Public investment policy, distribution, and growth: What levels of redistribution through public investment maximize growth?

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Abstract

This paper studies the distributional and the growth effects of public investment in a simple growth model with incomplete market where both growth and inequality are endogenously determined. Taxation lowers growth through distorting private investment, whereas public investment stimulates long run growth. Higher inequality corresponds to lower growth when the credit and insurance markets are missing as these prevent the efficient amount of investment to be undertaken in the economy. In this case, public investment could have additional efficiency effect through substituting for the missing markets. It serves as means to relax resource constraints that impede certain investment. The efficiency effect of complementary public investment, on the other hand, could be compromised as it aggravates it. The implication for growth maximizing tax rate is examined.

Keywords Public capital. elasticity of substitution. wealth mobility. growth. incomplete market

We may now proceed to examine more closely the things upon which the elasticity of substitution depends. ..., the mere extension of the use of instruments and methods of production from firms where they were previously employed to firms which could not previously afford them." Hicks, (1932, p.120)

1. Introduction

The recent economic and financial crisis have drawn infrastructure development into the highlight. The fiscal stimulus programs adopted by the industrialized countries followed by the "budget cuts to the bones" and pressures from the global credit shortages during and in the aftermath of the crisis, respectively, have made the future of many infrastructure projects uncertain now than ever leading to a growing concern in the effects on years progress in growth and development, particularly in developing countries.¹ This revives interests on the macroeconomic effects of infrastructure. The present paper attempts to address some of the issues related to growth and infrastructure investment with a particular focus on its distributional role in the economy.

Although the growth effects of public capital, especially infrastructure, have been the focus of endogenous growth literature over the past two decades,² its distributional aspects and the respective implication to growth, somewhat surprisingly, are less attended in this literature.³ However, many infrastructure services are arguably redistributive by nature. Depending on its type, infrastructure could benefit certain households more than proportionally. For instance, state provision of clean water, drought-tolerant variety of seeds, sanitation services, and public transport may benefit more those who lack these basic inputs (World Bank, 1994). On the other hand, construction of a new dam could be more beneficial for those who have access to high-yielding variety of seeds and fertilizer. The same goes to roads: they may benefit more those with bicycles, motorbikes, or even benefit most those who own trucks (Songco, 2002 and Estache et al., 2002). Evidences from empirical works also support a disproportional impact of infrastructure on growth. For instance, using cross country data, Calderon and Serven, (2004), Calderon and Chong, (2004), and Lopez, (2003) argue that infrastructure has a disproportionately positive impact on growth, whereas Khandker and Koolwal, (2007) find that paved roads enhance the income of the richest households in Rural Bangladesh.

This paper examines the distributional and growth effects of public capital, especially infrastructure, in a simple growth model where both growth and inequality are endogenously determined. Similar to Barro, (1990), the source of endogenous growth is public capital combined with private capital in a constant return to scale production function that makes the cumulative marginal product of capital in the long run constant. Endogenous inequality is rather generated due to missing credit and insurance markets, as in

¹Ahead of the African Union Summit of Heads of States and Government in Addis Ababa, scheduled for February 1 to 3, 2009, with the theme of "Infrastructure Development in Africa", the World Bank had stated that Africa risks a lost decade of underdevelopment if it neglects infrastructure development (Reuters, January 28, 2009).
³There is a vast literature on publicly provided private goods but it abstracts from growth (see, for example, Besley and Coate, 1991, Boadway and Marchand, 1995, Pirttila and Tuomala 2002).
Loury, (1981). When individuals cannot perfectly insure themselves from future income uncertainty and they are not allowed to borrow and lend among each other, wealth mobility and inequality emerge and persist. The dynamics of aggregate variables – public and private capital, and the growth rate of the economy – and wealth mobility are jointly determined.

In the model, diminishing returns to investment imply certain individuals, in particular, the poor have a higher marginal product than the rich. But, since they cannot borrow (from those who have a lower marginal product) and invest due to the credit market imperfection, Pareto efficiency cannot be achieved as often implicitly assumed in representative agent models, and hence a higher inequality leads to a greater inefficiency. When the equity efficiency trade-off is mitigated in this manner, redistributive policies could have a net positive growth or efficiency effects (Benabou, 2002).

The link between infrastructure provisions and inequality is established through the relationship between elasticity of substitution and factor shares. If the elasticity of substitution between public and private capital is greater than unity, for instance, an increase in public investment (or a decrease in private/public capital ratio) decreases the private capital’s income share, in a constant return to scale production function. This implies more opportunities for the poor who have high marginal productivity due to a lower private investment but cannot borrow (from other individuals who are less productive than them) and invest due to institutional constraints. They can relax some of their resource constraints through factor substitutions. This could in turn reduce inequality and hence increase growth through mitigating the inefficiency that arises due to the presence of inequality.

If the elasticity of substitution between public and private capital is less than unity, however, a relative increase in public investment will result in an increase in the relative income share of private capital. This rather creates more opportunity for the rich who have a lesser marginal product than the poor due to a higher level of private investment, as the public input is now more of a complement than a substitute. While the impact on distribution is negative, the net effect on growth could be positive. The increase in complementary public investment enhances the marginal productivity of private capital that in turn could have a positive net effect on growth although it creates inefficiency by aggravating existing inequality.

When the credit and capital markets are imperfect and agents are heterogenous in terms of initial wealth, thus, there could be a nonlinear relationship between growth and the degrees of complementarity (substitutability) between public and private capital. When there is greater complementarity (substitutability), an increase in public investment corresponds to a higher (a lower) marginal productivity effect on private capital. But, this aggravates (mitigates) inequality that in turn associates to a lower (a higher) efficiency

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4In fact, inequality emerges even if there is a perfect wealth equality at initial. See Aghion and Bolton, (1992), for a detailed discussion.

5Bertola, (1993) studies the implication of changes in income shares of "nonaccumulated" and "accumulated" factors to growth and distribution in endogenous growth models.

