Strategic Stability of Economic Decisions

Nikolay A. Zenkevich
Graduate School of Management, St. Petersburg University, St. Petersburg, Russia

mailto: zenkevich@gsom.pu.ru

DEGIT-XVI, St. Petersburg, Russia, September 9th, 2011
Outline

Introduction
1. Strategically Stable Cooperative Solution
   1.1. Differential Cooperative Game
   1.2. Optimal Cooperative Solution
   1.3. Dynamically Stable Cooperative Solution
   1.4. Payoff Distribution Procedure (PDP)
   1.5. Regularized Game
   1.6. Strategic Stability Theorem 1
   1.7. Strategic Stability

2. Strategically Stable Technological Alliance
   2.1. Technological Alliance Cooperative Game
   2.2. Dynamic Shapley Value
   2.3. Dynamic Stable Shapley Value
   2.4. Strategic Stability Theorem 2

Literature

DEGIT-XVI, St. Petersburg, Russia, September 9th, 2011
Introduction

Cooperation is a basic form of human behavior. And for many practical reasons it is important that cooperation stable on a time interval under consideration. There are three important aspects which must be taken into account when the problem of stability of long-range cooperative agreements is investigated.

1. *Time-consistency (dynamic stability)* of the cooperative agreements. Time-consistency involves the property that, as the cooperation develops cooperating partners are guided by the same optimality principle at each instant of time and hence do not possess incentives to deviate from the previously adopted cooperative behavior.

2. *Strategic stability*. The agreement is to be developed in such a manner that at least individual deviations from the cooperation by each partner will not give any advantage to the deviator. This means that the outcome of cooperative agreement must be attained in some Nash equilibrium, which will guarantee the strategic support of the cooperation.

The mathematical tool based on payoff distribution procedure (PDP) or imputation distribution procedure (IDP) is developed to deal with the above mentioned aspects of cooperation.
Introduction

1. Strategically Stable Cooperative Agreement

1.1. Differential Cooperative Game

Consider \( n \)-person differential game \( \Gamma(x_0, T-t_0) \) with prescribed duration and independent motions on the time interval \([t_0, T]\). Motion equations have the form:

\[
\begin{align*}
\dot{x}_i &= f_i(x_i, u_i), \quad u_i \in U_i \subset R^l, x_i \in R^n, \\
x_1(t_0) &= x_i^0, \quad i = 1, ..., n.
\end{align*}
\]  

(1)

It is assumed that the system of differential equations (1) satisfies all conditions necessary for the existence, prolongability and uniqueness of the solution for any \( n \)-tuple of measurable controls \( u_1(t), ..., u_n(t) \).

The payoff of player \( i \) is defined as:

\[
H_i(x_0, T-t_0; u_1(\cdot), ..., u_n(\cdot)) = \int_0^T h_i(x(\tau))d\tau,
\]

where \( h_i(x) \) is a continuous function and \( x(\tau) = \{x_1(\tau), ..., x_n(\tau)\} \) is the solution of (1) when open-loop controls \( u_1(t), ..., u_n(t) \) are used and \( x(t_0) = \{x_1(t_0), ..., x_n(t_0)\} = \{x_1^0, ..., x_n^0\} \)

Suppose that there exist an \( n \)-tuple of open-loop controls \( \bar{u}(t) = (\bar{u}_1(t), ..., \bar{u}_n(t)) \) and the trajectory \( \bar{x}(t) \), \( t \in [t_0, T] \), such that

\[
V(x_0, T-t_0; N) = \max_{u_1(t), ..., u_n(t)} \sum_{i=1}^n H_i(x_0, T-t_0; u_1(t), ..., u_n(t)) = \\
= \sum_{i=1}^n H_i(x_0, T-t_0; \bar{u}_1(t), ..., \bar{u}_n(t)) = \sum_{i=1}^n \int_0^T h_i(\bar{x}_i(\tau))d\tau,
\]

(2)

The trajectory \( \bar{x}(t) = (\bar{x}_1(t), ..., \bar{x}_n(t)) \) satisfying (2) we shall call “optimal cooperative trajectory”.

