Dynamic Optimization Techniques for Analyzing Proportions and Trends of Economic Growth: Optimal Investments in Capital and Labor Efficiency
This paper presents a methodology based on dynamic optimality principles for construction of solutions in economic growth models.

The basic technique is connected with the Pontryagin maximum principle and its generalizations for the problems with infinite horizon.

In the first part of the paper we present the series of economic models whose solutions are obtained with the help of algorithms elaborated within the developed technique:

- Simulation of Optimal Economic Growth
- Optimization of R&D Investment
- Technology Assimilation and Optimization of R&D Intensity
- Optimal Timing of Innovation
- Game Modeling of Energy Project Infrastructure
- Optimization of Investments in Capital and Labor Efficiency
- Optimal Dynamics of Functionality Development
- Optimal Feedbacks in Techno-Economic Dynamics
- Optimization of Trends in Resource Productivity

The second part of the paper we describe in details the economic growth model: Optimization of Investments in Capital and Labor Efficiency
Members:
Robert Ayres, Andrey Krasovskii, Alexander Tarasyev

Scientific institutes and organizations:
IIASA, INSEAD, IMM UrB RAS

Objectives and goals:
The main objectives of the research are to propose a methodology for modeling the long-term optimal economic development; to apply the elaborated approach to analysis of a real data representing aggregated macroeconomic indicators of economies; to present model modifications and case-studies for various economies in order to define the underlying laws driving the economic growth.

Main methods:
• economic growth theory (K. Arrow, F. Ramsey, R. Solow, K. Shell);
• optimal control theory (the maximum principle of L.S. Pontryagin for problems on infinite horizon);
• econometric analysis of the model;
• numerical simulation using elaborated software and forecasting of future scenarios.
Results and further steps:
The methodological scheme is elaborated for constructing synthetic optimal trajectories of economic growth on the basis of real data time series. It is applied to the case of the US and Japan economies using LINEX production function. Computer simulations show that the model adequately reflects the growth trends and dynamics of proportions of economic variables on the data for the US economy and can be used as an instrument for generating credible scenarios of optimal economic growth and investments. Theoretical results for the optimal control problem arising in the model are obtained. They provide sufficient conditions of optimality for the constructed solutions and estimate the numerical algorithm. Future steps are connected with application of the approach to multifactor model including models with exhausting energy resources. Further case studies for the economies of the UK, Austria and other countries will be performed.
Publications:

1. **Krasovskii, A.A.**

2. **A.A. Krasovskii and A.M. Tarasyev**
   Conjugation of Hamiltonian Systems in Optimal Control Problems. Proceedings of the 17th World Congress.

3. **Ayres, R., Krasovskii, A.A., Tarasyev, A.M.**
   Nonlinear Stabilizers of Economic Growth under Exhausting Energy Resources.
Optimization of R&D Investment

Members:
Chihiro Watanabe, Alexander Tarasyev

Scientific institutes and organizations:
IIASA, Tokyo Seitoku University, IMM UrB RAS

Objectives and goals:
The research is devoted to analysis of a nonlinear model of economic growth which involves production, technology stock, and their rates as the main variables. Two trends (growth and decline) in the interaction between the production and R&D investment are examined in the balanced dynamics. The optimal control problem of R&D investment is studied for the balanced dynamics and the utility function with the discounted consumption. The growth properties of the production rate, R&D, and technology intensities are examined on the generated trajectories.

Main methods:
• endogenous economic growth theory (K. Arrow, Grossman, Helpman);
• optimal control problems with infinite horizon;
• econometric analysis of the model;
• construction of model synthetic trajectories and their matching with the real data.
Results and further steps:

The constructed dynamical model of optimal economic growth is used for the comparison of catalogs of real econometric data and synthetic growth scenarios. The model is calibrated on a database of the Tokyo Institute of Technology. Special attention is paid to the aggregated data of the Japanese manufacturing industry in the period 1960–92. The next steps could deal with case studies for with particular economy’s sectors.
Optimization of R&D Investment

Publications:

1. Tarasyev, A. M., and Watanabe, C.
   *Optimal Dynamics of Innovation in Models of Economic Growth.*

   *Optimal Trajectories of the Innovation Process and Their Matching with Econometric Data.*
Members:
Bernadette Ane, Chihiro Watanabe, Alexander Tarasyev

Scientific institutes and organizations:
IIASA, Tokyo Seitoku University, IMM UrB RAS

Objectives and goals:
The research deals with a dynamic model of optimization of R&D intensity under the effect of technology assimilation. The model involves R&D investments, technology stock, production, and technology productivity as main variables. The technology stock is constructed as a function of indigenous and exogenous technology stocks and their growth rates. The research focuses on the issue of a reasonable balance between the indigenous technology stock and assimilated technology flow. Econometric linearization of the technology assimilation effect is used to construct a reasonable optimal control model.

Main methods:
• economic growth theory;
• Hamiltonian systems in optimal control problems;
• generalized solutions of Hamilton-Jacobi equations;
• econometric analysis of the model.
Technology Assimilation and Optimization of R&D Intensity

Results and further steps:

The existence of the value function for the problem of the optimal economic growth on the infinite horizon is proved and the basic features of the value function are outlined. The property of strong invariance for the main proportions of the model such as technology productivity and R&D intensity is proved. The model is calibrated on the aggregate data of the Japanese automotive industry. Trends of optimal R&D intensity are examined depending on the values of the model macroeconomic parameters and the feedback variables. The implemented analysis shows that additional investments and restructuring of these sources for knowledge absorption could have the effect of increasing returns and provide a strong leverage for reaching qualitatively higher levels of sales, technology development, and consumption index.

\[ T = T_d + zT_s \]

Dynamics of technology stock in the Japanese automobile industry for two scenarios:
1. under technology assimilation,
2. on the basis only indigenous technology
Technology Assimilation and Optimization of R&D Intensity

Dynamics of sales of Japanese automobile companies for two scenarios:
(1) under technology assimilation,
(2) on the basis only indigenous technology

Publications:

1. Ane, B.K., Tarasyev, A.M., Watanabe, C.,
   Construction of Nonlinear Stabilizer for Trajectories of Economic Growth.

2. Ane, B.K., Tarasyev, A.M., Watanabe, C.
Optimal Timing of Innovation

Members: C. Watanabe, A.A. Krasovskii, A.M. Tarasyev

Scientific institutes and organizations: IIASA, Tokyo Seiitsu University, IMM UrB RAS

Objectives and goals: The research is aimed at the analysis of a dynamic model of optimal timing and its application to the problem of optimization of an innovation process in a competitive market environment. The basic element of the model is a block for optimization of the stopping time of the investment process. The problem of optimal timing is solved in parallel with adjoint problems of assessment of the market potential innovation and optimal control synthesis of the investment process. For construction of the optimal control synthesis the basic elements of models of optimal economic growth are used. In the econometric block, a probabilistic model is applied for design of market trajectories described by distribution functions. These distribution functions determine a market share of commercialized projects and probability of presence of competitors on the market at the current moment of time. Stochastic models for specification of the price formation mechanism are constructed for a series of distribution functions.

Main methods:
• differential games;
• generalized solutions of Hamilton-Jacobi equations;
• optimal stopping time;
• econometric analysis of the model.
Results and further steps:

An algorithm for construction of the optimal commercialization time and optimal investment plan is proposed. The algorithm is based on qualitative analysis of extreme points of the utility function which correspond to intersection points of a market distribution function and marginal costs of the innovation process. The algorithms are realized in the program software for simulation of optimal investment plans. Econometric analysis of the model and identification of its parameters is implemented on the basis of the data provided by Tokyo Institute of Technology. In particular, the innovation process is studied for technology development of Canon laser printers.

