Numerical solution of dynamic equilibrium models under Poisson uncertainty

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Introduction

- Since their first appearance in the 1970s, dynamic stochastic general equilibrium (DSGE) models became the workhorse in dynamic macroeconomic theory.
- Successfully capture aggregate dynamics over the business cycle, but surprisingly quiet on the effects of rare events:
- Actual/potential rare events: economic events (Great Depression, financial crises), wartime (world wars), natural disasters (tsunamis, hurricanes, earthquakes, asteroid collisions), epidemics (Black Death, avian flu) (Barro 2006, *Quart. J. Econ.*).
Theory:

- rare events present in quality ladder, endogenous growth and matching models

Empirics:

- anecdotic and historical evidence for rare disasters

- empirical evidence for Poisson jumps in US output data
  (Posch 2009, *J. Econometrics*)
Introduction

- Caveat: no analytical solutions for DSGE models, need computational methods
- Powerful computational methods such as perturbation and projection methods (Judd 1992, *J. Econ. Theory*, 1998 MIT press)
- Highly accurate solving DSGE models under Normal uncertainty (at least locally) (Aruoba, Fernández-Villaverde, Rubio-Ramírez 2006, *J. Econ. Dynam. Control*)

Open questions:
- Effects of large shocks on approximation errors are largely unexplored
- No clear answer has been given on the effects of rare events on optimal decisions such as consumption or leisure (Barro-Rietz rare disaster hypothesis)
Introduction

This paper:

▶ proposes a simple and powerful method for determining the transition process in continuous-time DSGE models under Poisson uncertainty numerically

▶ shows how to extend DSGE models and existing solution methods in order to allow for the possibilities of rare events

▶ illustrates our approach for two popular methods computing numerical solutions to dynamic general equilibrium models, by solving

▶ the neoclassical growth model with disasters (extending backward integration)  
(Brunner, Strulik 2002, *J. Econ. Dynam. Control*)

▶ the Lucas’ model of endogenous growth with disasters (Relaxation algorithm)  
(Trimborn, Koch, Steger 2008, *Macroecon. Dynam.*)
Introduction

Our framework:
- continuous-time formulation of DGSE models, which imply an equilibrium system of controlled stochastic differential equations (SDEs) under Poisson uncertainty
  - relatively simple to obtain reduced forms using stochastic calculus
  - closed-form solutions available for reasonable parametric restrictions, which are important benchmark solutions to explore broader classes of models (Judd 1997, *J. Econ. Dynam. Control*)

Our message: solution method works
- obtain highly accurate policy functions even for the complete state space (globally) compared to the benchmark analytical solutions
Introduction

Plan of the talk

- The macroeconomics theory
- The numerical solution
- A neoclassical growth model with disasters
- Conclusion
The macroeconomic theory

- Consider the autonomous infinite horizon stochastic control problem

\[
\max E \int_0^\infty e^{-\rho t} u(x_t, c_t) dt
\]

s.t.
\[
dx_t = f(x_t, c_t) dt + g(x_{t-}, c_{t-}) dN_t
\]

given initial states \( x_0 = x, \ N_0 = z, \ (x, z) \in \mathbb{R}_+^2 \) and the control \( \{c_t\}_{t=0}^\infty \)

- \( \{N_t\}_{t=0}^\infty \) Poisson process with arrival rate \( \lambda = \lambda(x_t, c_t) \), we denote \( x_{t-} \equiv \lim_{s \to t} x_s \) as the left-limit of the variable at date \( t \)

- (Controlled SDE) For illustration, consider the stochastic control problem where \( g(x, c) = g(x), \ \lambda(x, c) = \lambda, \ u(x, c) = u(c) \) and \( f_c(x, c) = -1. \)
The macroeconomic theory

- Suppose that \( u''(c) \neq 0 \), then the reduced form system reads

\[
dc_t = \left( \rho - f_x(x_t, c_t) + \lambda - \frac{u'(c(x_t + g(x_t)))}{u'(c(x_t))}(1 + g_x(x_t))\lambda \right) \frac{u'(c_t)}{u''(c_t)} dt \\
+ (c(x_{t-} + g(x_{t-})) - c(x_{t-})) dN_t,
\]

\[
dx_t = f(x_t, c_t) dt + g(x_{t-}) dN_t,
\]

where \( c_t = c(x_t) \) denotes the optimal policy function.

