Endogenous Saving in a Model of Factor-Eliminating Technical Change

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1. Background

- Perpetual growth requires that the marginal products of reproducible factors of production be bounded away from zero (Jones and Manuelli, 1997).

- Virtually all theoretical studies satisfy this requirement for growth via the augmentation of non-reproducible factors of production.

- Peretto and Seater (2009), however, satisfy the requirement using a different mechanism: factor-eliminating technical progress.
  - Economy develops a technology that uses only reproducible factors of production.

- Any reason to continue pursuing this theory?
    *Factor Shares are not constant!
1. **Background (Continued)**

- **Intuition and Motivation**
  - No reason to believe that technical progress cannot manifest itself as a change in factor intensities/factor shares.
    - Factor Augmenting Technical Progress impacts the effectiveness or productivity of factors of production.
    - Is it not reasonable to think that technology could impact the intensity with which factors of production are used?
  - Provides a purely endogenous explanation for the transition from a primitive to an advanced economy
  - The theory of factor-eliminating technical progress has a superficial resemblance to the growth mechanism that can arise through factor substitution from a CES production function.
    - Boldrin and Levine (2002), Givon (2006), and Zeira (1998, 2006) tell the CES story where capital is substituted for labor. Such substitution only occurs if mankind has been endowed with a sufficiently high elasticity of substitution.
2 Endogenous Saving in a Model of Factor-Eliminating Change

• The Peretto and Seater theory assumes households save a fixed fraction of total income.

• I extend and enrich the theory by incorporating consumer optimization. The extension is analogous to moving from the Solow model to the Cass model.

• **Main Finding:** *All equilibrium paths lead to perpetual growth.*
2.1 Model-Existing Framework

• Composition of Economy
  – Households
  – Competitive Final Good Producers
  – Monopolistically Competitive Intermediate Good Producers

My extension alters the behavior of households
2.1.1 Final Good Producers

- Competitive and produce the final good, \( Y \), according to the technology

\[
Y = \left[ \int_{0}^{1} X_i \frac{\varepsilon-1}{\varepsilon} \, di \right] \frac{\varepsilon}{\varepsilon-1}, \quad \varepsilon > 1
\]

where \( X_i \) is the quantity of intermediate good \( i \) and \( \varepsilon \) is the elasticity of substitution between intermediate goods.

- There is a fixed continuum of intermediate goods ranging from 0 to 1.
- The final good is the numeraire so that \( P_Y \equiv 1 \).
- Denoting \( P_i \) as the price of \( X_i \), maximization of profit

\[
\pi_Y = Y - \int_{0}^{1} P_i X_i \, di
\]

subject to \( Y = \left[ \int_{0}^{1} X_i \frac{\varepsilon-1}{\varepsilon} \, di \right] \frac{\varepsilon}{\varepsilon-1} \) yields the demand function

\[
X_i = YP_i^{-\varepsilon}.
\]
2.1.2 Intermediate Good Producers

Production
- Produces output according to $X_i = AK_i^{q_i} L_i^{1-q_i}$

R&D
- Factor intensity parameter $q_i$ is chosen from a set of known technologies $[0, \alpha_i]$, $0 \leq \alpha_i \leq 1$ where $\alpha_i$ is the technology frontier
- R&D $R$ accumulates knowledge $Z$ according to $\dot{Z}_i = R_i$
- Knowledge is converted into technology via the relation $\alpha_i = h(Z_i)$
- Technology frontier increases in response to R&D according to
  \[
  \dot{\alpha}_i = \begin{cases} 
    f(\alpha_i) \cdot R_i & \alpha_i < 1 \\
    0 & \alpha_i = 1 
  \end{cases}
  \]

  where $f(\alpha_i) = h'[h^{-1}(\alpha_i)]$

Investment in Capital
- Capital accumulates according to $\dot{K}_i = I_i - \delta K_i$
2.1.2 Intermediate Good Producers

(Continued)