Model trajectories for two scenarios of commercialization (fast and slow) and the real data on technology investments of the Canon Company.
Publications:

1. **Tarasyev, A.M., Watanabe, C.**  
   *Dynamic Optimality Principles and Sensitivity Analysis in Models of Economic Growth.*  

2. **Krasovskii A.A., Tarasyev A.M.**  
   *Problems of Optimal Timing Control*  

3. **Krasovskii, A.A., Tarasyev, A.M.**  
   *An algorithm for construction of optimal timing solutions in problems with a stochastic payoff function.*  
Game Modeling of Energy Project Infrastructure

Members:
Ger Klaassen, Arkady Kryazhimskiy, Alexander Tarasyev

Scientific institutes and organizations:
IIASA, Tokyo Seitoku University, IMM UrB RAS

Objectives and goals:
The purpose of the research is to study an optimal infrastructure of a system of international gas pipelines competing for a gas market. A game-dynamic model is suggested for description of the operation of several interacting gas pipeline projects treated as players in the game. The model treats the projects' commercialization times as major players' controls. Current quantities of gas supply are modeled as approximations to Nash equilibrium points in the instantaneous "gas supply games", in which each player maximizes his current benefit due to the sales of gas.

Main methods:
- differential games;
- Hamiltonian systems of optimal control problems;
- optimal stopping time;
- econometric analysis of the model.
Results and further steps:

An algorithm is proposed for numerical searching Nash equilibrium commercialization policies for the entire group of the pipelines. The model is used to analyze the Caspian gas market, on which gas routes originating from Russia and Turkmenistan are competing. The simulations show the degrees, to which the planned regimes are not optimal compared to the Nash equilibrium ones. Another observation is that in equilibrium regimes the pipelines are not always being run at their full capacities, which implies that the proposed pipeline capacities might not be optimal. The simulation results turn out to be moderately sensitive to changes in the discount rate and highly sensitive to changes in the price elasticity of gas demand.
Table 1
Cost estimations and capacities for major projects.

<table>
<thead>
<tr>
<th>Pipelines</th>
<th>Gas capacity (bcm)</th>
<th>Length (miles)</th>
<th>Investments (low estimate, mln US $)</th>
<th>Investments (high estimate, mln US $)</th>
<th>Operation &amp; maintenance costs US $/bcm</th>
<th>Trans. fees US $/bcm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Transcasian</td>
<td>3.15</td>
<td>1200</td>
<td>2000</td>
<td>3000</td>
<td>0</td>
<td>16.0</td>
</tr>
<tr>
<td>Blue stream</td>
<td>14.16</td>
<td>1220</td>
<td>4000</td>
<td>6000</td>
<td>14.1</td>
<td>0</td>
</tr>
</tbody>
</table>

Fig. 1. Gas deliveries of competitive pipelines.

Fig. 2. Net present values of gas suppliers in the game scenario.
Game Modeling of Energy Project Infrastructure

Publications:

1. **Klaassen, G., Kryazhimskii, A.V., Tarasyev, A.M.**
   *Multi equilibrium game of timing and competition of gas pipeline projects*

2. **Krasovskii, A.A., Matrosov, I.V., Tarasyev, A.M.**
   *Optimal timing control in game modeling of an energy project infrastructure.*
Optimization of Investments in Capital and Labor Efficiency

Members:
Warren Sanderson, Alexander Tarasyev, Anastasia Usova

Scientific institutes and organizations:
IIASA, IMM UrB RAS

Objectives and goals:
The research is devoted to construction of optimal trajectories in the model which balances growth trends of investments in capital and labor efficiency. The model is based on three production factors: capital, educated labor and useful work. It is assumed that capital and educated labor are invested endogenously, and useful work is an exogenous flow. The level of GDP is described by an exponential production function of the Cobb-Douglas type. The utility function of the growth process is given by an integral consumption index discounted on the infinite horizon. The optimal control problem is posed to balance investments in capital and labor efficiency. The problem is solved on the basis of dynamic programming principles.

Main methods:
• economic growth theory (K. Arrow, F. Ramsey, R. Solow, K. Shell);
• optimal control theory for problems on infinite horizon;
• econometric analysis of the model;
• simulation using elaborated software and forecasting of future scenarios.
Optimization of Investments in Capital and Labor Efficiency

Results and further steps:
A series of Hamiltonian systems is examined including analysis of steady states, properties of trajectories and their growth rates. A novelty of the solution consists in constructing nonlinear stabilizers based on the feedback principle which lead the system from any current position to an equilibrium steady state. Growth and decline trends of the model trajectories are studied for all components of the system and their proportions including: dynamics of GDP, consumption, capital, labor efficiency, investments in capital and labor efficiency. The model parameters are calibrated on the basis of the macroeconomic data of the US economy. Computer experiments demonstrate quite realistic trends of the model trajectories which have a tendency of saturation around the steady state constructed in per capita variables.

Publications:

1. Sanderson, W., Tarasyev, A.M., Usova, A.A.
   *Capital vs. Education: Assessment of Economic Growth from Two Perspectives.*

2. Tarasyev, A.M., Usova, A.A.
   *Nonlinear Stabilizer Constructing for Two-Sector Economic Growth Model.*
Optimization of Investments in Capital and Labor Efficiency

Simulated trajectory of two-sector model, \( K(t) \)

Simulated trajectory of one-sector model, \( K(t) \)

Statistic data, \( K(t) \)

Simulated trajectory in two-sector model, \( Y(t) \)

Simulated trajectory in one-sector model, \( Y(t) \)

Statistic data, \( Y(t) \)

Growth trajectories for capital in one and two sectors model and real data.

Growth trajectories for labor in one and two sectors model and real data.
Optimal Dynamics of Functionality Development

Members:
Chihiro Watanabe, Jae-Ho Shin, Alexander Tarasyev

Scientific institutes and organizations:
IIASA, Tokyo Seitoku University, IMM UrB RAS

Objectives and goals:
The research deals with the economic growth model in which company’s indicator of functionality development stands for the basic optimization parameter. Dynamics of the output and functionality development factor are subject to differential equations describing both exponential and logistic growth trends. In the research various regimes are investigated for control of the functionality development indicator, including, constant control, control keeping a constant level of functionality development, control based on power, exponential and logistic functions, in order to identify possible production trends. For the dynamical system an optimal control problem is analyzed for maximization of the integral index of logarithmic utility. A problem is considered using the Pontryagin maximum principle for control problems on infinite horizon.

Main methods:
• logistic growth dynamics;
• optimal control theory for problems on infinite horizon;
• econometric analysis of the model;
• simulation using elaborated software and forecasting of future scenarios.
Results and further steps:
Analysis of properties of the steady states, optimal trajectories and stabilizing feedback strategies is fulfilled in the framework of the elaborated method of investigation of Hamiltonian systems arising in the maximum principle. The model is calibrated on econometrical data on two generations of mobile phones produced in Japan. Comparison of model and empirical growth trends of production is presented.

Publications:
   Optimization of Functionality Development.  
   doi:10.1016/j.amc.2010.03.069
2. Watanabe, C., Shin, J., Heikkinen, J., Tarasyev, A.  
   Optimal dynamics of functionality development in open innovation.  
   IFAC-PapersOnLine – The International Federation on Automatic Control, Conference Paper  
   Archive: Proceedings of the IFAC Workshop on Control Applications of Optimization,  
   Identifier: 10.3182/20090506-3-US-00032  
   Posted online: 03-29-2010.  
   http://www.ifac-papersonline.net/Detailed/41903.html
Dynamics of production in the case of constant level of functionality development.

Comparison of the model trajectory synthesized by the suboptimal stabilizer with trends of the real data.
Optimal Feedbacks in Techno-Economic Dynamics

Members:
Chihiro Watanabe, Bing Zhu, Alexander Tarasyev

Scientific institutes and organizations:
IIASA, Tokyo Seitoku University, Tsinghua University, IMM UrB RAS

Objectives and goals:
The objective of this work is to design control strategies which optimize composition of production, technology stock and their rates in a nonlinear model of economic growth. The optimal control problem of R&D investment is formulated for the balanced dynamics which binds production and technology and the discounted utility function which correlates the amount of sales and production diversity. The maximum principle of Pontryagin is applied for designing optimal nonlinear dynamics.