6The notion is first brought to our attention by Hicks, (1932, p. 117) who states: "An increase in the supply of any factor will increase its relative share (i.e., its proportion of the National Dividend) if its "elasticity of substitution" is greater than unity."
In the model, fiscal policy has a similar growth effect as in the literature of endogenous growth (see footnote two) on one hand. Taxation lowers growth through distorting private investment whereas public investment stimulates long run growth. Therefore, the relationship between long run growth and fiscal policy is non-monotonic. On the other hand, fiscal policy has an impact on growth through an indirect channel of its effect on the distribution of wealth. It could decrease (increase) efficiency by mitigating (aggravating) inequality depending on the elasticity of substitution between public and private capital.

This has implication to the growth maximizing public investment GDP share. If the elasticity of substitution between public and private capital is unity, public investment has no implication to the distribution of wealth and hence the growth maximizing tax rate is equal to the output elasticity of public capital, as in Barro, (1990). However, if the elasticity of substitution between public and private capital is greater (less) than unity, then the growth maximizing tax rate is greater (lesser) than the one that we find in a representative agent model with complete market.

The present paper builds a bridge between two strands of literature: the literature in public capital and endogenous growth (as listed in footnote one) and the ones in distribution and growth within imperfect credit market, (e.g., Loury, 1981, Galor and Zeira, 1993, Aghion and Bolton, 1997, Aghion, et al., 1999, and Benabou, 1996, 2000, 2002). The paper in particular is closely related to Benabou, (2000, 2002) that studied the distribution and growth effects of progressive education and income redistributive taxation using a log-linear income redistribution function. He showed that the progressively redistributive tax rates have a positive net growth effect through mitigating the cost of inequality that arises due to imperfection in the capital and credit market. But, Benabou does not address public investment.

The paper is also related to the recent literature in the relationship between public investment and wealth distribution. For instance, Garcia-Penalosa and Turnovsky, (2007), and Chatterjee and Turnovsky, (2010) have examined the impact of growth-enhancing fiscal policies on inequality. However, they study inequality within complete market following the line of Caselli and Ventura, (2000). The paper is also close to Getachew (2009; 2010) who have studied public investment and wealth mobility in endogenous growth models and imperfect credit market. But, the role of policy in wealth mobility is limited in these models. In addition, the growth effects of inequality are restricted to the transition as the models do not feature idiosyncratic uninsurable risks that generate nondegenerate wealth distribution. Another literature that this paper may be related to studies the role of the elasticity of substitution (between labour and capital) in endogenous growth (e.g., Klump and de la Grandville, 2000, Miyagiwa and Papageorgiou, 2003, Karagiannis et al., 2005 and Irmen and Klump, 2009) but this literature abstracts from redistribution issues.

The rest of the paper is organized as follows. Section 2 provides the model. Section 3 examines the dynamics and equilibrium of the model. In section 4, we calibrate and simulate the model and conduct some policy experiments. Section 5 concludes.

\[7\] There is also a vast literature that studies the relationship between public education and income distribution, which this paper may also be related to by virtue of common interest of examining the distributional effect of a productive public good in growth models, (e.g., Glomm and Ravikumar, 1992, 2003, Saint-Paul and Verdier, 1993, Eckstein and Zilcha, 1994 and Zhang, 1996).
2. The Model

2.1. Preferences and Technology

Suppose overlapping generations economy with two-period-lived agents in continuum heterogeneous households. Each household, \( i \in [0, 1] \), constitutes of a young – the offspring – and an adult – the parent – members at any time \( t \). Population size is thus constant. Each parent of the initial generation (at \( t = 0 \)) is endowed with a human capital wealth \( h^i_0 \), and a public infrastructure \( G_0 \) which is shared among others. The distribution of wealth takes a known initial probability distribution but evolves over time endogenously.

Individuals live two periods. In the first period, agents are young, they accumulate human capital while their consumptions are included in that of their parents, and do not take any decision. All economic decisions are thus done during the second period when children become parents. Parents supply human capital to privately-held firms. They operate in a neoclassical technology with constant returns to scale in private and public capitals and earn income. The government taxes income with a fixed flat rate tax in order to finance public investment, which will be used for the final goods production. Parents allocate after tax income for consumption and saving where the latter is used for the human capital accumulation of the offspring as a "joy of giving". At the end of the second period parents die. Thus, there are no any old age consumptions.

In a logarithmic preference, the individual utility function is defined as

\[
\ln c^i_t + \beta E_t [\ln h^i_{t+1}] \tag{1}
\]

subject to

\[
c^i_t + e^i_t = (1 - \tau)y^i_t \tag{2}
\]

where \( c^i_t, \beta, e^i_t, h^i_{t+1}, \tau \) and \( y^i_t \), denote the household consumption, the psychological discount factor, the household saving, the human capital of the child, the flat rate tax and household income, respectively. The human capital accumulation of the offspring and the production function of the firm are given by, respectively,

\[
h^i_{t+1} = B s^i_{t+1} (h^i_t)^{1-\eta} (e^i_t)^{\eta} \tag{3}
\]

and

\[
y_t = A \left( \alpha \exp \left( \rho \phi^i_t \right) + (1 - \alpha) \right)^{\frac{1}{\beta}} G_t \tag{4}
\]

\[
\rho \leq 1, \ 0 < \alpha < 1, \ 0 < \eta < 1, \ B > 0, \ A > 0 \tag{4'}
\]

This type of individual entrepreneurship is common in models with incomplete markets (see, for e.g., Loury, 1981, Benabou, 2000, 2002, and, Angeletos and Calvet, 2005, 2006).