Firm’s Optimization Problem

\[ \max_{\{a, P, L, I, R\}} \int_0^\infty \left( P_i X_{it} - w_i L_{it} - I_{it} - R_{it} \right) e^{-\gamma_t} dt \quad \text{s.t.} \quad X_i = YP_i^{-\epsilon} \]

\[ X_i = AK_i^{q_i} L_i^{1-q_i} \]

\[ \dot{\alpha}_i = \begin{cases} f(\alpha_i) \cdot R_i & \alpha_i < 1 \\ 0 & \alpha_i = 1 \end{cases} \]

\[ \dot{K}_i = I_i - \delta K_i \]

\[ I_{it} \geq 0 \]

\[ R_{it} \geq 0 \]

- Useful to think of intermediate firm as operating two divisions: production and investment.
- Solved in two steps:
  1. Production division faces a sequence of independent instantaneous profit maximization problems. Chooses \( q, P, L \) and \( K \).
  2. Investment division then chooses \( I \) and \( R \) subject to the solution presented by the production division.
2.1.2 Intermediate Good Producers
(Continued)

• Interesting twist to production division’s optimization problem.
  – Operates two plants – one using “primitive” technology \( q = 0 \) and one that uses “advanced” technology \( q = \alpha \). A labor allocation constraint arises and firm output can be expressed

\[
X = \begin{cases} 
A[L + m(\alpha)K] & K(1 - \alpha)^{1/\alpha} < L \\
AK^{\alpha} L^{1-\alpha} & K(1 - \alpha)^{1/\alpha} \geq L
\end{cases}
\]

where \( m(\alpha) \equiv \alpha(1 - \alpha)^{1-\alpha} \)

• Important properties of the function \( m(\alpha) \)

\[
m' > 0 \text{ and } m'' > 0 \text{ for all } \alpha \in [0,1] ; \ m(0) = 0 ; \ m(1) = 1 ; \ m'(0) = e^{-1} ; \ m'(1) = +\infty
\]
2.1.2 Intermediate Good Producers
(Continued)

- Optimization results

\[ w = \begin{cases} \frac{1}{Y^\varepsilon X^{-\varepsilon}} \left(1 - \frac{1}{\varepsilon}\right) A (1 - \alpha) \left(\frac{K}{L}\right)^\alpha & \quad K \left(1 - \alpha\right)^{\frac{1}{\alpha}} < L \\ \frac{1}{Y^\varepsilon X^{-\varepsilon}} \left(1 - \frac{1}{\varepsilon}\right) A (1 - \alpha) \left(\frac{K}{L}\right)^\alpha & \quad K \left(1 - \alpha\right)^{\frac{1}{\alpha}} \geq L \end{cases} \]

\[ r_\alpha \equiv \begin{cases} \frac{1}{Y^\varepsilon X^{-\varepsilon}} \left(1 - \frac{1}{\varepsilon}\right) AK m' (\alpha) f (\alpha) & \quad K \left(1 - \alpha\right)^{\frac{1}{\alpha}} < L \\ \frac{1}{Y^\varepsilon X^{-\varepsilon}} \left(1 - \frac{1}{\varepsilon}\right) A L \left(1 + \ln \frac{K}{L}\right) \left(\frac{K}{L}\right)^\alpha f (\alpha) & \quad K \left(1 - \alpha\right)^{\frac{1}{\alpha}} \geq L \end{cases} \]
2.2 Households

• Identical and infinitely lived

• Endowed with a fixed amount of labor $L$, which they supply inelastically in a competitive market.

• Own firms and hold assets in the form of corporate equity and loans

• Loans and corporate equity are assumed to be perfect substitutes as stores of value; thus, the two assets pay the same real rate of return, $r_t$

• Economy is closed, so no assets are traded internationally
2.2 Households (Continued)

- Let $E$ denote total equity held by households.
- Value of the firm, $V$, at time zero is

$$V(0) = \int_{0}^{\infty} D_t e^{-r_t} dt$$

where $D_t = P_t X_t - w_t L_t - I_t - R_t$ is the dividend payment.
- Total value of equity is given by $EV$
- Since mass of intermediate firms is fixed at 1, $E$ is fixed
- Normalize $E$ to 1 so that the value of equity holdings per worker is $\frac{V}{L}$
- Let $b$ represent loans per worker, and let $a$ represent assets per worker.
- It follows that $a \equiv \left( b + \frac{V}{L} \right)$. 
2.2 Households (Continued)