Main methods:
• optimal control theory for problems on infinite horizon, construction of feedback control strategies;
• econometric analysis of the model;
• sensitivity analysis.
**Results and further steps:**
Optimality principles are interlinked with equilibrium properties of the Hamiltonian system. Quasi-optimal feedbacks of the rational type for balancing the dynamical system are constructed. Properties of quasi-optimal feedbacks and techno-economic trajectories are indicated for different slopes of R&D intensities. These properties correspond to the trends of economic growth intrinsic to the real econometric data and are illustrated by statistical analysis. On the basis of the theoretical analyses, empirical analyses are attempted to demonstrate a practical significance of this approach. By utilizing the developed approach, evaluations of R&D intensity in major Japanese manufacturing sectors are conducted by comparing optimal and actual levels identifying “pseudo innovation” in high-technology and its sources.

**Publications:**

1. **Tarasyev, A.M., Watanabe, C.**

2. **Tarasyev, A.M., Watanabe, C., Zhu, B.**
   *Optimal Feedbacks in Techno-Economical Dynamics.*
Optimal Feedbacks in Techno-Economic Dynamics

Optimization of Trends in Resource Productivity

Members:
Bing Zhu, Alexander Tarasyev

Scientific institutes and organizations:
IIASA, Tsinghua University, IMM UrB RAS

Objectives and goals:
In this research, a dynamic optimization model of investment in improvement of the resource productivity index is analyzed for obtaining balanced economic growth trends including both the consumption index and natural resources use. The research is closely connected with the problem of shortages of natural resources stocks, the security of supply of energy and materials, and the environmental effectiveness of their consumption. The main idea of the model is to introduce an integrated environment for elaboration of a control policy for management of the investment process in development of basic production factors such as capital, energy and material consumption. An essential feature of the model is providing the possibility to invest in economy’s dematerialization. Another important construction is connected with the price formation mechanism which presumes the rapid growth of prices on exhausting materials. The balance is formed in the consumption index which negatively depends on growing prices on materials. The optimal control problem for the investment process is posed and solved within the Pontryagin maximum principle.
Main methods:

- optimal control theory for problems on infinite horizon, construction of feedback control strategies;
- qualitative analysis of Hamiltonian trajectories;
- generalized solutions of Hamilton-Jacobi equations;
- econometric analysis of the model.

Results and further steps:

The growth and decline trends of the Hamiltonian trajectories are examined for the optimal solution. It is proved that for specific range of the model parameters there exists the unique steady state of the Hamiltonian system. The steady state can be interpreted as the optimal steady trajectory along which investments in improving resource productivity provide raising resource efficiency and balancing this trend with growth of the consumption index. The comparison analysis is implemented for optimal model trends and historical trends of real econometric data. As a result of system analysis and modeling, one can elaborate investment strategies in economy’s dematerialization, resource and environmental management for improving the resource productivity index and, consequently, for shifting the economic system from non-optimal paths to the trajectory of sustainable development.
Optimization of Trends in Resource Productivity

Publications:

1. **Tarasyev, A.M., Watanabe, C., Zhu, B.**
   "Optimal Feedbacks in Techno-Economical Dynamics."

2. **Optimization of Resource Productivity for Sustainable Economic Development.**
   Research Proposal on the International Joint NSFC Project between Tsinghua University and IIASA.
Economic Growth Model is based on three production factors: capital, educated labor (human capital) and useful work.

Capital and Educated Labor are invested endogenously, and Useful Work is an exogenous flow.

The level of GDP is represented by a production function.

The utility function of the growth process is given by an Integral Consumption Index discounted on the infinite horizon.

The optimal control problem is posed to balance investments in capital and labor efficiency and solved on the basis of dynamic programming principles.

A novelty of the solution consists in constructing nonlinear stabilizers based on the feedback principle which leads the system from any current position to an equilibrium steady state.

Growth and decline trends of the model trajectories are studied for all components of the system and their proportions including: dynamics of GDP, consumption, capital, labor efficiency, investments in capital and labor efficiency.
Two-sector economic Growth Model

Consumption $C(t)$

GDP $Y(t)$

Investments $S(t)$ in Capital stock, $S(t) = s(t)Y(t)$

Investments $R(t)$ in Human capital, $R(t) = r(t)Y(t)$

Production Function $F(K, L, U)$

Capital Stock $K(t)$

Useful work $U(t)$

Human Capital $L(t)$
Production Factors: Capital Stock, $K(t)$

Solow Model:

$$\dot{K}(t) = S(t) - \delta K(t) = s(t)Y(t) - \delta K(t), \quad K_0 = K(t_0)$$

$\delta$ – positive capital depreciation rate.

Control variable:

$s(t)$ – investments share in Capital $K(t)$ satisfying restrictions

$$0 \leq s(t) \leq a_s < 1$$

$S(t)$ – investments in Capital $K(t)$

$a_s$ - upper bound of the capital investment.
Production Factors: Human capital, $L(t)$

Production Function:
$$F[K(t), L(t), U(t)]$$

GDP:
$$Y(t)$$

Investments:
$$bR(t)$$

Human capital changes:
$$\dot{L}(t)$$

Sanderson’s Model:
$$L(t) = E(t)P(t)$$

$E(t)$ - Labor Efficiency

$P(t)$ - Labor Force that is proportional to Population size and has exponential dynamics with the positive growth rate $\rho$: 
$$\dot{P}(t) = \rho P(t), \quad P_0 = P(t_0)$$

Human Capital Dynamics:

is proportional to investments in the Labor Efficiency:
$$\dot{L}(t) = bR(t) = br(t)Y(t), \quad L_0 = L(t_0)$$

$b$ – marginal effectiveness of investments in Human Capital $L(t)$

Control variable:

$r(t)$ – investments share in Human Capital $L(t)$ satisfying restrictions
$$0 \leq r(t) \leq a_r < 1$$

$R(t)$ – investments in Human Capital $L(t)$

$a_r$ – upper bound of human capital investment.
Exogenous Production Factor: Useful work, $U(t)$

Logistic dynamics of Useful Work:

$$\dot{U}(t) = aU(t) \left(1 - \frac{U(t)}{B}\right), \quad U_0 = U(t_0)$$

- $a$ – maximal changes of useful work, $a > 0$
- $B$ – useful work capacity, $B > 0$
Production Function, $Y(t)$

Gross Domestic Product (GDP) $Y(t)$ at time $t$ depends on three Production Factors: Capital Stock $K(t)$, Human Capital $L(t)$ and Useful Work $U(t)$:

$$Y(t) = F(K(t), L(t), U(t))$$

Function $F(\cdot)$ is the Production Function which is homogeneous with the unitary degree of homogeneity:

$$F(\nu K, \nu L, \nu U) = \nu F(K, L, U), \quad \forall \nu \geq 0$$

Cobb-Douglas production function

$$F(K, L, U) = AK^\alpha L^\beta U^{1-\alpha-\beta}$$

with nonnegative elasticity coefficients

$$\alpha \geq 0, \quad \beta \geq 0, \quad 1 - \alpha - \beta \geq 0$$

Positive scaled factor $A$ is the total factor productivity.
Balance Equation and Consumption, $C(t)$

We consider the closed economic system in which GDP $Y(t)$ can be spent on investments $S(t)$, $R(t)$ and consumption $C(t)$ only:

$$ Y(t) = S(t) + R(t) + C(t) \quad \text{– balance equation} $$

Investments should not exceed the GDP level at time $t$:

$$ 0 \leq S(t) + R(t) = (s(t) + r(t))Y(t) < Y(t) $$

$$ 0 \leq s(t) + r(t) < 1 $$

It is assumed that the following balance relations take place:

$$ 0 \leq s(t) \leq a_s < 1 \quad \text{and} \quad 0 \leq r(t) \leq a_r < 1 \quad \text{and} \quad 0 \leq a_s + a_r < 1 $$