These assumptions of a unitary inter-temporal elasticity of substitution utility function of altruistic agents with a "joy of giving" motive are widely used in the literature of income distribution dynamics (see, for instance, Glomm and Ravikumar, 1992, Galor and Zeira, 1993, Saint-Paul and Verdier 1993, and Benabou, 2000).
where $\phi_t^i \equiv \ln \frac{h_t^i}{C_t^i}$, $\phi_t \equiv \ln \frac{H_t}{C_t}$, and $\phi \equiv \ln \frac{H}{C}$; $H_t$ and $G_t$ represent the individual and the aggregate private and public capital, respectively; and, $\xi_{t+1}^i$ denotes the uninsured idiosyncratic shock related to the ability of individuals to accumulate human capital.\(^\text{10}\) It is assumed to be lognormal with mean one, $\ln \xi_{t+1}^i \sim N(-\nu^2/2, \nu^2)$. The distribution of human capital is also assumed to be lognormal: $\ln h_0^i \sim N(\mu_0, \sigma_0^2)$. The markets for insurance and credit are missing. Though these are extreme forms of market incompleteness what matters is that existence of some forms of incompleteness (see, for e.g., Aghion et al., 1999 and Benabou, 2000, and 2002).\(^\text{11}\)

The government provides inputs for goods production, and collects proportional taxes on marketed outputs. Its budget is always balanced. Public capital accumulation thus takes the form, with complete depreciation,

$$G_{t+1} = \tau E [y_t^i] \equiv \tau Y_t$$  \hspace{1cm} (5)

where $Y_t$ and $G_t$ represent the aggregate income and infrastructure service.\(^\text{12}\) Therefore, public capital is treated as a flow rather a stock variable.\(^\text{13}\)

### 2.2. Individuals’ optimal decision rule

The individual of generation $t$ solves the following problem, which is derived by substituting (2) and (3) into (1),

$$\max_{e_t^i} \left\{ \ln(1-\tau)y_t^i - e_t^i \right\} + \beta E_t \left[ \ln B\xi_{t+1}^i \left( h_t^i \right)^{1-\eta} \left( e_t^i \right)^{\eta} \right]$$

taking as given $\tau, G_t, h_t^i$ and $\xi_{t+1}^i$.

The first order condition yields,

$$e_t^i = a(1-\tau)y_t^i$$  \hspace{1cm} (6)

where $a = \frac{\beta \eta}{1+\beta \eta}$. Eq. (6) is the parent’s optimal saving, which is simply a fraction of his/her after tax income. As common in logarithmic preference functions, the saving rate is independent of the rate of return on investment.\(^\text{14}\)

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\(^{10}\)Note that in all the texts, small letters and/or superscript $i$ denote individual variables while capital letters are used to denote aggregate variables. Variables without time subscript represent equilibrium values. Subscripts (other than $t$) with prime denote derivatives with respect to the variable: e.g., $\frac{d}{dx}x = x'$.\(^{11}\)The credit market could be imperfect when “children cannot be held responsible for the debts of their parents” (Benabou 2000, p. 101).\(^{12}\)As $h_t^i$ is a random variable with a log-normal distribution, we frequently applies the expectation (E) and variance (var) operators in deriving aggregate (average) and distribution variables: $E \left[ h_t^i \right] = \int h_t^i \equiv H_t$ and $\text{var} \left[ \ln h_t^i \right] = \sigma_t^2$.\(^{13}\)The model can be extended to public capital stock, for instance, by considering adjustment cost $\nu \neq 1$, $G_{t+1} = (\tau Y_t)^{1-\nu} (G_t)^{\nu}$. But, it is simpler to work with $\nu = 1$, without losing important insight. Besides, even with complete depreciation and zero adjustment cost, the model features transitional dynamics at the individual level due to persistent inequality and/or inter-generational spillover.\(^{14}\)According to De La Croix and Michel, (2002, p. 13-14), this is because that when inter-temporal elasticity of substitution is unity, income effects exactly compensate substitution effects.
3. Wealth mobility, macroeconomic dynamics, and public investment

We begin characterizing the dynamics and equilibrium of the economy at both the individual and aggregate levels with the special case when the production function is Cobb-Douglas. Although a unitary elasticity of substitution is too restrictive for dealing with distributional problem, the Cobb-Douglas is important in providing a closed form analytical solution. We can use this as a benchmark and to make comparisons and contrasts with the results we find in the general CES case, which is rather less tractable though less restrictive.

3.1. Cobb-Douglas technology

The production function is Cobb-Douglas implies that \( \rho = 0 \) in eq. (4). In this case, the dynamics of capital accumulation that corresponds to the individual’s optimal behavior, from (3), (4), and (6), is given by:

\[
\ln h_{i+1} \ = \ \ln (Aa(1 - \tau)) \theta BG_t + \ln \xi_{t+1} + (1 - \eta(1 - \alpha)) \phi_t
\]

By simply taking the variance of (7a), we get the dynamics for wealth distribution,

\[
\sigma^2_{i+1} = \sigma^2 + (1 - (1 - \alpha) \eta)^2 \sigma^2_t
\]

Wealth accumulation differs among individuals due to differences in uninsured idiosyncratic ability \( \xi_{t+1} \) and initial wealth ratio \( \phi_t \). Individuals with relatively lower \( \phi_t \) rapidly accumulate wealth due to their relatively high marginal productivity, which, in turn, is due to the presence of diminishing returns on investment coupled with imperfect credit market. In this case, the economy features a declining wealth mobility along the transition to a steady state. However, it is the uninsured idiosyncratic risk that derives the long run distribution of wealth. Given (4'), this dynamics converges to a long run distribution of wealth

\[
\sigma^2 = \frac{\nu^2}{1 - (1 - \eta(1 - \alpha))^2}
\]

We obtain the aggregate private capital by aggregating (7a), and using (8a),

\[
\ln H_{t+1} = \ln (Aa(1 - \tau)) \theta BG_t + (1 - (1 - \alpha) \eta) \phi_t
\]

\[
\quad - \frac{1}{2} \sigma^2_t (1 - (1 - \alpha) \eta)(1 - \alpha) \eta
\]

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15 Although Solow, (1960) believes that a Cobb-Douglas may do fairly well in tracking observed changes in production, he has a reservation: "as long as no deep distributive meaning is read into the results".

16 Computing the distribution of income (vis-a-vis wealth) is rather simple. Suppose \( \sigma_{i,y}^2 \equiv \text{var} \{\ln y_i\} \). Then, the distribution of income compared to that of capital, considering (4) and \( \rho = 0 \), is given by \( \sigma_{i,y}^2 = \alpha^2 \sigma_{i+1}^2 \), which is smaller than that of wealth distribution.