- Assume log utility
- Household’s optimization problem is

\[
\max_{\{c_t\}} \int_0^{\infty} \log c_t e^{-\rho t} \quad s.t. \quad \dot{a}_t = r a_t + w_t - c_t
\]

- Setup yields the well known result:

\[
\frac{\dot{C}}{C} = r - \rho.
\]
2.3 General Equilibrium

- Four Markets: Final Good, Intermediate Good, Labor, and Asset
  - Equilibrium in Intermediate Good Market $\Rightarrow$ $Y = X$
  - Equilibrium in Labor Market
    \[
    w = \begin{cases}
    \left(1 - \frac{1}{\varepsilon}\right)A & K(1 - \alpha)^{\frac{1}{\alpha}} < L \\
    \left(1 - \frac{1}{\varepsilon}\right)A(1 - \alpha)\left(\frac{K}{L}\right)^{\alpha} & K(1 - \alpha)^{\frac{1}{\alpha}} \geq L
    \end{cases}
    \]
  - Equilibrium in Final Goods Market
    \[
    I + R = Y - C
    \]
    \[
    \dot{K} + \delta K + \dot{Z} = \dot{K} + \delta K + \frac{\dot{\alpha}}{f(\alpha)} = \begin{cases}
    A[L + m(\alpha)K] - C & K(1 - \alpha)^{\frac{1}{\alpha}} < L \\
    AK^{\alpha}L^{1-\alpha} - C & K(1 - \alpha)^{\frac{1}{\alpha}} \geq L
    \end{cases}
    \]
2.3 General Equilibrium (Continued)

Equilibrium in Final Goods Market (Continued)

• Saving is allocated between \( I \) and \( R \) based on how \( r_k \) relates to \( r_\alpha \).

\[
r_k \equiv \begin{cases} 
\left(1 - \frac{1}{\varepsilon}\right)Am(\alpha) - \delta & K(1 - \alpha)^{\frac{1}{\alpha}} < L \\
\left(1 - \frac{1}{\varepsilon}\right)\alpha A \left(\frac{K}{L}\right)^{\alpha-1} - \delta & K(1 - \alpha)^{\frac{1}{\alpha}} \geq L
\end{cases}
\]

\[
r_\alpha \equiv \begin{cases} 
\left(1 - \frac{1}{\varepsilon}\right)AKm'(\alpha)f(\alpha) & K(1 - \alpha)^{\frac{1}{\alpha}} < L \\
\left(1 - \frac{1}{\varepsilon}\right)AL \left(1 + \ln \left(\frac{K}{L}\right)\right)^{\alpha} f(\alpha) & K(1 - \alpha)^{\frac{1}{\alpha}} \geq L
\end{cases}
\]

• Three combinations for the \((I, R)\) pair:

\[
\begin{align*}
& r_k > r_\alpha \iff I > 0, R = 0 \\
& r_k = r_\alpha \iff I > 0, R > 0 \\
& r_k < r_\alpha \iff I = 0, R > 0
\end{align*}
\]
2.3 General Equilibrium (Continued)

- Equilibrium in Asset Market
  - If $r_K > r_\alpha$:

\[
\frac{\dot{C}}{C} = \begin{cases}
\left(1 - \frac{1}{\varepsilon}\right)A m(\alpha) - \delta - \rho & K(1 - \alpha)^{\frac{1}{\alpha}} < L \\
\left(1 - \frac{1}{\varepsilon}\right)\alpha A \left(\frac{K}{L}\right)^{\alpha - 1} - \delta - \rho & K(1 - \alpha)^{\frac{1}{\alpha}} \geq L
\end{cases}
\]

- If $r_K < r_\alpha$:

\[
\frac{\dot{C}}{C} = \begin{cases}
\left(1 - \frac{1}{\varepsilon}\right)A K m'(\alpha)f(\alpha) - \rho & K(1 - \alpha)^{\frac{1}{\alpha}} < L \\
\left(1 - \frac{1}{\varepsilon}\right)A L \left(1 + \ln \frac{K}{L}\right)\left(\frac{K}{L}\right)^{\alpha} f(\alpha) - \delta - \rho & K(1 - \alpha)^{\frac{1}{\alpha}} \geq L
\end{cases}
\]
2.4 Dynamic Analysis