Consumption $C(t)$ is defined by the relation:

$$ C(t) = Y(t) - S(t) - R(t) = (1 - s(t) - r(t))Y(t) $$

It is supposed that the product $s(t)r(t)$ is much smaller than $s(t)$ and $r(t)$ and can be neglected, and consumption $C(t)$ is found by formula:

$$ C(t) \approx (1 - s(t))(1 - r(t))Y(t) $$
Per capita variables and production function

Let us introduce per capita variables normalizing GDP $Y$, consumption $C$, capital $K$, educated labor force $L$, and useful work $U$, with respect to labor force $P$:

$$y = \frac{Y}{P}, \quad k = \frac{K}{P}, \quad l = \frac{L}{P}, \quad u = \frac{U}{P}, \quad c = \frac{C}{P}$$

The per capita production function $y = F(k, l, u)$ due to the property of homogeneous of degree one is represented as

$$y = \frac{Y}{P} = \frac{F(K, L, U)}{P} = F\left(\frac{K}{P}, \frac{L}{P}, \frac{U}{P}\right) = F(k, l, u)$$

Dynamics of per capita variables

$$\dot{k}(t) = s(t)F(k(t), l(t), u(t)) - (\delta + \rho)k(t), \quad k(t_0) = \frac{K_0}{P_0} = k_0$$

$$\dot{l}(t) = br(t)F(k(t), l(t), u(t)) - \rho l(t), \quad l(t_0) = \frac{L_0}{P_0} = l_0$$

$$\dot{u}(t) = u(t)\left(a - \rho - a \frac{U(t)}{B}\right), \quad u(t_0) = \frac{U_0}{P_0} = u_0$$
Per capita useful work, $u(t)$

The per capita useful work has the following dynamic:

$$\dot{u}(t) = u(t) \left( a - \rho - a \frac{U(t)}{B} \right), \quad u(t_0) = \frac{U_0}{P_0} = u_0$$

In the current version of the model it is supposed

1. The capacity level $B$ of the useful work is high enough and ratio $\frac{U(t)}{B}$ can be neglected
2. The maximal growth rate of the useful work $a$ is close to the growth rate of the labor force $\rho$

Due to these assumptions $\dot{u}(t) \approx 0$. Hence, the per capita useful work $u(t)$ is constant and is equal to its average level $\hat{u}$.

The production function with the constant level $\hat{u}$ of the per capita useful work:

$$y(t) = F(k(t), l(t), u(t)) \approx F(k(t), l(t), \hat{u}) = f(k(t), l(t))$$
Utility function, $J$

The utility function $J(t_0; k_0, l_0)$ is represented as the integrated logarithmic consumption index discounted on the infinite time horizon

$$J(t_0; k_0, l_0) = \int_{t_0}^{+\infty} e^{-\lambda t} \ln c(t) \, dt$$

The per capita consumption $c(t) = (1 - s(t))(1 - r(t))f(k(t), l(t))$

Parameter $\lambda$ is the discount rate.

Substitution of the per capita consumption $c(t)$ into the utility function provides:

$$J(t_0; k_0, l_0) = \int_{t_0}^{+\infty} e^{-\lambda t} \left( \ln(1 - s(t)) + \ln(1 - r(t)) + \ln f(k(t), l(t)) \right) dt$$
Optimal control problem, CP

The optimal control problem (CP) is to maximize the utility function

$$J(t_0; k_0, l_0) = \int_{t_0}^{+\infty} e^{-\lambda t} \left( \ln(1 - s(t)) + \ln(1 - r(t)) + \ln f(k(t), l(t)) \right) dt$$

over trajectory $k(\cdot), l(\cdot), s(\cdot), r(\cdot)$ of the system

$$\begin{cases} 
\dot{k}(t) = s(t)f(k(t), l(t)) - (\delta + \rho)k(t), \\
\dot{l}(t) = br(t)f(k(t), l(t)) - \rho l(t)
\end{cases}$$

with control parameters $(s(\cdot), r(\cdot))$ subject to constraints

$$0 \leq s(t) \leq a_s < 1 \text{ and } 0 \leq r(t) \leq a_r < 1 \text{ and } 0 \leq a_s + a_r < 1$$

and phase variables $(k(\cdot), l(\cdot))$ satisfying initial conditions

$$k(t_0) = k_0, \quad l(t_0) = l_0$$

Production function $f(k, l)$ for all positive values of phase variables $k$ and $l$

1. is positive $f(k, l) > 0$,
2. has positive first derivatives: $\frac{\partial f(k,l)}{\partial k} > 0, \quad \frac{\partial f(k,l)}{\partial l} > 0$,
3. is strongly concave: $\frac{\partial^2 f(k,l)}{\partial k^2} < 0, \quad \frac{\partial^2 f(k,l)}{\partial l^2} < 0$
The Hamiltonian function corresponds to the optimal control problem (CP):
\[
\tilde{H}(t; s, r; k, l; \tilde{\psi}_1, \tilde{\psi}_2) = e^{-\lambda t}(\ln(1-s) + \ln(1-r) + \ln f(k, l)) + \\
\tilde{\psi}_1(sf(k, l) - (\delta + \rho)k) + \tilde{\psi}_2(brf(k, l) - \rho l)
\]

Excluding the time term by the change of variables
\[
\psi_1 = e^{\lambda t} \tilde{\psi}_1, \quad \psi_2 = e^{\lambda t} \tilde{\psi}_2, \quad \tilde{H} = e^{\lambda t} \tilde{H}
\]

The Hamiltonian function looks as follows
\[
\tilde{H}(s, r; k, l; \psi_1, \psi_2) = \ln(1-s) + \ln(1-r) + \ln f(k, l) + \\
\psi_1(sf(k, l) - (\delta + \rho)k) + \psi_2(brf(k, l) - \rho l)
\]

**Property of the Hamiltonian function \( \tilde{H} \)**

The Hamiltonian function \( \tilde{H}(\cdot) \) is strictly concave in control variables \( s \) and \( r \).
**Theorem (by S.M. Aseev, A.V. Kryazhimskiy, 2007)**

Let \((s^0, r^0)\) be an optimal process. Then there exist adjoint variables \(\tilde{\psi}_1\) and \(\tilde{\psi}_2\) corresponding to the process \((s^0, r^0; k^0, l^0)\) and satisfying to adjoint equations:

\[
\dot{\tilde{\psi}}_1 = -\frac{\partial \tilde{H}(t; k^0, l^0; s^0, r^0; \tilde{\psi}_1, \tilde{\psi}_2)}{\partial k}, \quad \dot{\tilde{\psi}}_2 = -\frac{\partial \tilde{H}(t; k^0, l^0; s^0, r^0; \tilde{\psi}_1, \tilde{\psi}_2)}{\partial l}
\]

such that the optimal process \((s^0, r^0; k^0, l^0)\) and adjoint variables \(\tilde{\psi}_1, \tilde{\psi}_2\)

1. **satisfy the conditions of the Pontryagin maximum principle**

   \[
   \tilde{H}(t; k^0, l^0; s^0, r^0; \tilde{\psi}_1, \tilde{\psi}_2) = \max_{s \in [0; \alpha_s]} \max_{r \in [0; \alpha_r]} \tilde{H}(t; k^0, l^0; s, r; \tilde{\psi}_1, \tilde{\psi}_2)
   \]

2. **meet the stationarity condition**

   \[
   \tilde{H}(t; k^0, l^0; s^0, r^0; \tilde{\psi}_1, \tilde{\psi}_2) = \lambda \int_t^{+\infty} e^{-\lambda \tau} \ln \left( (1 - s^0(\tau))(1 - r^0(\tau)) f(k^0(\tau), l^0(\tau)) \right) d\tau
   \]

3. **for all** \(t \geq t_0\) **adjoint variables are positive** \(\tilde{\psi}_1(t) > 0\) **and** \(\tilde{\psi}_2(t) > 0\)

4. **satisfy the transversality condition**

   \[
   \lim_{t \to \infty} \left( \tilde{\psi}_1(t) k^0(t) + \tilde{\psi}_2(t) l^0(t) \right) = 0
   \]
Optimal controls, \((s^0, r^0)\)