17 With respect to the aggregation, note that first \( \text{E} \{\ln h_{i+1}^1\} = \int \ln h_{i+1} \). Second, \( \text{E} \{\ln h_{t+1}\} = \ln \text{E} [h_{t+1}] - \frac{1}{2} \text{var} [\ln h_{t+1}] = \ln H_{t+1} - \frac{1}{2} \sigma_{t+1}^2 \) where \( H_{t+1} \equiv \text{E} [h_{t+1}] \).
From (4), (recall $\rho = 0$), and (6), we have the difference equation for public capital,

$$\ln G_{t+1} = \ln \tau AG_t + \alpha \phi_t - \frac{1}{2} \alpha (1 - \alpha) \sigma_t^2$$  (11a)

The dynamic path of the capital ratio at the individual level is given by, from (7a) and (11a),

$$\phi_{t+1} = \ln A^{\eta^{-1}} (a(1 - \tau))^\eta \tau^{-1} B + \ln \xi_{t+1}^i + (1 - \eta (1 - \alpha)) \phi_t - \alpha \phi_t + \frac{1}{2} \alpha (1 - \alpha) \sigma_t^2$$  (12a)

We obtain the dynamics of the aggregate capital ratio from (10a) and (11a),

$$\phi_{t+1} = \ln BA^{\eta^{-1}} (a(1 - \tau))^\eta \tau^{-1} + (1 - \eta) (1 - \alpha) \phi_t - \frac{1}{2} \sigma_t^2 (1 - \alpha) ((\eta - \alpha) - (1 - \alpha) \eta^2)$$  (13a)

In case of a one sector $AK$ economy, i.e., $\eta = 1$, the individual and the aggregate capital ratios are given by, from (12a) and (13a), respectively,

$$\phi_{t+1}^i = \ln (a(1 - \tau)) \tau^{-1} B + \ln \xi_{t+1}^i + \alpha (\phi_t^i - \phi_t) + \frac{1}{2} \alpha (1 - \alpha) \sigma_t^2$$  (12a')

$$\phi_{t+1}^A = \phi = \ln (a(1 - \tau)) \tau^{-1} B$$  (13a')

The system with eqs. (8a) and (12a) determines the dynamic path of individuals while the one with eqs. (8a) and (13a) determines the dynamics of the aggregate economy. It is shown in Appendix B.1. that both systems’ steady states are sink path stable.

An interesting feature of the model is that even when the capital ratio at aggregate becomes constant along the transition path (typical of $AK$ models), the capital ratio at individual features transitional dynamics. In eq. (13a'), the aggregate capital ratio is always at equilibrium whereas in (12a') individuals are at transitional dynamics along with the distribution of wealth (8a). The intuition is that the distribution of wealth determines the accumulation of public capital (11a) that dictates the capital ratio dynamics at individual (12a'). In contrast, the dynamics of capital ratio washed at aggregate since public and private capitals grow at the same rate, when $\eta = 1$.

Fig. 1. illustrates the qualitative features of (8a), (12a'), and (13a'). It shows the transitional dynamics of two different individuals, the $j^{th}$ and $k^{th}$ persons, and their respective steady states, $E_j$ and $E_k$, denoted by the solid and dashed lines, respectively.

$^{18}$The path of idiosyncratic uncertainty at individual is implicitly assumed: $\xi_{t+1}^i = \xi_t^i = \xi^i$, whereas at aggregate, it is vanished.
The individuals evolve towards their unique steady states, according to (8a) and (12a'), whereas the economy’s capital ratio $\phi$ remains constant, (13a'). The two phase lines are parallel as their slopes are the same and positive while the intercept is determined by the idiosyncratic shock and the initial distribution of wealth.

Figure 1. Case 1: $\eta = 1$. Dynamic paths and steady states of the capital ratios associated to heterogenous individuals and the economy. Individuals’ capital ratios $\phi_i^j$ feature transitional dynamics even if the technology is $AK$ and hence the aggregate capital ratio remains constant along the path. The solid and the dashed arrows illustrate the $j^{th}$ and the $k^{th}$ persons’ dynamic paths towards their respective steady states $E_j$ and $E_k$ whereas the representative individual has a constant capital ratio $\phi$.

With respect to the general case that $\eta \neq 1$, there is no much change with respect to the individuals’ dynamic behavior. But, the aggregate economy features now transitional dynamics. Fig. 2. examines (8a), (12a) and (13a) qualitatively. The dynamic paths associated to individuals $k$ and $j$ are denoted by the dashed phase lines and arrows with their respective steady states $E_k$ and $E_j$. The solid phase line and arrows show the transitional dynamics.

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19It is shown in Appendix B.2 that the steady state associated to each individual exists and is unique.

20Since, from (12a'), $\frac{\partial \phi_{i+1}^j}{\partial \sigma^2_I} = \frac{\partial \phi_{i+1}^j}{\partial \sigma^2_I} > 0$. 

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dynamic path of the representative individual. Its slope is positive now but still less steeper than that of the individuals.\textsuperscript{21}

Figure 2. Case 2. $\eta \neq 1$. In contrast to the $AK$ model earlier, \textit{both} the individuals and aggregate capital ratios have transitional dynamics. The dashed loci and arrows illustrate the dynamic paths of two different individuals, $k$ and $j$, heading towards their equilibria, $E_k$ and $E_j$, respectively, whereas the solid line and arrows illustrate the transition path of the representative person. $E$ is the balanced growth path (steady state) of the economy. The slope of the locus for the representative individual is positive similar to the individuals’ but smaller.

3.1.1. Inequality and the aggregate economy

Both the persistence and the steady state of inequality is determined by the technological parameters in the production and accumulation sectors ($\alpha$ and $\eta$, respectively) as shown in (8a) and (9a). A greater public factor share $1 - \alpha$ and/or a smaller inter-generational spillover $\eta$ lead to a more equal society. But, note that the dynamics and

\begin{equation*}
\text{From (12a) and (13a), } \frac{\partial \phi_{t+1}}{\partial \sigma_t^2} > \frac{\partial \phi_{t+1}}{\partial \sigma_t^2}.
\end{equation*}
equilibrium of distribution are shown in this model to be independent of the capital ratio and policy \( \tau \). However, we see below, when \( \rho \neq 1 \), the aggregate capital ratio and public investment policy become important to both the transition and the long run distribution of wealth.