DEGIT-XVI, St. Petersburg, Russia, September 9th, 2011
1.2. Optimal cooperative solution

Let \( N = \{1, \ldots, n\} \) be the set of players. Define in \( \Gamma(x_0, T - t_0) \) characteristic function in a classical way:

\[
V(x_0, T - t_0; N) = \sum_{i=1}^{n} \int_{0}^{\tau} h_i(\bar{x}_i(\tau))d\tau,
\]

\[
V(x_0, T - t_0; O) = 0,
\]

\[
V(x_0, T - t_0; S) = \text{Val} \Gamma_{S,N\setminus S}(x_0, T - t_0),
\]

where \( \text{Val} \Gamma_{S,N\setminus S}(x_0, T - t_0) \) is a value of zero-sum game played between coalition \( S \) acting as first player and coalition \( N \setminus S \) acting as player 2, with payoff of player \( S \) equal to:

Define \( L(x_0, T - t_0) \) as imputation set in the game \( \Gamma(x_0, T - t_0) \) (see Neumann and Morgenstern (1947))

\[
L(x_0, T - t_0) = \{ \alpha = (\alpha_1, \ldots, \alpha_n) : \\
\alpha_i \geq V(x_0, T - t_0; \{i\}), \sum_{i \in N} \alpha_i = V(x_0, T - t_0; N) \}
\]

The set \( W(x_0, T - t_0) \subset L(x_0, T - t_0) \) will be called as optimal solution.
1.3. Dynamically stable cooperative decision

Cooperative differential game \( \Gamma(x_0, T-t_0) \) has dynamically stable solution \( W(x_0, T-t_0) \), if all imputations \( \xi(x_0, T-t_0) \in W(x_0, T-t_0) \) are dynamically stable.

We have proved under general conditions that the procedure \( B(t), [t_0, T] \) (PDP) leading to dynamic stable cooperative solution exist and realizable (Petrosjan, Zenkevich, 1996).
Fig. 1. Dynamically Stable Cooperative Decision (PDP)
1.4. Payoff Distribution Procedure (PDP)

A payoff distribution procedure (PDP) proposed by (Petrosyan, 1997) will be formulated so that the agreed upon dynamically stable imputations can be realized. Let the payoff player \( i \) receives over the time interval \([t_0, t]\) on the optimal trajectory be expressed as:

\[
\phi_i[x_0(\cdot), T-t_0; x^*(\cdot), t-t_0] = \int_{t_0}^{t} B_i(s) ds,
\]

where

\[
\sum_{j \in N} B_j(s) = \sum_{j \in N} g^j[s, x^*(s), u^*(s)], \text{ for } t_0 \leq s \leq t \leq T
\]

Therefore

\[
B_i(t) = - \frac{d\eta_i}{dt}, \quad \text{or} \quad \frac{d\phi_i}{dt} = B_i(t)
\]

This quantity may be interpreted as the instantaneous payoff of the player \( i \) at the moment \( t \). Hence it is clear the vector \( B(t) = [B_1(t), B_2(t), \ldots, B_n(t)] \) prescribes distribution of the total gain among the members of the coalition \( N \). By properly choosing \( B(t) \), the players can ensure the desirable outcome that at each instant \( t \in [t_0, T] \) there will be no objection against realization of the original agreement (the imputation \( \xi(x_0, T-t_0) \)) as shown on Fig. 1, i.e. the imputation \( \xi(x_0, T-t_0) \) is dynamically stable.
1.5. Regularized game

For every $\alpha \in L(x_0, T-t_0)$ define the noncooperative game $\Gamma_\alpha(x_0, T-t_0)$ which differs from the game $\Gamma(x_0, T-t_0)$ only by payoffs defined along optimal cooperative trajectory $\bar{x}(\tau)$, $\tau \in [t_0, T]$.

Let $\alpha \in L(x_0, T-t_0)$. Define the imputation distribution procedure (IDP) (see Petrosjan (1993)) as function $\beta(\tau) = \{\beta_1(\tau), \ldots, \beta_n(\tau)\}$, $\tau \in [t_0, T]$ such that

$$\alpha_i = \int_0^T \beta_i(\tau) d\tau,$$

(5)

Denote by $H_i^\alpha(x_0, T-t_0; u_1(\cdot), \ldots, u_n(\cdot))$ in the game $\Gamma_\alpha(x_0, T-t_0)$ and by $x(\tau)$ the corresponding trajectory, then

$$H_i^\alpha(x_0, T-t_0; u_1(\cdot), \ldots, u_n(\cdot)) = H_i(x_0, T-t_0; u_1(\cdot), \ldots, u_n(\cdot))$$

If there does not exist $t \in [t_0, T]$ such that $x(\tau) = \bar{x}(\tau)$ for $\tau \in (t_0, t]$. Let

