2, \ldots, f_n$ have numerical values. Thus $y_i = f_i(x) \in R$ for every $x \in X$ and all $i = 1, 2, \ldots, m$. This is sufficient to consider a multicriteria choice problem within a mathematical framework.

However, for any applied multicriteria problem the numerical value of criterion is a result of measuring on a scale. For instance, if the criterion expresses cost of a project, profit, or expenses then its values are measured in rubles, millions of rubles, dollars, euro or other currency units. Measuring the length of objects, we use meters, inches, feet, yards, and so forth. To indicate a period of time, hours and
seconds, years, millions of years are eligible. Thus, solving a concrete applied problem, we should use different scales and units of measurement.

There are some types of measurement scales. To count a number of objects, people, properties, and so on, an absolute scale is used. This scale has fixed the origin of reference [a “true” zero point]. Two different [measuring] persons should obtain identical results, if they perform their measurement of the same quantities on this scale regardless of each other. It is also may be said that there is the only unit of measurement [a “true” one] for all measuring persons that use an absolute scale.

To measure a physical property such as mass of a thing, various units of measurement are used. It is well known that mass could be expressed in kilograms, pounds, tons, and so on. In such a case, for all measuring persons a fixed object is the origin of reference [a “true” zero point] that corresponds to absence of any mass, but units of measurement may vary. Two results of measurement \( y'_i \) and \( y''_i \) of the same object for two different measuring persons that use different units of measurement differ by definite positive factor \( \alpha_i \), i.e. \( y'_i = \alpha_i y''_i \). In other words the results of measurement are defined up to the transformation \( \varphi(y_i) = \alpha_i y_i, \, \alpha_i > 0 \). A scale of this type is called a ratio scale.

The name of this scale is associated with the fact that rations of measurements are the same regardless of the using units. Indeed, if we have two numbers [measurements] \( y'_i \) and \( y''_i \) for two different objects in one unit of measurement and two numbers \( \tilde{y}'_i \) and \( \tilde{y}''_i \) for the same two objects in other unit of measurement then
\[
\frac{\tilde{y}'_i}{\tilde{y}''_i} = \frac{\alpha_i y'_i}{\alpha_i y''_i} = \frac{y'_i}{y''_i}
\]
since \( \tilde{y}'_i = \alpha_i y'_i \) and \( \tilde{y}''_i = \alpha_i y''_i \).

For instance, if the first measuring person concludes that mass of one object is twice as much mass of the other object then the second measuring person by his/her own unit of measurement must obtain the same result.

This means that the statement "value \( A \) is \( x \) times greater [or less] than \( B \)" is appropriate for \( A \) and \( B \) measured on a ratio scale.

Obviously, values of criterion that expresses cost of a project, profit or an expense also should be measured on the ratio scale.

Another measurement scale is characterized by settled the unit of measurement and no fixed the reference origin; an example of such a scale is a chronology scale. Namely, a scale of differences is a scale the results of measurement in which are invariant with respect to the transformation \( \varphi(y_i) = y_i + c_i \), where \( c_i \) is a fixed number. Measurements on such scales are characterized by invariance of the differences between two values of measurement. In other words the statement “a value \( A \) is greater [or less] than \( B \) with the difference \( x \)” is appropriate for measurements on a scale of differences. For instance, Nicholas the Second’s duration of governing, evaluated according to Gregorian and Julian calendar, are the same.
An interval scale is a scale for which results of measurement are invariant with respect to positive linear transformation \( \phi(y_i) = \alpha_i y_i + c_i \), where \( \alpha_i > 0 \) and \( c_i \) are fixed numbers. Typical example of such a scale is that of temperatures. It is known that there are Celsius and Fahrenheit scales to measure temperature. Conversion of temperature from degrees Celsius to degrees Fahrenheit can be realized by the formula \( F = \frac{9}{5} C + 32 \), which determines some positive linear transformation.

Besides quantitative scales there are qualitative ones. Typical representative of such a scale is an ordinal scale the results of measurement in which are invariant with respect to transformation of the following type: \( \phi(y_i) \), where \( \phi \) is a strictly increasing function. Examples of such scales are Moh's scale for hardness of minerals, grades for academic performance, and blood sedimentation rate as a measure of intensity of pathology.

The statement "A is greater [or less] than B" is appropriate for A and B measured on an ordinal scale, since an order of values is essential. Both the statements "A is x times greater [or less] than B" and "A is greater [or less] than B with the difference x" are unavailable because of the reference origin as well as unit of measurement are not fixed and might vary on the ordinal scale.

It should be noted that there are other quantitative scales (see, e.g., [10], [27]). All statements concerning to the measurements performed on qualitative scale take place for quantitative scales but not wise versa. That is why quantitative scales seem to be richer in content than qualitative ones. For quantitative scales, the obtained results can be more informative although they might be applied to a narrower class of problems.

**Invariance of Pareto Set**

Recall that a Pareto set \( P(Y) \) is defined by

\[
P(Y) = \{ y' \in Y \mid \text{there does not exist } y \in Y \text{ such that } y \geq y' \}
\]

where \( y \geq y' \) is equivalent to \( y_i \geq y_i' \) for all \( i \in I \) and \( y_s > y_s' \) for at least one \( s \in I \).

Let \( \phi \) be a strictly increasing numerical function of one variable defined everywhere, i.e.

\[
y_i > y_i' \iff \phi(y_i) > \phi(y_i') \quad \text{for all } y, y' \in R.
\]

Evidently, \( y = y' \) is equivalent to \( \phi(y) = \phi(y') \) for any strictly increasing function \( \phi \).