Since the last terms in (10a) and (11a) are less than zero, the effect of inequality on the dynamics of aggregate capital accumulation is negative. The competitive equilibrium is not Pareto optimal in this economy as individuals that are heterogenous in terms of initial wealth and hence productivity are not be able to make efficient transactions due to the credit constraints. A greater initial inequality is thus associated to forgoing more efficient opportunities. Moreover, a higher inequality corresponds to a greater private-public capital ratio dynamics at both individual and aggregate levels but for a slightly different reason. At individual, inequality does not affect the accumulation of private capital but public capital. At aggregate, it affects both but the effect on aggregate private capital is partly mitigated as \( \eta < 1 \).

The equilibrium of the economy is characterized with a balanced growth path where \( H_t, G_t \) and \( Y_t \) grow at the same rate. The dynamics of the capital ratio converges to its steady state value

\[
\phi = \ln \xi + \ln \left( \frac{(a(1 - \tau))^\eta}{\tau} \right)^{1-\frac{1}{(\alpha-1)(\eta-1)}} + \chi \sigma^2 \tag{13a''}
\]

where \( \chi \equiv \frac{1}{2} (\alpha - 1) \frac{\eta((\alpha-1)(\eta+1)-\alpha)}{a-\eta(\alpha-1)} ; \xi \equiv (A^{\eta-1}B)^{1-\frac{1}{(\alpha-1)(\eta-1)}} \). From (9a), (11a) and (13a''), the growth rate of the economy along a balanced growth path is given by,

\[
\gamma = \ln \tau A + \ln \xi^\alpha \left( \frac{(a(1 - \tau))^\eta}{\tau} \right)^{1-\frac{\alpha}{(\alpha-1)(\eta-1)}} + \psi \sigma^2 \tag{14a}
\]

where \( \sigma^2 \) is given by (9a), and \( \psi \equiv \frac{1}{2} \alpha \eta (\alpha - 1) \left( \frac{(\alpha-1)(\eta-1)+1}{\alpha-\eta(\alpha-1)} \right) < 0 \).

The last term in (14a) is negative implying a negative effect of inequality on growth. It does not occur in a representative version of the model. The intuition is similar to what we mentioned earlier. If credit market is missing, inequality could be persistent since individuals’ investment opportunities wholly depend on initial wealth. Then, more productive investment opportunities could be forgone by certain individuals who are more productive than others. This leads to an inefficient accumulation of aggregate capital. Because of the differences in idiosyncratic shock among individuals, inequality and its effects last in the long term.

From (14a), the growth maximizing tax rate \( \tau^* \) is

\[
\tau^* = 1 - \alpha \tag{15a}
\]

Thus, \( \tau^* \) is similar to what is found by Barro (1990) that the growth maximizing tax rate is equivalent to the social rate of return of public capital.

We summarize the above results with the following proposition:
Proposition 1  (1) The greater is the public capital elasticity of output $1 - \alpha$ is and/or the lesser is the inter-generational spillover $\eta$ is, the lesser is the inequality persistent and equilibrium. (2) A higher inequality corresponds to a lower capital accumulation and long run growth but to a higher private-public capital ratio.

3.2. CES technology

In this section, we examine wealth mobility and dynamics when the elasticity of substitution is different from unity, i.e., $\rho \neq 0$ in eq. (4). From (3), (4) and (6), capital accumulation is given by,

$$
\ln h_{t+1}^i = \ln (Aa(1-\tau))^\eta BG_t + \ln \xi_{t+1}^i + (1-\eta) \phi_t^i
$$

$$
+ \frac{\eta}{\rho} \ln \left( \alpha \exp \left( \rho \phi_t^i \right) + (1-\alpha) \right)
$$

(7b)

In contrast to (7a), the last term to the right of (7b) is nonlinear and correlates to the second term from the last. We therefore use its log-linear approximate in deriving the dynamics of wealth and aggregate variables, following Campbell (1994).\textsuperscript{22} Notably, we make all the results from the CES with approximation easily comparable to those that are based on the Cobb-Douglas.\textsuperscript{23} We further assume an $AK$ technology, that is, set $\eta = 1$.\textsuperscript{24}

We have seen earlier that the generalization $\eta \neq 1$ is important in providing the aggregate economy transitional dynamics. However, there is still transitional dynamics at individual when $\eta = 1$ but at aggregate where the economy always remains at equilibrium.

The log-linear equivalent of (7b) is given by (see Appendix A),

$$
\ln h_{t+1}^i \approx \ln (Aa(1-\tau)) BG_t + \phi + \ln \xi_{t+1}^i + (1-\omega) \phi_t^i
$$

(7b')

From which we derive the distributional dynamics,

$$
\sigma_{t+1}^2 = v^2 + (1-\omega)^2 \sigma_t^2
$$

(8b)

where $\omega \equiv \frac{1-\alpha}{\alpha \rho \phi + 1}$. Simply, $\omega$ is the output elasticity of public capital weighed by $\rho$ and $\phi$. If $\rho = 0$, then $\omega = 1 - \alpha$ and (7b) is equivalent to (7a). When $\rho \neq 0$, however, the elasticity of output to public capital $(1 - \alpha)$ is adjusted by the factor $\alpha \rho \phi + 1$. For small

\textsuperscript{22}Turnovsky and Garcia-Penalosa, (2007, 2009) also linearize their models around steady state in characterizing the evolution of wealth in Ramsey growth models. In their models, however, individuals have identical growth rate and rate of return due to complete market that allow easy aggregation.

\textsuperscript{23}To make comparison easier, we label similar equations from the Cobb-Douglas and the CES with identical numbers but different letters. For instance, eqs. (7a), and (7b) denote the individuals human capital accumulation functions in the Cobb-Douglas and the CES case, respectively.

\textsuperscript{24}This avoids further complications without much additional insight. But, for a robustness sake, we have made a numerical analysis in the general case that when $\eta$ is different but close to unity, $\eta \neq .99$, to find similar results.
\(\rho\phi\), the log-linearized system, \((7b')\) and \((8b)\), is stable.\(^{25}\). In particular, for the stability, the weighed public capital elasticity of output should satisfy \(\omega \in (0, 1)\), or

\[\phi \rho > -1\]  \hspace{1cm} (4'')

\(^{26}\)The steady state wealth distribution is easily obtained from \((8b)\),

\[\sigma^2 = \frac{v^2}{1 - (1 - \omega)^2}\]  \hspace{1cm} (9b)

Given \((4')\) and \((4'')\), from \((8b)\) and \((9b)\), we then have the following proposition:

**Proposition 2** If \(\rho > 0\) \((\rho < 0)\), then a greater \((a smaller)\) \(\phi\) corresponds to both a higher steady state and persistence of inequality.