$t = \sup\{t' : x(\tau) = \bar{x}(\tau), \tau \in [t_0, t']\}$ and $t > t_0$ then

$$H_i^\alpha(x_0, T-t_0; u_1(\cdot), \ldots, u_n(\cdot)) =$$

$$= \int_0^t \beta_i(\tau) d\tau + H_i(\bar{x}, T-t; u_1(\cdot), \ldots, u_n(\cdot)) =$$

$$= \int_0^t \beta_i(\tau) d\tau + \int_t^T h_i(x(\tau)) d\tau.$$

In a special case, when $x(\tau) = \bar{x}(\tau)$, $\tau \in [t_0, T]$ (if $x(\tau)$ is an optimal cooperative trajectory in the sense of Eq. (2), we have:

DEGIT-XVI, St. Petersburg, Russia, September 9th, 2011
By the definition of payoff function in the game $\Gamma_\alpha(x_0,T-t_0)$ we get that the payoffs along the optimal trajectory are equal to the components of the imputation $\alpha=(\alpha_1,\ldots,\alpha_i,\ldots,\alpha_n)$.

Consider the current subgames (see Neumann and Morgenstern (1947)) - $\Gamma(\bar{x}(t),T-t)$ along $\bar{x}(t)$ and current imputation sets $L(\bar{x}(t),T-t)$. Let $\alpha(t)\in L(\bar{x}(t),T-t)$. Suppose that $\alpha(t)$ can be selected as differentiable function of $t$, $t\in[t_0,T]

**Definition 1.** The game $\Gamma_\alpha(x_0,T-t_0)$ is called regularization of the game $\Gamma(x_0,T-t_0)$ ($\alpha$-regularization) if IDP $\beta$ is defined in such a way that

$$\alpha_i(t) = \int_{t_0}^T \beta_i(\tau) d\tau$$

or

$$\beta_i(t) = -\alpha_i'(t).$$  \tag{6}$$

From (6) we get

$$\alpha_i = \int_0^T \beta_i(\tau) d\tau + \alpha_i(t),$$  \tag{7}$$

where $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_n)\in L(x_0,T-t_0)$, and $\alpha(t)=(\alpha_1(t),\alpha_2(t),\ldots,\alpha_n(t))\in L(\bar{x}(t),T-t)$.
1.6. Strategic Stability Theorem 1

Consider now the problem of strategic stability of cooperative agreements. Based on imputation distribution procedure $\beta$, satisfying (5) we can prove the following basic theorem.

**Theorem 1.** In the regularization of the game $\Gamma_\alpha(x_0, T-t_0)$ for every $\varepsilon > 0$ there exist an $\varepsilon$-Nash Equilibrium (Nash(1951)) with payoffs $\alpha = (\alpha_1, \ldots, \alpha_i, \ldots, \alpha_n)$.

Proof. The proof is based on actual construction of the $\varepsilon$-Nash equilibrium in piecewise open-loop (POL) strategies with memory.
Suppose now that
\[ W(x_0, T-t_0) \subseteq L(x_0, T-t_0) \]
is some optimality principle in the cooperative version of the game \( \Gamma(x_0, T-t_0) \) and
\[ M(\bar{x}(t), T-t) \subseteq L(\bar{x}(t), T-t) \]
is the same optimality principle defined in the subgames \( \Gamma(\bar{x}(t), T-t) \) with initial coalitions on the optimal trajectory.

\( M \) can be \( c \)-core, HM-solution, Shapley value, Nucleous etc.
If \( \alpha \in M(x_0, T-t_0) \), and \( \alpha(t) \in M(\bar{x}(t), T-t) \) the condition (7) gives us the time consistence of the chosen imputation \( \alpha \), or the chosen optimality principle.

Then we have the time consistency (dynamic stability) of the chosen cooperative agreement.
1.7. Strategic Stability

\[ \Gamma(x_0, T - t_0); \quad \bar{x}(t) = (\bar{x}_1(t), \ldots, \bar{x}_i(t), \ldots, \bar{x}_n(t)) \]

\[ Sh(x_0) = (Sh_1(x_0), \ldots, Sh_i(x_0), \ldots, Sh_n(x_0)) \]

\[ \Gamma(\bar{x}(\tau), T - \tau); \quad \{N - k\} \rightarrow k; \]

\[ \Gamma_{k,N\backslash k}(\bar{x}(\tau), T - \tau); \quad \nu_k(\bar{x}(\tau), T - \tau) \]

\[ \int_{\tau}^{T} \beta_k(t) dt \geq \nu_k(\bar{x}(\tau), T - \tau); \quad Sh_i(\bar{x}(\tau)) = \int_{\tau}^{T} \beta_i(t) dt \geq \nu_i(\bar{x}(\tau), T - \tau) \]
2. Strategically Stable Technological Alliance

2.1. Technological Alliance Cooperative Game

Consider a technological alliance in which involved \( n \) firms (\( i \in \mathbb{N} \)).