- The appearance of \( c(x_t + g(x_t)) \) means that at date \( \tau_1 \) the solution at another date \( \tau \in \mathbb{R} \) influences the slopes \( dc_t \) and \( dx_t \).

- Economically: consumers are aware of the possibility of jumps in their optimal consumption and take into account the level of consumption if such events occur.
The numerical solution

- Technically, we solve

\[
dc_t = f_1(c_t, x_t, \vec{c}, \vec{x}) \, dt + g_1(c_{t-}, x_{t-}) \, dN_t,
\]

\[
dx_t = f_2(c_t, x_t, \vec{c}, \vec{x}) \, dt + g_2(c_{t-}, x_{t-}) \, dN_t,
\]

augmented with boundary conditions for the beginning and the end of the horizon.

- \(\vec{c}\) and \(\vec{x}\) are the unknown optimal pairs for the entire solution of the system above, need to compute on the solution on the entire domain of \(x_t\), usually \(x \in (0, \infty)\)

- solve using the idea of Waveform Relaxation
The numerical solution

- **Step 1 (pathwise continuous):**
  - compute the policy function implied by the (conditional) deterministic system
    \[
    dc_t = f_1 (c_t, x_t, \vec{c}, \vec{x}) \, dt \\
    dx_t = f_2 (c_t, x_t, \vec{c}, \vec{x}) \, dt
    \]

- **Step 2 (discontinuities):**
  - the stochastic paths can be obtained by augmenting the solution by realizations of the stochastic process \( N_t \), making use of the entire solution \( \vec{c} \)
The numerical solution

- **Step 1a (initial guess):**
  - provide an initial guess of optimal pairs $(\vec{c}_0, \vec{x}_0)$, the system reduces to
    \begin{align*}
    dc_t &= \tilde{f}_1(c_t, x_t) \, dt \\
    dx_t &= \tilde{f}_2(c_t, x_t) \, dt
    \end{align*}
  as the feedback is neglected and solve by standard algorithms to obtain $(\vec{c}_1, \vec{x}_1)$

- **Step 1b (update guess and iterate):**
  - take the trial solution $(\vec{c}_0, \vec{x}_0)$ as given and define for each iteration $i = 1, \ldots, n$
    \begin{align*}
    dc_i &= f_1(c_i(t), x_i(t), \vec{c}_{i-1}, \vec{x}_{i-1}) \, dt \\
    dk_i &= f_2(c_i(t), x_i(t), \vec{c}_{i-1}, \vec{x}_{i-1}) \, dt
    \end{align*}
The numerical solution

- Technically: we construct a fix-point iteration for the operator $\mathcal{N}$ such that the desired solution $z : \mathbb{R} \rightarrow \mathbb{R}^2$ is a fix point of this operator: $\mathcal{N}(z) = z$

- we start with a trial solution $z_0$ and iterate by evaluating $\mathcal{N}$ until $\|z_i - z_{i-1}\|$ is sufficiently small

- Remark: cubic spline interpolation,
A neoclassical growth model with disasters

- One-good production economy with constant returns to scale technology
  (Merton 1975, *Rev. Econ. Stud.*)

\[ Y_t = AK_t^\alpha L^{1-\alpha} \]

where

\[ dK_t = (I_t - \delta K_t) \, dt - \gamma K_t \, dN_t \]

- \( N_t \) number of (natural) disasters up to time \( t \) at arrival rate \( \lambda \geq 0 \)
- \( K_t \) capital stock, \( I_t \) investment, \( C_t \) consumption