**Proof.** If \(\rho > 0\), then \(\omega\) increases at \(\phi\). Conversely, it decreases at \(\phi\) if \(\rho < 0\). From \((8b)\) and \((9b)\), the greater is \(\omega\), the lower is inequality persistence and equilibrium, respectively.

In Section 3.1, from \((8a)\) and \((9a)\), we have seen earlier that wealth mobility depends on the value of the public capital elasticity of output \((1 - \alpha)\). Whereas, from \((8b)\) and \((9b)\), it also depends on the value of the capital ratio \(\phi\) and the sign and magnitude of \(\rho\). This new result opens a path for policy interventions. If \(\rho > 0\), one may decrease \(\phi\) (increase public investment) to a level that is required to mitigate inequality persistence, for instance.

The dynamics and stability analysis for the rest of the variables are determined in a similar way to the previous section while the results are also comparable to the ones we have found there. Thus, in deriving the dynamics for the aggregate private capital, we aggregate \((7b')\),

\[\ln H_{t+1} = \ln \left(Aa(1 - \tau)\right)^\eta BG_t + \varrho + (1 - \omega) \phi_t - \frac{1}{2} \sigma_t^2 (1 - \omega) \omega\]  \hspace{1cm} (10b)

The dynamics for public capital is given by, from \((4)\) and \((5)\), after first log-linearly approximating it in a similar fashion to \((7b)\) (see Appendix A),

\[\ln G_{t+1} \approx \ln AG_t \tau + \varrho + (1 - \omega) \phi_t - \frac{1}{2} \sigma_t^2 (1 - \omega) \omega\]  \hspace{1cm} (11b)

\(^{25}\)In fact, there exists a unique globally stable steady state even with respect to the non-linearized eq. \((7b)\). In order to see this, suppose \(x'_{t+1} = \ln h^*_{t+1}\) while \(x'_{t+1} = g(x_i)\) represents \((7b)\), which is stable since \(|g'_{x_i}(x_i)| < 1\) (see Galor, 2007, Corollary 1.14).

\(^{26}\)Note that \((7b')\) is stable if \(0 < 1 - \omega < 1 \Rightarrow 0 < \omega < 1\). Thus, \(\alpha \rho \phi > -1 \& \rho \phi > -1\), where the latter is the intersection of the two.
Then, from (7b') and (11b), the log-linear approximate capital ratio dynamics corresponds to individual $i$ is given by,

$$
\phi_{i+1}^{t} = \ln BA^{\eta-1} (a(1 - \tau))^\eta \tau^{-1} + (1 - \omega) \phi_t^i \\
+ (1 - \omega) \phi_t + \frac{1}{2} \sigma_t^2 (1 - \omega) \omega
$$

(12b)

The capital ratio with respect to the aggregate economy, from (10b) and (11b), is given by

$$
\phi(\tau) = \ln B (a(1 - \tau)) \tau^{-1}
$$

(13b)

where $\phi$ is simply the capital ratio of a representative economy with a simple $AK$ technology.\(^{27}\)

Considering (13b), we can rewrite (8b) and (9b), respectively,

$$
\sigma_{t+1}^2 = v^2 + (1 - \omega (\tau))^2 \sigma_t^2
$$

(8b')

$$
\sigma^2 (\tau) = \frac{v^2}{1 - (1 - \omega (\tau))^2}
$$

(9b')

where we redefined: $\omega (\tau) \equiv \frac{1-\alpha}{\alpha\phi(\tau)+1}$; where $\phi (\tau)$ is given by (13b).

First note that when $\rho = 0$, then $\omega = 1 - \alpha$. In this case, (and when $\eta = 1$) eqs. (9b)-(13b) will be equivalent to (9a)-(13a), respectively. Second, now it is explicitly shown that policy can influence inequality persistence, the long run distribution of wealth and hence the long run growth rate of the economy indirectly.\(^{28}\) From (8b'), (9b') and (13b), we thus make the following proposition:

**Proposition 3** If $\rho > 0$ ($\rho < 0$), then higher $\tau$ lowers (increases) the steady state and persistence of inequality.

**Proof.** See Appendix C. ■

Depending on the sign of $\rho$, $\sigma^2 (\tau)$ is strictly increasing or decreasing at $\tau$. Therefore, only a corner solution for $\tau$ could exist in minimizing (8b') or (9b'). If $\rho > 0$ ($\rho < 0$), for instance, (9b') will be minimized when $\tau$ approaches unity (zero). This features the linear relationship between public investment and inequality in the model.

From (9b'), (11b) and (13b), we derive the steady state growth rate of the economy, $\gamma$

\(^{27}\)Note that (13b) is identical to (13a'), which is rewritten here only for the sake of convenience.

\(^{28}\)Note that $\rho$ itself could be considered as a policy rather than a structural parameter. Similar to $\tau$ or $\phi$, the government could choose the magnitude and the sign of $\rho$. For instance, public investment in road construction (in contrast to mas transit) may correspond to $\rho > 0$. 


\[
\gamma(\tau) = \ln A \tau + (1 - \omega(\tau)) \phi(\tau) - \frac{v^2}{2} \frac{1 - \omega(\tau)}{2 - \omega(\tau)} \quad (14b)
\]

Similar to (14a), (14b) implies a higher initial inequality have a negative impact on long run growth. However, since \( \omega(\tau) \) is now a function of \( \tau \), unlike the Cobb-Douglas case, policy could affect growth through an indirect channel of influencing inequality.