Control variables \((s, r)\) maximizing the Hamiltonian function \(\widehat{H}(s, r; k, l; \psi_1, \psi_2)\) have the following structures:

\[
\begin{align*}
  s^0 &= \begin{cases} 
    s_1 = 0, & (k, l; \psi_1) \in \Delta^1_s \\
    s_2 = 1 - \frac{1}{\psi_1 f(k, l)}, & (k, l; \psi_1) \in \Delta^2_s \\
    s_3 = a_s, & (k, l; \psi_1) \in \Delta^3_s
  \end{cases} \\
  r^0 &= \begin{cases} 
    r_1 = 0, & (k, l; \psi_2) \in \Delta^1_r \\
    r_2 = 1 - \frac{1}{b \psi_2 f(k, l)}, & (k, l; \psi_2) \in \Delta^2_r \\
    r_3 = a_r, & (k, l; \psi_2) \in \Delta^3_r
  \end{cases}
\end{align*}
\]

where domains \(\Delta^i_s\) and \(\Delta^j_r\) \((i, j = 1, 2, 3)\) are defined by inequalities:

\[
\begin{align*}
  \Delta^1_s &= \{(k, l, \psi_1): \psi_1 f(k, l) \leq 1\}, \\
  \Delta^2_s &= \{(k, l, \psi_1): 1 \leq \psi_1 f(k, l) \leq \frac{1}{1-a_s}\}, \\
  \Delta^3_s &= \{(k, l, \psi_1): \psi_1 f(k, l) \geq \frac{1}{1-a_s}\}, \\
  \Delta^1_r &= \{(k, l, \psi_2): \psi_2 f(k, l) \leq \frac{1}{b}\}, \\
  \Delta^2_r &= \{(k, l, \psi_2): \frac{1}{b} \leq \psi_2 f(k, l) \leq \frac{1}{b(1-a_r)}\}, \\
  \Delta^3_r &= \{(k, l, \psi_2): \psi_2 f(k, l) \geq \frac{1}{b(1-a_r)}\}.
\end{align*}
\]
Maximized Hamiltonian function, \( H(k, l; \psi_1, \psi_2) \)

The maximized Hamiltonian function \( H(k, l; \psi_1, \psi_2) \) is defined by the following relation:

\[
H(k, l; \psi_1, \psi_2) = \max_{s \in [0, a_s], r \in [0, a_r]} \hat{H}(s, r; k, l; \psi_1, \psi_2) = \hat{H}(s^0, r^0; k, l; \psi_1, \psi_2)
\]

has 9 nine domains of definition:

\[
D_{ij} = \{(k, l; \psi_1, \psi_2) : (k, l; \psi_1, \psi_2) \in \Delta^i_s \cap \Delta^j_r\}, \quad i, j = 1, 2, 3
\]

Branches \( H_{ij} \) of the maximized Hamiltonian function \( H \) in the domain \( D_{ij} \) are descried as

\[
H_{ij} = H_{ij}(k, l; \psi_1, \psi_2) = \left\{ H(k, l; \psi_1, \psi_2) : (k, l; \psi_1, \psi_2) \in D_{ij} \right\}, \quad i, j = 1, 2, 3
\]

The pair of controls \((s_i, r_j)\) acts in the domain \( D_{ij} \), \( i, j = 1, 2, 3 \)
Sufficient optimality conditions

Smoothness of the maximized Hamiltonian function $H$

The maximized Hamiltonian function $H(k, l; \psi_1, \psi_2)$ is smoothly pasted out of branches $H_{ij}(k, l; \psi_1, \psi_2)$ in all variables on borders of domains $D_{ij}, i,j = 1,2,3$.

Concavity of the maximized Hamiltonian function $H$

The maximized Hamiltonian function $H(k, l; \psi_1, \psi_2)$ is a strictly concave function in phase variables $k, l$ for all positive values of conjugate variables $\psi_1, \psi_2$ if the following conditions are satisfied:

$$\left(\frac{\partial f(k, l)}{\partial k}\right)^2 + f(k, l) \frac{\partial^2 f(k, l)}{\partial k^2} < 0,$$

$$\left(\frac{\partial f(k, l)}{\partial l}\right)^2 + f(k, l) \frac{\partial^2 f(k, l)}{\partial l^2} < 0$$

Theorem (by A.A. Krasovskii, A.M. Tarasyev, 2008)

If the maximized Hamiltonian $H(k, l; \psi_1, \psi_2)$ has properties of smoothness in variables $k, l, \psi_1, \psi_2$ and strictly concavity in phase variables $k, l$ for all positive values of adjoint variables $\psi_1, \psi_2$, then the Pontryagin maximum principle ensures sufficient optimality conditions in the considered optimal control problem.
Qualitative analysis. Hamiltonian system

The Hamiltonian system:

\[
\begin{align*}
\dot{k} &= \frac{\partial H(k, l; \psi_1, \psi_2)}{\partial \psi_1} \\
\dot{\psi}_1 &= \lambda \psi_1 - \frac{\partial H(k, l; \psi_1, \psi_2)}{\partial k} \\
\dot{\psi}_2 &= \lambda \psi_2 - \frac{\partial H(k, l; \psi_1, \psi_2)}{\partial l} \\
\end{align*}
\]

The change of adjoint variables: \( z_1 = k\psi_1 \) and \( z_2 = l\psi_2 \)

\[
\begin{align*}
\dot{z}_1 &= \dot{k}\psi_1 + k\dot{\psi}_1 \\
\dot{z}_2 &= \dot{l}\psi_2 + l\dot{\psi}_2 \\
\end{align*}
\]
Domain $D_{22}$ with transient control regime

Domain $D_{22}$ is described by the following inequalities:

$$k \leq z_1 f(k, l) \leq \frac{k}{1 - a_s}, \quad \frac{l}{b} \leq z_2 f(k, l) \leq \frac{l}{1 - a_r}$$

Optimal controls $(s^0, r^0)$ in the domain $D_{22}$ are equal to

$$s^0 = s_2 = 1 - \frac{k}{z_1 f(k, l)}, \quad r^0 = r_2 = 1 - \frac{l}{b z_2 f(k, l)}$$

The maximized Hamiltonian function in the domain $D_{22}$ has the form:

$$H_{22} = - \ln \left( \frac{b z_1 z_2 f(k, l)}{k l} \right) + \left( \frac{z_1}{k} + \frac{b z_2}{l} \right) f(k, l) - (\delta + \rho) z_1 - \rho z_2 - 2$$

The Hamiltonian system in domain $D_{22}$ looks like:

$$\dot{k} = f(k, l) - (\delta + \rho) k - \frac{k}{z_1} = F_1(k, l; z_1, z_2)$$

$$\dot{l} = b f(k, l) - \rho l - \frac{l}{z_2} = F_2(k, l; z_1, z_2)$$

$$\dot{z}_1 = \left( \lambda - \frac{\partial f(k, l)}{\partial k} + \frac{f(k, l)}{k} \right) z_1 - b \frac{k}{l} \frac{\partial f(k, l)}{\partial k} z_2 + \frac{k}{f(k, l)} \frac{\partial f(k, l)}{\partial k} - 1 = F_3(k, l; z_1, z_2)$$

$$\dot{z}_2 = -\frac{l}{k} \frac{\partial f(k, l)}{\partial l} z_1 + \left( \lambda - b \frac{\partial f(k, l)}{\partial l} + b \frac{f(k, l)}{l} \right) z_2 + \frac{l}{f(k, l)} \frac{\partial f(k, l)}{\partial l} - 1 = F_4(k, l; z_1, z_2)$$
Steady state

The Cobb-Douglas production function: \( f(k, l) = \mu k^\alpha l^\beta \)

Auxiliary variables: \( a_k = \frac{f(k, l)}{k} \) and \( a_l = b \frac{f(k, l)}{l} \)