• Three loci are required

• Construct a sequence of two dimensional phase diagrams in $(\alpha, K)$ space, each corresponding to a different fixed value of $C$

• Important characteristics of my approach:
  – Posit the functional form $f(\alpha) = 1 - \alpha$ for the productivity of R&D
  – Focus the dynamic analysis on the unconstrained case where
    
    $$K(1 - \alpha)^{\frac{1}{\alpha}} < L$$
2.4.1 Equilibrium Loci

Proposition 1 Arbitrage Locus. The arbitrage locus in \((\alpha, K)\) space is

\[
K = \begin{cases} 
0 & \text{if } 0 \leq \alpha \leq \bar{\alpha} \\
\frac{1}{m'(\alpha)(1-\alpha)} \left[ m(\alpha) - \frac{\delta \varepsilon}{A(\varepsilon - 1)} \right] & \text{if } \bar{\alpha} < \alpha \leq 1
\end{cases}
\]

where, assuming \( \frac{\delta \varepsilon}{A(\varepsilon - 1)} < 1 \), \( \bar{\alpha} \in (0, 1) \) solves \( m(\alpha) = \frac{\delta \varepsilon}{A(\varepsilon - 1)} \). The locus starts at zero and lies on the \( \alpha \)-axis for \( 0 \leq \alpha \leq \bar{\alpha} \). For \( \bar{\alpha} < \alpha \leq 1 \), the locus is positive and increases to \(+\infty\) as \( \alpha \to 1 \).
Arbitrage Locus

\[ r_\alpha = r_K \]
2.4.1 Equilibrium Loci (Continued)

Proposition 2 Stationarity Locus. The stationarity locus in \((\alpha, K)\) space is

\[
K = \frac{AL - C}{\delta - Am(\alpha)}.
\]

Assuming \(\frac{\delta}{A} < 1\), the locus has an asymptote at \(\tilde{\alpha} \in (0,1)\), where \(\tilde{\alpha}\) solves \(\delta = Am(\alpha)\).

The shape and position of the locus in \((\alpha, K)\) space depends on the value of C. There are three possibilities.

i. \(0 \leq C < AL\): For \(\alpha < \tilde{\alpha}\), \(K > 0\), and the locus starts at \(\frac{AL - C}{\delta} > 0\) and goes asymptotically to \(+\infty\) as \(\alpha \to \tilde{\alpha}\). For \(\alpha > \tilde{\alpha}\), \(K < 0\), and the locus starts at \(-\infty\) and rises to \(\frac{AL - C}{\delta - A} < 0\) at \(\alpha = 1\).

ii. \(C = AL\): The \(\alpha\)-axis and the vertical line \(\alpha = \tilde{\alpha}\) form the stationarity locus.

iii. \(C > AL\): For \(\alpha < \tilde{\alpha}\), \(K < 0\), and the locus starts at \(\frac{AL - C}{\delta} < 0\) and goes asymptotically to \(-\infty\) as \(\alpha \to \tilde{\alpha}\). For \(\alpha > \tilde{\alpha}\), \(K > 0\), and the locus starts at \(+\infty\) and falls to \(\frac{AL - C}{\delta - A} > 0\) at \(\alpha = 1\).
2.4.1 Equilibrium Loci (Continued)

Stationarity Locus, $C = 0$

Stationarity Locus, $0 < C < AL$
2.4.1 Equilibrium Loci (Continued)

Stationarity Locus, $C = AL$

Stationarity Locus, $C > AL$
2.4.1 Equilibrium Loci (Continued)

**Proposition 3** \( \dot{C} = 0 \) **Locus.** \( \dot{C} = 0 \) locus in \((\alpha, K)\) space is the vertical line \( \alpha = \alpha^* \) where, assuming \( \rho \leq \left( \frac{\varepsilon - 1}{\varepsilon} \right) A - \delta \), \( \alpha \in (0,1] \) solves

\[
m(\alpha) = \frac{(\rho + \delta)\varepsilon}{A(\varepsilon - 1)}.\]

For \( r_K > r_\alpha \), the \( \dot{C} = 0 \) locus in \((\alpha, K)\) space is

\[
K = \frac{\rho}{\left(1 - \frac{1}{\varepsilon}\right) A m'(\alpha)(1 - \alpha)}.
\]