The state dynamics of the firm \( i \) is characterized by the vector-valued differential equations:

\[
\dot{x}_i(s) = f_i^i[s, x_i(s), u_i(s)], \quad x_i(t_0) = x_i^0, \quad i \in \mathbb{N},
\]

where \( x_i \in X_i \subset \mathbb{R}^{n_i} \) denotes the state variables of player \( i \), \( u_i \in U^i \) is the control vector of firm \( i \).

The state of firm \( i \) includes its capital stock, level of technology, special skills and productive resources.

The objective of firm \( i \) is:

\[
\int_{t_0}^{T} g_i^i[s, x_i(s), u_i(s)] \exp \left[ -\int_{t_0}^{s} r(y) dy \right] ds + \exp \left[ -\int_{t_0}^{T} r(y) dy \right] q_i^i(x_i(T)),
\]

where \( \exp \left[ -\int_{t_0}^{s} r(y) dy \right] \) is the discount factor, \( g_i^i[s, x_i(s), u_i(s)] \) the instantaneous profit, and the \( q_i^i(x_i(T)) \) terminal payment.

In particular, \( g_i^i[s, x_i(s), u_i(s)] \) and \( q_i^i(x_i(T)) \) are positively related to the level of technology \( x_i \).
Consider a coalition $K \subseteq N$. The participating firms can gain core skills and technology that would be very difficult for them to obtain on their own, and hence the state dynamics of firm $i$ in the coalition $K$ becomes

$$
\dot{x}_i(s) = f^K_i s, x_K(s), u_i(s) \ , \ x_i(t_0) = x_i^0 \quad i \in K
$$

(9)

where $x_K(s)$ is the concatenation of the vectors $x_j(s)$ for $j \in K$.

In particular, $\frac{\partial f^K_i [s, x_i, u_K]}{\partial u_j} \geq 0$, for $j \neq i$ (synergy effect). Thus positive effects on the state of firm $i$ could be derived from the technology of other firms within the coalition.

At time $t_0$, the profit of the coalition $K \subseteq N$:

$$
W^{(t_0)K}(t_0, x_K^0) = \max \int \sum_{j \in K} g^j \left[ s, x_j(s), u_j(s) \right] \exp \left[ - \int_{t_0}^{s} r(y) \, dy \right] ds + \sum_{j \in K} \exp \left[ - \int_{t_0}^{T} r(y) \, dy \right] q^j (x_j(T))
$$

(10)
To compute the profit of the coalition $K$ we have to consider the optimal control problem $\mathcal{O}[K; t_0, x_K^0]$ which maximizes (10) subject to (9).

For notational convenience, we express (9) as:

$$\dot{x}_K(s) = f^K s, x_K(s), u_K(s), \quad x_K(t_0) = x_K^0$$

$(11)$

Where $u_K$ is the set of $u_j$ for $u_j$, $j \in K$; $f^K t, x_K, u_K$ is a column vector containing $f^K_j t, x_K, u_K$ for $j \in K$.

The dynamics of the optimal state trajectory of the grand coalition (technological alliance) can be obtained as:

$$\dot{x}_j(s) = f_j^N \left[ s, x_N(s), \psi_j^{(t_0)N_*}(s, x_N(s)) \right], \quad x_j(t_0) = x_j^0, \quad j \in N$$

which can also be expressed as

$$\dot{x}_N(s) = f^N \left[ s, x_N(s), \psi^{(t_0)N_*}_N(s, x_N(s)) \right], \quad x_N(t_0) = x_N^0$$

$(12)$

Let $x_N^*(t) = \left[ x_1^*(t), x_2^*(t), \ldots, x_n^*(t) \right]$ denote the solution to (12).
2.2. Dynamic Shapley Value

Consider Shapley value as imputation rule for sharing the cooperative profit among the members in a coalition as:

$$\Phi^i_v = \sum_{k \subseteq N} \frac{(k-1)!}{n!} \frac{n-k!}{n!} v(K) - v(K \setminus i), \quad i \in N$$  \hspace{1cm} (13)

where $K \setminus i$ is the relative complement of $i$ in $K$,

$v(K)$ is a profit of the coalition $K$,

$v(K) - v(K \setminus i)$ is the marginal contribution of the firm $i$ to the coalition $K$.

To maximize the joint venture’s profits the firms would adopt the control vector $\psi^{(t_0)N^*}_{t_0}(t, x^*_N)_{t_{t_0}}$ over the time interval $t_0, T$, and the corresponding optimal state trajectory $x^*_N(t)_{t_{t_0}}$ in (5) would result.