- Closed economy

\[ Y_t = C_t + I_t \]
A neoclassical growth model with disasters

- Benevolent planner maximizes

\[ U_0 \equiv E \int_0^\infty e^{-\rho t} u(C_t)dt, \quad u' > 0, u'' < 0 \]

subject to

\[ dK_t = (Y_t - C_t - \delta K_t)dt - \gamma K_t dN_t \]

- Preferences

\[ u(C_t) = \frac{C_t^{1-\theta}}{1-\theta} \]
A neoclassical growth model with disasters

- Reduced form

\[
dC_t = \frac{r_t - \rho - \delta - \lambda + \lambda(1 - \gamma)\tilde{C}(K_t)^{-\theta}}{\theta}C_t dt
- (1 - \tilde{C}(K_{t-}))C_{t-}dN_t
\]

\[
dK_t = (AK_t^\alpha L^{1-\alpha} - C_t - \delta K_t)dt - \gamma K_{t-}dN_t
\]

- defining \(1 - \tilde{C}(K_{t-}) \equiv (C(K_t) - C((1 - \gamma)K_t))/C(K_t)\)

percentage change of optimal consumption after a disaster
A neoclassical growth model with disasters

Closed-form solutions

- **constant-saving-function**

  \[
  \rho = ((1 - \gamma)^{1 - \alpha \theta} - 1) \lambda - (1 - \alpha \theta) \delta
  \]

  \[
  \Rightarrow C_t = C(K_t) = (1 - s) AK_t^\alpha L^{1-\alpha}
  \]

  calibration \((\alpha, \theta, \delta, \lambda, \gamma) = (0.5, 2.5, 0.05, 0.2, 0.1) \Rightarrow \rho = 0.0178, s = 0.4\)

- **linear-policy-function**

  \[
  \alpha = \theta \Rightarrow C_t = C(K_t) = b K_t
  \]

  calibration \((\alpha, \theta, \delta, \lambda, \gamma) = (0.5, 0.5, 0.05, 0.2, 0.1), \rho = 0.0178 \Rightarrow b = 0.1\)
A neoclassical growth model with disasters

- Step 1a (initial guess):
  - solve the deterministic system
    \[
    dC_t = \frac{r_t - \rho - \delta}{\theta} C_t dt
    \]
    \[
    dK_t = (AK_t^\alpha L^{1-\alpha} - C_t - \delta K_t) dt
    \]

- Step 1b (update guess and iterate):
  - define for each iteration \( i = 1, \ldots, n \)
    \[
    dC_i = \frac{r_i - \rho - \delta - \lambda + \lambda(1 - \gamma)\tilde{C}_{i-1}(K_i)^{-\theta}}{\theta} C_i dt
    \]
    \[
    dK_i = (AK_i^\alpha L^{1-\alpha} - C_i - \delta K_i) dt
    \]
A neoclassical growth model with disasters ($\theta = 2.5$)

Optimal policy functions: deterministic (dashed) vs. stochastic (solid) compared to the analytical benchmark solution (dotted)
A neoclassical growth model with disasters ($\theta = 2.5$)

Optimal jump functions: numerical calculated solution (solid) compared to the analytical benchmark solution (dotted)
A neoclassical growth model with disasters ($\theta = 2.5$)
A neoclassical growth model with disasters ($\theta = 1$)

Optimal policy functions: deterministic (dashed) vs. stochastic (solid)
A neoclassical growth model with disasters ($\theta = 1$)

Optimal jump functions: numerical calculated solution (solid)
A neoclassical growth model with disasters ($\theta = 1$)
A neoclassical growth model with disasters ($\theta = 0.5$)

Optimal policy functions: deterministic (dashed) vs. stochastic (solid) compared to the analytical benchmark solution (dotted)
A neoclassical growth model with disasters ($\theta = 0.5$)

Optimal jump functions: numerical calculated solution (solid) compared to the analytical benchmark solution (dotted)
A neoclassical growth model with disasters ($\theta = 0.5$)
Conclusion

This paper

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- shows how to extend DSGE models and existing solution methods in order to allow for the possibilities of rare events
- illustrates our approach for two popular methods computing numerical solutions to dynamic general equilibrium models