As mentioned earlier, in the literature of public capital and endogenous growth, taxation has two opposing effects in the economy. On one hand, a higher taxation lowers growth rates through distorting private investment on the other public investment raises the efficiency of private investment and increases growth. Both of these effects are captured by the first two terms in the right hand side of (14b). They represent the representative economy with complete market version of the model, \( \sigma^2 = \nu^2 = 0 \), where inequality has no long term impact. The last term in (14b), however, features a new channel through which public investment could affect long run growth by mitigating or aggravating inequality where the latter is shown to have a detrimental impact on growth.

3.2.1. Growth maximizing redistributive tax rate

From (14b), we can compute the growth maximizing tax rate \( \tau^* \), which is defined implicitly,

\[
\epsilon(\tau^*, \rho, \alpha, v^2) \equiv \frac{\partial \gamma}{\partial \tau} = 0 \quad (15b)
\]

Not only \( \tau^* \) balances the distortionary and efficiency effects of taxation and public investment, respectively, but also has a redistributive feature, in contrast to many Barro-type endogenous growth models. Barro’s (1990) result that the growth maximizing tax rate is equivalent to the output elasticity of the public capital, \( 1 - \alpha \), holds when \( \rho = 0 \) as in (15a). A sensible question to ask is: How \( \tau^* \), in this economy, is different from what is predicted by a representative agent model with complete market? In comparing \( \tau^*|_{\sigma^2=0} \), the growth maximizing tax rate in the representative agent economy with complete market, to \( \tau^*|_{\sigma^2\neq0} \), the one in incomplete market with heterogenous agents, we have the following proposition:

**Proposition 4** If \( \rho \gtrless 0 \), then \( \tau^*|_{\sigma^2=0} \lesssim \gtrsim \tau^*|_{\sigma^2\neq0} \).

**Proof.** The proof is in Appendix C.

The growth maximizing tax rate is relatively higher (lower) in our economy if the elasticity of substitution between public and private capital is greater (lesser) than unity. The intuition is related to the additional effects of taxation on growth in a heterogenous economy with incomplete market that we have mentioned earlier. Because inequality is negatively related to growth and public investment is going to aggravate (mitigate) this when the elasticity of substitution is lesser (greater) than unity, in this economy, optimality conditions (in terms of growth maximizing) require lesser (greater) allocation of resource to the public input.

\[\text{In Appendix B.4, we examine the behavior of (15b); particularly, we specify conditions where the second derivative becomes negative.}\]
4. Calibration and simulations

In this section, we numerically examine the role of fiscal policy in wealth mobility and long run growth under various assumptions of elasticity of substitution between public and private capital. We first look into the Propositions and then conduct policy experiments. We start with calibrating a benchmark economy. We construct parameter values that most of them are conventional and reasonably reflect real economies.

To calculate α, we first consider (4) as income net of the cost of physical capital \( k_t^i \) in a constant return to scale production function following Benabou, (2002). We then set the elasticities of output of physical and public capital at 0.3 and 0.14, respectively. \(^\text{30}\) This gives us \( \alpha = 0.56/0.7 = 0.8 \), and \( 1 - \alpha = 0.14/0.7 = 0.2 \). \(^\text{31}\)

We set \( \nu^2 = 0.25 \) that implies about 0.69 and 0.44 variance of log wealth and income, respectively (see footnote 16). Assuming a lognormal distribution of income, the mean-median ratio implies 0.44 average log-income variance for the United States for the years 1991, 1994, 1997, and 2000, based on Luxembourg Income Study (UNU-WIDER, 2007). We set \( \tau = 0.15 \). \(^\text{32}\) Assuming a psychological discount factor of 0.96 , we set \( \beta = 0.96\approx 0.3 \), in a period of 30 years (de la Croix and Michel, 2002, p.255). \(^\text{33}\) There is little known about the empirical value of elasticity of substitution between public and private capital. We, therefore, experiment by considering a variety of values similar to Baier and Glomm, (2001) while considering \( \rho = 0 \) as benchmark. The choice for \( A \) and \( B \), 2.75 and 2.5, are made to match a long run output-capital ratio of about 2.1.

Fig. 3 illustrates the relationship between infrastructure investment and the long run distribution of wealth that corresponds to Proposition 3. The steady state wealth distribution \( (9b') \) is shown to be decreasing (increasing) at public investment GDP ratio in two cases where the elasticity of substitution between public and private capital is greater (lesser) than unity. \(^\text{34}\) As \( |\rho| \) increases the slopes become steeper implying stronger effect of public investment on inequality. For instance, when \( \rho = 0.5 \), a 1.33% increase in public investment GDP share from its benchmark value decreases the steady state inequality by 0.39% whereas it decreases by only 0.24% when \( \rho = 0.25 \). In contrast, when \( \rho = -0.5 \), the same increase in public investment GDP share leads to 0.91% increase in steady state inequality compared to 0.35% when \( \rho = -0.25 \).

We now compare the infrastructure investment-growth relationship \( (14b) \) in two different scenarios, when \( \sigma_0^2 = 0 \) and \( \sigma_0^2 \neq 0 \). Fig. 4 illustrates this relationship for a variety of elasticity of substitutions. First, note that for every public investment policy, long run growth is much greater when \( \sigma_0^2 = 0 \) compared to \( \sigma_0^2 \neq 0 \) (Proposition 1). For instance, steady state growth rates in the former are higher than the latter by 5.76 , 5.56

\(^{30}\)Munnell, (1990) estimated the elasticity of output of public capital about 0.15 for the United States.

\(^{31}\)If (4) represents individuals’ income net of the cost of physical capital \( k_t^i \), then, when \( \rho = 0 \), gross and net incomes are given by \( \tilde{y}_t^i = A(k_t^i)^{\lambda_1}(h_t^i)^{\lambda_2}(G_t)^{\lambda_3} \) and \( y_t^i = (h_t^i)^{\alpha}(G_t)^{1-\alpha} \), respectively. That is, \( y_t^i = \tilde{y}_t^i - \frac{\partial \tilde{y}_t^i}{\partial k_t^i}k_t^i = \tilde{y}_t^i(1-\lambda_1) \). Since \( \frac{\partial \tilde{y}_t^i}{\partial k_t^i} = \frac{\partial \tilde{y}_t^i}{\partial \lambda_1} \), it follows that \( \alpha = \frac{\lambda_2}{1-\lambda_3} \) and \( 1 - \alpha = \frac{\lambda_3}{1-\lambda_3} \).