Summarizing, we have

\[
y \geq y' \iff (y_1, \ldots, y_{i-1}, \phi(y_i), y_{i+1}, \ldots, y_n) \geq (y_1', \ldots, y_{i-1}', \phi(y_i'), y_{i+1}', \ldots, y_n').
\]
This equivalence implies that a Pareto set is invariant with respect to any strictly increasing transformation of criteria; in other words a concept of the Pareto set might be used if all criteria are measured on ordinal [in particular, on quantitative] scales.

**Invariance of Theorem 2.5**

The main theoretical result of this chapter is Theorem 2.5 that shows how information on the relative importance of criteria might be used to reduce the Pareto set. As it was shown in the previous section, a basis of this reducing is

\[
\text{Sel} Y \subset \tilde{P}(Y) \subset P(Y)
\]

(2.9)

where

\[
\tilde{P}(Y) = f(P_{\hat{f}}(X))
\]

\(P_{\hat{f}}(X)\) is a set of Pareto-optimal alternatives with respect to \(\hat{f}\)

\[
\hat{f} = (\hat{f}_1, \hat{f}_2, ..., \hat{f}_m)
\]

and also

\[
\hat{f}_j = w^*_j f_{i_j} + w^*_j j_f_j, \quad \hat{f}_s = f_{s} \quad \text{for all} \quad s \in I \setminus \{j\}.
\]

According to a quantitative approach, which has accepted throughout this book, we suppose that values of all criteria are measured on quantitative scales. In this case, the following theorem substantives applications of Theorem 2.5 to multicriteria choice problems under consideration.

**Theorem 2.6.** The inclusions (2.9), (2.17) are invariant with respect to positive linear transformation of criteria.

**Proof.** Recall that the concepts of a set of selected vectors as well as a set of all nondominated vectors are not based on criteria \(f_{1}, f_{2}, ..., f_{m}\). Consequently, the inclusion \(\text{Sel} Y \subset \text{Ndom} Y\) is invariant with respect to any [in particular, positive linear] transformation of the criteria.

Invariance of \(P(Y)\) was established in the previous subsection. To finish the proof it remains to prove the invariance property of \(\tilde{P}(Y)\).

To this end, it is sufficient to establish the invariance property of the inequality

\[
\hat{f}_j = w^*_j y_{i_j} + w^*_j y_{j_j} > w^*_j y_{i_j} + w^*_j y_{j_j} = \tilde{f}_j.
\]

(2.19)

Recall that

\[
w^*_j = y^*_i - y^*_i, \quad w^*_j = y^*_j - y^*_j
\]

where \(y^*_i = f_k(x^*_k), \quad y^*_j = f_k(x^*_k), \quad k = i, j\); and also \(w^*_j, w^*_j\) are fixed positive numbers.
For $k = i, j$ replace $y_k$ by $\tilde{y}_k = \alpha_k y_k + c_k$ ($\alpha_k > 0$) in $\tilde{f}_j = w'_j f_j + w'_i f_j$, taking into account that $y_k = f_k$. As a result we obtain a “transformed” criterion

$$
\tilde{f}_j = (\alpha_j y'_j + c_j - \alpha_j y'_j - c_j) (\alpha_i y_i + c_i) + (\alpha_i y'_i + c_i - \alpha_i y'_i - c_i) (\alpha_j y'_j + c_j).
$$

After simplifying, we get

$$
\tilde{f}_j = \alpha_i \alpha_j w'_j y_i + \alpha_i \alpha_j w'_i y_j + C
$$

where the constant

$$
C = \alpha_i \alpha_j w'_i c_i + \alpha_i \alpha_j w'_j c_j
$$

does not depend on $y_i, y_j$.

Assume that (2.19) is valid for arbitrary $y_i, y_j, \bar{y}_i, \bar{y}_j$. Multiply (2.19) by the positive number $\alpha_i \alpha_j$ and add $C$ to both the sides. By (2.20), we obtain

$$
\tilde{f}_j = \alpha_i \alpha_j \tilde{f}_j + \alpha_i \alpha_j w'_i \bar{y}_j + C.
$$

(2.21)

Thus (2.19) implies (2.21).

Remark 2.1. Simple analysis demonstrates that the relative importance coefficient is not invariant with respect to positive linear transformation of criteria. Moreover, it is not invariant with respect to both the transformations $\tilde{y}_k = a_k y_k$ and $\tilde{y}_k = y_k + c_k$ ($k = i, j$).

Remark 2.1 provides that the relative importance coefficient might be different for decision makers, even if they have the same preferences and perform their measurements on scales of the same type. This is no wonder, since these decision makers may use different units of measurement for the same criteria.

Example 2.2. Let two decision makers have the same preferences. Assume that they measure values of the first criterion on the ratio scale [in currency units] but values of the second criterion on an absolute scale. One of them uses dollars whereas the other prefers rubles. Let the first criterion be more important than the second one and $w'_1 = 1000, w'_2 = 10$ for the first DM. Then the corresponding relative importance coefficient is

$$
\theta'_{12} = \frac{10}{1000 + 10} \approx 0.01.
$$

Since the decision makers have the same preferences and the currency rate is about USD/RUR 30, for the second DM we have $w'_1 = 30000, w'_2 = 10$. Therefore,

$$
\theta'_{12} = \frac{10}{30000 + 10} \approx 0.00033.
$$
that is significantly less than $\theta_i$. And it is no wonder, because $1 is more “expensive” than 1 ruble.