The locus starts at \( \frac{\rho}{\left(1 - \frac{1}{\varepsilon}\right) Ae^{-1}} > 0 \) for \( \alpha = 0 \) and increases to \( +\infty \) as \( \alpha \to 1 \).
2.4.1 Equilibrium Loci (Continued)

\[ \alpha K_0 \leq \dot{C} \leq \alpha K_r \]

- \( r_\alpha < r_k \) branch of \( \dot{C} = 0 \) Locus
- \( r_\alpha > r_k \) branch of \( \dot{C} = 0 \) Locus
2.4.2 Phase Diagrams

- Diagrams are two dimensional and constructed in \((\alpha, K)\) space, but there are three variables that are evolving.
- Each diagram reveals the laws of motion for \(\alpha, K\) and \(C\) at all possible \((\alpha, K)\) combinations assuming some specific value for the current level of \(C\).
- Assume \(\frac{\delta \varepsilon}{A(\varepsilon - 1)} < 1\) and \(0 < \rho < \frac{A(\varepsilon - 1)}{\varepsilon} - \delta\)
- The relation \(\alpha^* > \bar{\alpha} > \bar{\alpha}\) always holds.
- The two branches of the \(\dot{C} = 0\) locus always intersect each other and the arbitrage locus at \(\alpha^*\).
- Only the stationarity locus changes from one diagram to the next.
2.4.2 Phase Diagrams (Continued)

Figure 3.1.1: Phase Diagram; $C = 0$, $0 < \rho < \frac{A(\varepsilon - 1)}{\varepsilon} - \delta$

Notes:
Region 1: $\dot{K} + \dot{Z} < 0$  Region 2: $\dot{K} + \dot{Z} < 0$  Region 3: $\dot{K} + \dot{Z} > 0$  Region 4: $\dot{K} + \dot{Z} > 0$

$r_a > r_k$  $r_a > r_k$  $r_a > r_k$  $r_a > r_k$

$\dot{C} > 0$  $\dot{C} < 0$  $\dot{C} < 0$  $\dot{C} > 0$

Region 5: $\dot{K} + \dot{Z} > 0$  Region 6: $\dot{K} + \dot{Z} > 0$  Region 7: $\dot{K} + \dot{Z} > 0$  Region 8: $\dot{K} + \dot{Z} > 0$

$r_a < r_k$  $r_a > r_k$  $r_a < r_k$  $r_a < r_k$

$\dot{C} < 0$  $\dot{C} > 0$  $\dot{C} > 0$  $\dot{C} > 0$
2.4.2 Phase Diagrams (Continued)

Figure 3.1.2: Phase Diagram; $ALC_0 < \rho < \frac{A(e-1)}{\varepsilon} - \delta$

Notes:
Region 1: $\dot{K} + \dot{Z} < 0$ Region 2: $\dot{K} + \dot{Z} < 0$ Region 3: $\dot{K} + \dot{Z} > 0$ Region 4: $\dot{K} + \dot{Z} > 0$
- $r_a > r_K$ $r_a > r_K$ $r_a > r_K$ $r_a > r_K$
- $\dot{C} > 0$ $\dot{C} < 0$ $\dot{C} < 0$ $\dot{C} > 0$

Region 5: $\dot{K} + \dot{Z} > 0$ Region 6: $\dot{K} + \dot{Z} > 0$ Region 7: $\dot{K} + \dot{Z} > 0$ Region 8: $\dot{K} + \dot{Z} > 0$
- $r_a < r_K$ $r_a > r_K$ $r_a < r_K$ $r_a < r_K$
- $\dot{C} < 0$ $\dot{C} > 0$ $\dot{C} > 0$ $\dot{C} > 0$
2.4.2 Phase Diagrams (Continued)

Figure 3.1.3: Phase Diagram; 
\[ \text{ALC} = , \quad \left( \delta \varepsilon \right)_{1 \varepsilon A \rho 0} - < < \]

Notes:

Region 1: \( \dot{K} + \dot{Z} < 0 \)  Region 2: \( \dot{K} + \dot{Z} < 0 \)  Region 3: \( \dot{K} + \dot{Z} > 0 \)  Region 4: \( \dot{K} + \dot{Z} > 0 \)
\[ r_{a} > r_{K} \quad r_{a} > r_{K} \quad r_{a} > r_{K} \quad r_{a} > r_{K} \]
\[ \dot{C} < 0 \quad \dot{C} < 0 \quad \dot{C} < 0 \quad \dot{C} < 0 \]

Region 5: \( \dot{K} + \dot{Z} > 0 \)  Region 6: \( \dot{K} + \dot{Z} > 0 \)  Region 7: \( \dot{K} + \dot{Z} > 0 \)  Region 8: \( \dot{K} + \dot{Z} > 0 \)
\[ r_{a} < r_{K} \quad r_{a} > r_{K} \quad r_{a} < r_{K} \quad r_{a} < r_{K} \]
\[ \dot{C} < 0 \quad \dot{C} > 0 \quad \dot{C} > 0 \quad \dot{C} > 0 \]
2.4.2 Phase Diagrams (Continued)

Figure 3.1.4: Phase Diagram; $C > AL$, $0 < \rho < \frac{A(e-1)}{\varepsilon} - \delta$

Notes:
Region 1: $\dot{K} + \dot{Z} < 0$  Region 2: $\dot{K} + \dot{Z} < 0$  Region 3: $\dot{K} + \dot{Z} < 0$  Region 4: $\dot{K} + \dot{Z} < 0$  Region 5: $\dot{K} + \dot{Z} > 0$
- $r_a > r_k$
- $\dot{C} > 0$
Region 6: $\dot{K} + \dot{Z} > 0$  Region 7: $\dot{K} + \dot{Z} > 0$  Region 8: $\dot{K} + \dot{Z} > 0$  Region 9: $\dot{K} + \dot{Z} > 0$  Region 10: $\dot{K} + \dot{Z} > 0$
- $r_a > r_k$
- $\dot{C} < 0$
2.4.2 Phase Diagrams (Continued)

Figure 3.1.5: Phase Diagram; 
$\phi > \phi$, 
$\left( \delta - \alpha \right) < 0$

Notes:
Region 1: $\dot{K} + \dot{Z} < 0$ Region 2: $\dot{K} + \dot{Z} < 0$ Region 3: $\dot{K} + \dot{Z} < 0$ Region 4: $\dot{K} + \dot{Z} < 0$ Region 5: $\dot{K} + \dot{Z} < 0$
$r_a > r_k$ $r_a > r_k$ $r_a < r_k$ $r_a < r_k$ $r_a < r_k$
$\dot{C} > 0$ $\dot{C} < 0$ $\dot{C} < 0$ $\dot{C} > 0$ $\dot{C} > 0$

Region 6: $\dot{K} + \dot{Z} < 0$ Region 7: $\dot{K} + \dot{Z} > 0$ Region 8: $\dot{K} + \dot{Z} > 0$ Region 9: $\dot{K} + \dot{Z} > 0$ Region 10: $\dot{K} + \dot{Z} > 0$
$r_a > r_k$ $r_a > r_k$ $r_a > r_k$ $r_a < r_k$ $r_a < r_k$
$\dot{C} > 0$ $\dot{C} > 0$ $\dot{C} > 0$ $\dot{C} < 0$ $\dot{C} < 0$
2.4.3 Impact of $\rho$

- Equilibrium Dynamics: $\rho = 0$
  - All equilibrium paths eventually lead to perpetual growth in $\alpha$ and $K$.
  - $C$ always grows
- Equilibrium Dynamics: $\rho = \frac{A(\varepsilon - 1)}{\varepsilon} - \delta$
  - All equilibrium paths eventually lead to perpetual growth in $\alpha$ and $K$, but $C$ falls to zero as $t \to \infty$.
    - The preferences of households make it optimal for consumption to fall to zero as output rises.
    - Though the “high $\rho$” scenario is feasible from a theoretical perspective, it makes little sense from a practical standpoint. The economy disappears because consumption goes to zero and households cease to exist. Can be ruled out because it does not correspond to economic history.
2.5 Conclusion

- Extend Peretto and Seater model by incorporating consumer optimization.
- **All** equilibrium paths lead to an economy that is asymptotically AK
  - Saving rate chosen by optimizing households is always high enough to yield perpetual growth.