Steady state coordinates:

\[
    a_k^* = \frac{(\lambda + (1 - \beta)\rho)(\delta + \lambda + \rho)}{\alpha(\lambda + \rho)}, \quad \quad a_l^* = \frac{(\lambda + (1 - \alpha)(\delta + \rho))(\lambda + \rho)}{\beta(\delta + \lambda + \rho)}
\]

\[
    k^* = \left( \frac{\mu}{a_k^*} \left( \frac{b a_k^*}{a_l^*} \right)^{\beta} \right)^{\frac{1}{1-\alpha-\beta}}, \quad \quad l^* = \left( \frac{\mu}{a_k^*} \left( \frac{b a_k^*}{a_l^*} \right)^{1-\alpha} \right)^{\frac{1}{1-\alpha-\beta}}
\]

\[
    z_1^* = \frac{1}{a_k^* - \delta - \rho}, \quad \quad z_2^* = \frac{1}{a_l^* - \rho}
\]

Conditions

The steady state belongs to the domain \( D_{22} \) if

1. The maximized Hamiltonian function is strongly concave, i.e. \( 0 < \alpha < 0.5, 0 < \beta < 0.5 \)
2. Maximum levels of investments \( a_s \) and \( a_r \) meet restrictions:

\[
    a_s \geq \frac{\alpha(\delta + \rho)(\lambda + \rho)}{(\lambda + (1 - \beta)\rho)(\delta + \lambda + \rho)}; \quad a_r \geq \frac{\beta(\delta + \lambda + \rho)\rho}{(\lambda + (1 - \alpha)(\delta + \rho))(\lambda + \rho)}
\]
Jacoby matrix at the steady state

Columns $I_j$ ($j = 1, ..., 4$) of the Jacobi matrix $I$ at the steady state $(k^*, l^*; z_1^*, z_2^*)$ is evaluated by formulas:

$$I_1 = \left\{ \frac{\partial F_i(k^*, l^*; z_1^*, z_2^*)}{\partial k} \right\}_{i=1}^4,$$
$$I_2 = \left\{ \frac{\partial F_i(k^*, l^*; z_1^*, z_2^*)}{\partial l} \right\}_{i=1}^4,$$
$$I_3 = \left\{ \frac{\partial F_i(k^*, l^*; z_1^*, z_2^*)}{\partial z_1} \right\}_{i=1}^4,$$
$$I_4 = \left\{ \frac{\partial F_i(k^*, l^*; z_1^*, z_2^*)}{\partial z_2} \right\}_{i=1}^4.$$

Coefficients of the Jacobi matrix $I = \{I_{ij}\}_{i,j=1}^4$ at the steady state $(k^*, l^*; z_1^*, z_2^*)$ are presented analytically:

$$
\begin{pmatrix}
-(1-\alpha)a_k^* & \frac{\beta a_i^*}{b} & k^*(a_k^* - \delta - \rho)^2 & 0 \\
\alpha b a_k^* & -(1-\beta)a_l^* & 0 & l^*(a_i^* - \rho)^2 \\
-(1-\alpha)^2a_k^* & \alpha^2a_l & \lambda + (1-\alpha)a_k^* & -\alpha a_i^* \\
-\frac{(1-\alpha)^2a_k^*}{k^*(a_k^* - \delta - \rho)} + \frac{\alpha^2a_l}{k^*(a_l^* - \rho)} & \frac{(1-\alpha)\beta a_k^*}{l^*(a_k^* - \delta - \rho)} + \frac{\alpha(1-\beta)a_l}{l^*(a_l^* - \rho)} & -\beta a_k^* & \lambda + (1-\beta)a_i^* \\
\end{pmatrix}
$$
**Requirements**

The nonlinear stabilizer has to:
1. exclude adjoint variables from the Hamiltonian system
2. preserve the steady state
3. stabilize the Hamiltonian system at the steady state.

**Assumptions**

1. The Jacobi matrix $I$ has four different real eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$: two eigenvalues $\lambda_1, \lambda_2$ are negative and two others $\lambda_3, \lambda_4$ are positive, i.e. $\lambda_1 < \lambda_2 < 0 < \lambda_3, < \lambda_4$. Eigenvector $h_i = \{h_{ij}\}_{j=1}^4$ corresponds to eigenvalue $\lambda_i$ ($i = 1, ..., 4$), i.e. $I h_i = \lambda_i h_i$.
2. Coordinates of eigenvectors $h_1, h_2$ corresponding to negative eigenvalues $\lambda_1, \lambda_2$ satisfy the condition:

\[ h_{11} h_{22} \neq h_{12} h_{21} \]
Algorithm for constructing the nonlinear stabilizer

Step 1. Eigenplane constructing

To construct the plane $\pi(h_1, h_2)$ basing on two eigenvectors $h_1$ and $h_2$ corresponding to negative eigenvalues $\lambda_1, \lambda_2$:

\[ k - k^* = \mu_1 h_{11} + \mu_2 h_{21}, \quad z_1 - z_1^* = \mu_1 h_{13} + \mu_2 h_{23}, \]
\[ l - l^* = \mu_1 h_{12} + \mu_2 h_{22}, \quad z_2 - z_2^* = \mu_1 h_{14} + \mu_2 h_{24}. \]

Step 2. Adjoint variables deriving

To derive adjoint variables $z_1 = z_1(k, l)$ and $z_2 = z_2(k, l)$ from eigenplane $\pi(h_1, h_2)$ equations:

\[ z_1(k, l) = z_1^* + \gamma_{11}(k - k^*) + \gamma_{12}(l - l^*), \quad z_2(k, l) = z_2^* + \gamma_{21}(k - k^*) + \gamma_{22}(l - l^*). \]

Step 3. Stabilized Hamiltonian system

To substitute representation of adjoint variables $z_1(k, l)$ and $z_2(k, l)$ to the first two equations of the Hamiltonian system:

\[ \dot{k} = f(k, l) - (\delta + \rho)k - \frac{k}{z_1(k, l)}, \quad \dot{l} = b f(k, l) - \rho l - \frac{l}{z_2(k, l)} \]

Step 4. Nonlinear stabilizer

To substitute values of adjoint variables $z_1(k, l)$ and $z_2(k, l)$ into the optimal control relations:

\[ \hat{s}(k, l) = 1 - \frac{k}{z_1(k, l)f(k, l)}, \quad \hat{r}(k, l) = 1 - \frac{l}{b z_2(k, l)f(k, l)} \]
Properties of the nonlinear stabilizer

**Theorem** *(by A.M. Tarasyev, A.A. Usova, 2010)*

Let the Hamiltonian system corresponding to the domain $D_{22}$ is linearized in the steady state $(k^*, l^*, z_1^*, z_2^*)$ neighborhood. The Jacobi matrix $I$ satisfies conditions
1. It has four different real eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$: two eigenvalues $\lambda_1, \lambda_2$ are negative and two others $\lambda_3, \lambda_4$ are positive, namely $\lambda_1 < \lambda_2 < 0 < \lambda_3 < \lambda_4$.
2. Coordinates of eigenvectors $h_1, h_2$ corresponding to negative eigenvalues $\lambda_1, \lambda_2$ meet the restriction:

   $$h_{11} h_{22} \neq h_{12} h_{21}$$

Then there exists the nonlinear stabilizer

$$\hat{s}(k, l) = 1 - \frac{k}{z_1(k, l)f(k, l)} , \quad \hat{r}(k, l) = 1 - \frac{l}{b z_2(k, l)f(k, l)}$$

generating the system which has the following properties:
1. It is a closed system with respect to the phase variables $(k, l)$.
2. It has the unique steady state with coordinates $(k^*, l^*)$.
3. It is stabilized at the steady state, i.e. the Jacobi matrix evaluated at the steady state has two negative eigenvalues coinciding with $\lambda_1, \lambda_2$.
4. It has reduced eigenvectors $\hat{h}_1 = (h_{11}, h_{12})$ and $\hat{h}_2 = (h_{21}, h_{22})$ with respect to the original system.
Preliminary conclusions

1. The solution of the system generated by the nonlinear stabilizer can be considered as the first approximation of optimal trajectories and used for preliminary assessment of growth trends.