At time $t_0$ with the state $x^*_N$, the firms agree that firm $i$’s share of profits be:

$$v^{(t_0)}(t_0, x^*_N) = \sum_{k \subseteq N} \frac{(k-1)!}{n!} \frac{n-k!}{n!} \left[ W^{(t_0)K}_{t_0}(t_0, x^*_N) - W^{(t_0)K_{\cup i}}_{t_0}(t_0, x^*_N) \right]$$ \hspace{1cm} (14)

However, the Shapley value has to be maintained throughout the venture horizon $t_0, T$.

In particular, at time $\tau \in t_0, T$ with the state being $x^*_N$, the following imputation principle has to be maintained:

$$v^{(\tau)}(\tau, x^*_N) = \sum_{k \subseteq N} \frac{(k-1)!}{n!} \frac{n-k!}{n!} \left[ W^{(\tau)K}_{t_0}(\tau, x^*_N) - W^{(\tau)K_{\cup i}}_{\tau}(\tau, x^*_N) \right]$$ \hspace{1cm} (15)

where $i \in N$ and $\tau \in t_0, T$.
2.3. Dynamic Stable Shapley Value

Let $B_i(t)$ denote the PDP procedure, where

$$
\psi^{(t_0)}(t_0, x_N^0) = \sum_{k \in N} \frac{(k-1)!}{n!} \left[ W^{(t_0)}(t_0, x_K^0) - W^{(t_0)}(t_0, x_K^0) \right] = \\
= \int_{t_0}^{t} B_i(s) \exp \left[ - \int_{y}^{s} r(y) \, dy \right] ds + q_i(t_0) \exp \left[ - \int_{t_0}^{t} r(y) \, dy \right] 
$$

The following formula describes the rule $B_i(\tau)$ for distribution Shapley value in the time, providing time consistency of Shapley value.

$$
B_i(\tau) = \sum_{k \in N} \frac{(k-1)!}{n!} \left[ W_i^{(\tau)}(t, x_K^0) |_{t=\tau} \right] - \left[ W_{\bar{i}}^{(\tau)}(t, x_K^0) |_{t=\tau} \right] \\
+ \sum_{j \in K} \left[ W_{x_j}^{(\tau)}(t, x_K^0) |_{t=\tau} \right] f_j^N \left[ \tau, x_N^0, \psi_j^{(\tau)}(\tau, x_N^0) \right] - \sum_{h \in K \cup} \left[ W_{x_h}^{(\tau)}(\tau, x_K^0) |_{t=\tau} \right] f_h^N \left[ \tau, x_N^0, \psi_h^{(\tau)}(\tau, x_N^0) \right] \\
= \sum_{k \in N} \frac{(k-1)!}{n!} \left[ W_i^{(\tau)}(t, x_K^0) |_{t=\tau} \right] - \left[ W_{\bar{i}}^{(\tau)}(t, x_K^0) |_{t=\tau} \right] \\
+ \left[ W_{x_i}^{(\tau)}(t, x_K^0) |_{t=\tau} \right] f_i^N \left[ \tau, x_N^0, \psi_i^{(\tau)}(\tau, x_N^0) \right] - \left[ W_{x_i}^{(\tau)}(\tau, x_K^0) |_{t=\tau} \right] f_i^N \left[ \tau, x_N^0, \psi_i^{(\tau)}(\tau, x_N^0) \right] (17)
$$

where $f_i^N \left[ \tau, x_N^0, \psi_i^{(\tau)}(\tau, x_N^0) \right]$ is a column vector containing $f_i^N \left[ \tau, x_N^0, \psi_i^{(\tau)}(\tau, x_N^0) \right], i \in K$. 

DEGIT-XVI, St. Petersburg, Russia, September 9th, 2011
2.4. Strategic Stability Theorem 2

Consider now the problem of strategic stability of cooperative agreements.

**Theorem 2.** In the regularization of the game $\Gamma_\alpha(x_0, T - t_0)$ for every $\varepsilon > 0$ there exist an $\varepsilon$-Nash Equilibrium with payoffs $\alpha = (\alpha_1, \ldots, \alpha_i, \ldots, \alpha_n)$. 
Literature

• Petrosjan L. A., Zakharov V. V. 1997. Mathematical models in ecology. SPb., Izd-vo SPbSU.
Literature


Literature

Literature

Literature

- Zenkevich N. A. and Petrosyan L.A. 2009. Stable cooperation properties // Large systems control, N 26.1,