2. It can be considered as the basis of the algorithm for constructing optimal trajectories. The lack of initial data for variables \( z_1, z_2 \) impedes the solution of the original Hamiltonian system.

3. According to the results of the qualitative theory of differential equations [Ph. Hartman. Ordinary Differential Equations, 1964] the trajectory of the nonlinear Hamiltonian dynamics converges to the steady state \( (k^*, l^*, z_1^*, z_2^*) \) tangentially to the plane generated by eigenvectors \( h_1, h_2 \) corresponding to negative eigenvalues \( \lambda_1, \lambda_2 \) of the Jacobi matrix.

4. Approximate values of the initial position \( (k^T, l^T, z_1^T, z_2^T) \) for the backward time integration of the Hamiltonian system can be taken from the vicinity of points situated on trajectories of the stabilized Hamiltonian system. If the computed solution comes to the original initial point \( (k_0, l_0) \) then this trajectory is the required optimal solution.
Nonlinear stabilizer and optimal solutions

There exist algorithms for solution of similar control problems

based on results of the qualitative theory of differential equations [see Ph. Hartman. Ordinary Differential Equations, 1964], that is

the trajectory of the nonlinear Hamiltonian dynamics converges to the steady state tangentially to the plane $\pi$ generated by eigenvectors corresponding to negative eigenvalues of the Jacobi matrix evaluated at the equilibrium point.

These approaches use the backward time integration of the Hamiltonian system with an initial position located on the plane $\pi$ in the steady state neighborhood.

Application of the nonlinear stabilizer provides essential advantages:
- The trajectories generated by the stabilizer have similar properties as optimal trajectories locally at the steady state including rates and directions of convergence.
- These properties allow to use the nonlinear stabilizer for localizing the search of an optimal trajectory.
The algorithm of optimal trajectories constructing

Step 1. Nonlinear stabilizer

To get the stabilized Hamiltonian system basing on the algorithm for construction of the nonlinear stabilizer

\[
\dot{k} = f(k, l) - (\delta + \rho)k - \frac{k}{z_1(k, l)}, \quad \dot{l} = bf(k, l) - \rho l - \frac{l}{z_2(k, l)},
\]

\[k(t_0) = k_0, \quad l(t_0) = l_0\]

Step 2. Solution of the stabilized system

To find solution of the stabilized system: \(k(t)\) and \(l(t)\)
The algorithm of optimal trajectories constructing

Step 3. Preparation to the backward time integration

The computed solution located in the plane $\pi$ comes to the steady state. This step implies
1. to take the small $\varepsilon^*$ neighborhood of the equilibrium point $(k^*, l^*, z_{1e}^*, z_{2e}^*)$
2. to find the point $(k_{e}^*, l_{e}^*, z_{1e}^*, z_{2e}^*)$ of intersection of the trajectory generated by the nonlinear stabilizer with $\varepsilon^*$ neighborhood of the steady state. Since this point is located on the plane $\pi$ then coordinates of adjoint variables are given by formulas:

$$z_{1e}^* = z_{1}^* + \gamma_{11}(k_{e}^* - k^*) + \gamma_{12}(l_{e}^* - l^*), \quad z_{2e}^* = z_{2}^* + \gamma_{21}(k_{e}^* - k^*) + \gamma_{22}(l_{e}^* - l^*)$$

3. to chose the small $\varepsilon$ vicinity of the point $(k_{e}^*, l_{e}^*, z_{1e}^*, z_{2e}^*)$ on the plane $\pi$.
4. to select initial positions $(k_{e}^*, l_{e}^*, z_{1e}^*, z_{2e}^*)$ for the backward time integration of the Hamiltonian system from the $\varepsilon$ neighborhood of the point $(k^*, l^*, z_{1e}^*, z_{2e}^*)$, i.e. $(k_{e}^*, l_{e}^*, z_{1e}^*, z_{2e}^*) \in \partial O_{\varepsilon}((k^*, l^*, z_{1e}^*, z_{2e}^*))$
**Step 4. Backward time integration of the Hamiltonian system**

This step deals with solution of the original Hamiltonian system in the backward time. We solve the original Hamiltonian system in the reverse time starting from the selected position \((k_ε, l_ε, z_{1ε}, z_{2ε})\) located on the plane \(π\). Integration is performed until one of three alternatives happens:

1. If the integrated trajectory reaches the initial point \((k_0, l_0)\) in the domain \(D_{22}\) then the algorithm is stopped and the trajectory is assumed to be constructed.

2. If the integrated trajectory crosses one of sewing surfaces between adjoining domains \(D_{22}\), \(D_{12}\), or \(D_{23}\), or \(D_{32}\), before it reaches the initial point \((k_0, l_0)\), then the Hamiltonian system is switched to the corresponding Hamiltonian system defined for a new domain.

3. If the integrated trajectory does not achieve the original initial point or boundaries between domains, then we exhaustively search for another initial point \((k_ε, l_ε, z_{1ε}, z_{2ε})\) according to the following scheme:
   3.1. we search over points from the fixed \(ε\) neighborhood of the position \((k_ε^*, l_ε^*, z_{1ε}^*, z_{2ε}^*)\) by varying the angle in the plane.
   3.2. if the solution can not be found in the angle varying procedure then we change the value of radius \(ε\) and return back to the beginning of the fourth step.
Estimation of the algorithm accuracy, $\Delta$. Conclusions

**Theorem (by A.M. Tarasyev, A.A. Usova, 2011)**

Estimation of the algorithm accuracy $\Delta$ by the utility functional depends on the precision $\varepsilon$ of the approximation of initial conditions in the algorithm. Let symbol $\nu$ denote the Lipschitzian module of the Hamiltonian system and parameter $\lambda$ be the discount rate of the model, then the algorithm accuracy has the following order:

$$\Delta = \Delta_{\varepsilon}(\nu, \lambda) = \begin{cases} 
\varepsilon^2, & \nu < \lambda \\
\varepsilon^2 \ln \left( \frac{1}{\varepsilon^2} \right), & \nu = \lambda \\
(\varepsilon^2)^\lambda \nu \ln \left( \frac{1}{\varepsilon^2} \right), & \nu > \lambda 
\end{cases}$$

**Conclusions**

The algorithm approximates the unique optimal trajectory converging to the steady state, since this algorithm is constructed within the framework of the Pontryagin maximum principle for optimal control problems with infinite horizon. It means that the optimal trajectory

(1) exists,

(2) satisfies the maximum principle including the transversality conditions at the infinite time,

(3) is unique due to the property of strict concavity of the Hamiltonian function in phase variables.

These results combined with the exhaustive search in a neighborhood of the equilibrium point within the eigenplane provide the fact of definite search of the optimal solution.
Numerical experiments. Data calibration

Numerical experiments are implemented for the US economic data in the period since 1900 to 2004.

- Production function parameters:
  - Total factor of productivity, $\mu = 2.19$
  - Elasticity coefficients, $\alpha = 0.3$, $\beta = 0.1$

- Utility functional parameters:
  - Discount rate, $\lambda = 0.03$

- Maximal investments levels:
  - For capital stock investments $s$, $a_s = 0.3$
  - For human capital investments $r$, $a_r = 0.2$

- Other model parameters:
  - Labor growth rate, $\rho = 0.013$
  - Capital depreciation level, $\delta = 0.2$
  - Marginal effectiveness of investments in labor efficiency, $b = 0.31$

- Initial position:
  - Per capita capital, $k_0 = 1$
  - Labor efficiency, $l_0 = 1$
Numeric experiments. Results of qualitative model analysis

The qualitative model analysis provides the following results:

- Steady state coordinates, \((k^*, l^*, z_1^*, z_2^*) = (5.75; 5.2; 1.758; 3.822)\)
- The Jacobi matrix evaluated at the steady state:
  \[
  J = \begin{pmatrix}
  -0.54744 & 0.10031 & 1.86023 & 0.00000 \\
  0.07459 & -0.24305 & 0.00000 & 0.35605 \\
  -0.13362 & 0.07506 & 0.57744 & -0.08248 \\
  0.06788 & -0.16146 & -0.09072 & 0.27305 \\
  \end{pmatrix}
  \]
- Eigenvalues: \((\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (-0.268; -0.094; 0.124; 0.298)\)
  It means that the steady state has the saddle character. Eigenvectors \(h_1, h_2\) corresponding to negative eigenvalues \(\lambda_1, \lambda_2\) meet restrictions: \(h_{11} h_{22} \neq h_{12} h_{21}\).
- Equations for the eigenplane \(\pi\):
  \[
  z_1(k, l) = 1.7581295719 + 0.1476485627 (k - k^*) - 0.0626475369 (l - l^*) \\
  z_2(k, l) = 3.8215908104 - 0.0566551638 (k - k^*) + 0.4329057207 (l - l^*)
  \]
- Values of the optimal control at the steady state: \(s^* = 0.2795, r^* = 0.0379\)
Stabilized and optimal solutions of per capita capital, $k(t)$
Stabilized and optimal solutions of labor efficiency, $l(t)$
Phase trajectories of stabilized and optimal solutions, $k(l)$

Tangent to the phase trajectory of the optimal solution, $k(l)$

Tangent to the phase trajectory of the stabilized solution, $k(l)$

Phase trajectory of the optimal solution, $k(l)$

Phase trajectory of the stabilized solution, $k(l)$
Simulated trajectory of the capital stock and statistic data, $K(t)$
Simulated trajectory of the human capital and statistic data, $L(t)$
Optimal investments share in capital stock, $s(t)$
Optimal investments share in human capital, $r(t)$
Ratios of investments

Relation of investments share in capital $s(t)$ to investments share in human capital $r(t)$

Relation of investments share in human capital $r(t)$ to investments share in capital $s(t)$
Summary on numerical experiments

1. Numerical experiments show that the initial point \((k_\varepsilon, l_\varepsilon, z_{1\varepsilon}, z_{2\varepsilon})\) for the backward system integration is located very close to trajectories of the stabilized system. The radius of the vicinity of the point \((k^*_\varepsilon, l^*_\varepsilon, z^*_{1\varepsilon}, z^*_{2\varepsilon})\) taken from the solution of the stabilized Hamiltonian system is about \(1.1 \cdot 10^{-5}\). One can see that the stabilized system solution and the original one are very close to each other in the neighborhood of the steady state. They have the similar tangential behavior with respect to the eigenplane.

2. Experiments demonstrate that the synthetic trajectory adequately reflects trends of the real data and can be used for forecasting future scenarios of growth.

3. Optimal trajectories of per capita variables \(k(t)\) and \(l(t)\) have S-shape graphs which are provided by restrictions \(a_s\) and \(a_r\) on investments \(s(t)\) and \(r(t)\) respectively. Graphs demonstrate the saturation tendency of growth.
One-sector economic growth model

Consumption \( C(t) \)

GDP \( Y(t) \)

Investments \( S(t) \)
in Capital stock,
\[ S(t) = s(t)Y(t) \]

Production Function \( F(K, L) \)

Capital Stock \( K(t) \)

Labor force \( L(t) \)
Production Function in one-sector economic growth model, $Y(t)$

Gross Domestic Product (GDP) $Y(t)$ at time $t$ depends on two Production Factors: Capital Stock $K(t)$ and Labor Force $L(t)$:

$$Y(t) = F(K(t), L(t))$$

Function $F(\cdot)$ is the Production Function which is homogeneous with the unitary degree of homogeneity:

$$F(\nu K, \nu L, \nu U) = \nu F(K, L), \quad \forall \nu \geq 0$$

Cobb-Douglas production function

$$F(K, L) = \alpha K^\beta L^{1-\beta}$$

with nonnegative elasticity coefficients

$$\beta \geq 0, \quad 1 - \beta \geq 0$$

Positive scaled factor $\alpha$ is the total factor productivity.
Balance Equation and Consumption, $C(t)$

We consider the closed economic system in which GDP $Y(t)$ can be spent on investments $S(t)$ and consumption $C(t)$ only:

$$Y(t) = S(t) + C(t)$$  – balance equation

Investments should not exceed the GDP level at time $t$:

$$0 \leq S(t) = s(t)Y(t) < Y(t)$$
$$0 \leq s(t) < 1$$

It is assumed that the following balance relations take place:

$$0 \leq s(t) \leq \alpha_s < 1$$

Consumption $C(t)$ is defined by the relation:

$$C(t) = Y(t) - S(t) = (1 - s(t))Y(t)$$
Per capita variables and production function

Let us introduce per capita variables normalizing GDP $Y$, consumption $C$, capital $K$ with respect to the labor force $L$:

$$y = \frac{Y}{L}, \quad k = \frac{K}{L}, \quad c = \frac{C}{L}$$

The per capita production function $y = F(k)$ due to the property of homogeneous of degree one is represented as

$$y = \frac{Y}{L} = \frac{F(K, L)}{L} = F\left(\frac{K}{L}, 1\right) = f(k)$$

Dynamics of per capita variables

Solow model: $\dot{K}(t) = s(t)F(K(t), L(t)) - \delta K(t), \quad K(t_0) = K_0$

Labor force dynamics: $\dot{L}(t) = \rho L(t), \quad L(t_0) = L_0$

$\dot{k}(t) = s(t)f(k(t)) - (\delta + \rho)k(t), \quad k(t_0) = \frac{K_0}{L_0} = k_0$
Utility function, $J$

The utility function $J(t_0; k_0)$ is represented as the integrated logarithmic consumption index discounted on the infinite time horizon

$$J(t_0; k_0, ) = \int_{t_0}^{+\infty} e^{-\lambda t} \ln c(t) \, dt$$

The per capita consumption $c(t) = (1 - s(t)) f(k(t))$
Parameter $\lambda$ is the discount rate.

Substitution of the per capita consumption $c(t)$ into the utility function provides:

$$J(t_0; k_0) = \int_{t_0}^{+\infty} e^{-\lambda t} \left( \ln(1 - s(t)) + \ln f(k(t)) \right) dt$$
Optimal control problem for the one-sector model, CP

The optimal control problem (CP) is to maximize the utility function

$$J(t_0; k_0) = \int_{t_0}^{+\infty} e^{-\lambda t} \left( \ln(1 - s(t)) + \ln f(k(t)) \right) dt$$

over trajectory $k(\cdot), s(\cdot)$ of the system

$$\dot{k}(t) = s(t)f(k(t)) - (\delta + \rho)k(t),$$

with the control parameter $s(\cdot)$ subject to the constraint

$$0 \leq s(t) \leq a < 1$$

and phase variables $k(\cdot)$ satisfying the initial condition

$$k(t_0) = k_0$$

The production function $f(k)$ for all positive values of the phase variable $k$

1. is positive $f(k) > 0$,
2. has positive first derivative: $f'(k) = \frac{df(k)}{dk} > 0$,
3. is strongly concave: $f''(k) = \frac{d^2f(k)}{dk^2} < 0$
Comparison of the optimal trajectories in one- and two-sectors models

Simulated trajectory of two-sector model, $K(t)$

Simulated trajectory of one-sector model, $K(t)$

Statistic data, $K(t)$

Capital stock in one-sector model $K(\cdot)$, two-sector model $K(\cdot)$ and statistical data $K(\cdot)$
Comparison of the optimal trajectories in one- and two-sectors models

Gross domestic product in one-sector model $Y(\cdot)$, two-sector model $Y(\cdot)$ and statistical data $Y(\cdot)$
Summary on the comparison of one- and two-sectors models

The human capital $L(t)$ is a very important factor in the economical growth. Investments in increasing of the educational level of citizens have an essential influence on the labor efficiency and, as a consequence, on the economical growth. Numerical experiments have shown that adding factor of the human capital $L(t)$ in the model improves model results.
Dynamic Optimization Techniques for Analyzing Proportions and Trends of Economic Growth:
Optimal Investments in Capital and Labor Efficiency

Thank you for your attention!