\(^{32}\)A psychological discount factor of 0.96 matches a 4.17 percent rate of time preference \( \vartheta \) in an infinite lived agent model. That is, \( \beta = 1/(1 + \vartheta) = 1/(1 + .0417) = 0.96 \).

\(^{33}\)Similar results could be found for other values of elasticity of substitutions as far as the restrictions in \( (4') \) and \( (4'') \) hold.
and 5.43 percents when $\rho = 0.4$, 0.2 and $-0.15$, respectively, at the benchmark. The intuition is that, as we have argued earlier, a greater inequality corresponds to a lower long run growth when the credit and the insurance markets are missing as these prevent the efficient amount of investment to be undertaken in the economy. Second, regardless of wealth distribution, a higher elasticity of substitution leads to a higher growth in line with Klump and de la Grandville, (2000). Note that when $\rho > 0$, the growth rates often remain positive in the figure under different values of $\tau$. For instance, when $\rho = 0.2$, (and $\sigma_0^2 \neq 0$), about two-third of the observations of $\gamma$ are positive, whereas more than 85 percent are negative when $\rho = -0.15$.

Third, however, the effect of policy in growth is more pronounced when there is a lower substitutability (or higher complementarity) between public and private capital. This is demonstrated in the figure where the slopes of the curves associated to smaller values of elasticity of substitution are more steeper than those with larger values. For instance, a 1.33 percent increase in tax from its benchmark value, when $\sigma_0^2 \neq 0$ and $\sigma_0^2 = 0$, will increase growth only by 0.31 and 0.15 percents when $\rho = 0.2$, respectively compared to 22.26 and 2.56 percents when $\rho = -0.05$. Moreover, comparing between $\sigma_0^2 \neq 0$ and $\sigma_0^2 = 0$, the growth effect of taxation is much stronger in the former due to the additional channel through which policy could influence growth. The intuition for that policy could have a more profound effect when the elasticity of substitution is lower is simple. A higher taxation lowers growth through distorting private activities though public investment raises efficiency. From growth maximizing perspective, whenever the elasticity of substitution between public and private capital is higher, it may be efficient to substitute factors and avoid higher taxation that creates inefficiency.

Fig. 5 corresponds to Proposition 4. We plot here the growth maximizing tax rates
Figure 4. Growth rate versus public investment-GDP share under a variety of elasticity of substitutions
that correspond to a range of values of elasticity of substitutions; \( \rho \) ranges within \(-0.4\) and \(0.4\) in 0.1 interval. The figure plots \( \tau^* \) versus \( \rho \) for the two cases, when \( \sigma_0^2 = 0 \) and \( \sigma_0^2 \neq 0 \). First, note that the growth maximizing tax rates in both cases are equal to 0.2, – the value of the public capital elasticity of output \((1 - \alpha)\), in line with Barro (1990), – when \( \rho = 0 \). Second, in both cases the growth maximizing taxes are decreasing at \( \rho \): when \( \rho \leq 0 \), then \( \tau^* \geq 0.2 \). Third, \( \tau^*|_{\sigma_0^2 \neq 0} \) is higher (lower) than \( \tau^*|_{\sigma_0^2 = 0} \) when \( \rho > 0 \) (\( \rho < 0 \)). For instance, \( \tau^*|_{\sigma_0^2 \neq 0} = 7.73\% \) (26.9\%) compared to \( \tau^*|_{\sigma_0^2 = 0} = 7.58\% \) (27.1\%), when \( \rho = 0.4 \) (\(-0.4\)).

![Graph showing growth maximizing tax rates versus \( \rho \) for different \( \sigma_0^2 \) values]

Figure 5. Growth maximizing tax rates and elasticity of substitutions

Finally, we conduct policy experiments to examine the dynamic effects of a 5\% increase in public investment GDP share on the economy. Figs. 6-8 depict and compare the transitional dynamics of the benchmark economy pre- and post-shock under a variety of elasticity of substitutions, \( \rho = 0 \), \(-0.15\) and \(0.15\). The left panels of the figures show the pre-shock dynamics of GDP, wealth distribution, output and consumption capital ratios, whereas the right panels show the post-shock values of these variables.

From Fig. 6, when \( \rho = 0 \), following the policy change, growth, output and consumption capital ratios all increase by more than 40.6\%, 1.15\% and 0.27\%, respectively, whereas there is no change in the steady state wealth distribution as expected. But, when \( \rho = 0.15 \), the increase in the steady state growth rate due to the shock is only 0.2\% while output and consumption ratios increase by only 0.88\% and 0.01\%, respectively (Fig 7). But inequality decreases by 0.54\% this time. The effects from the shock are relatively smaller compared to the case when \( \rho = 0 \) because public investment is now closer to its growth maximizing level. The growth maximizing tax rates when \( \rho = 0 \) and 0.15 are 0.2 and 0.154, respectively. On the other hand, from Fig. 8, when \( \rho = -0.15 \), the steady state
growth rate, output and capital consumption ratios increase by about 21%, 1.54%, and 0.66% though inequality also increases by 0.7%, following the shock. It is interesting to note that even if public investment is now more complementary than the previous two cases and thus that we expect it to have a stronger growth effect, the effect is lower than the case when $\rho = 0$. The reason is that in the latter the change in policy has no impact on inequality but growth. But, in the former, policy affects both growth and inequality simultaneously where the latter also affects growth. Therefore, in the case $\rho = -0.15$, the increase in growth due to the complementarity effect is compromised by the rise of inefficiency from the increase in inequality.

Figure 6. Effects of a 5% increase in public investment, when $\rho = 0$. 
Figure 7. Effects of a 5% increase in substitutable public investment, $\rho = 0.15$. 
Conclusion

All governments provide productive public goods that could serve mainly as complementary or substitutable to the private sector. Public policies associated to them are often identified on their efficiency and distributional role in the economy, respectively. In fact, certain public goods (such as roads and dams) mostly complement private investment (such as fertilizer, high-yielding variety of seeds, and trucks), and hence could have a positive impact on growth. Other public goods (such as public health system, and public transport) do rather mainly substitute private goods (such as private health system and private transport services), and have impact on income distribution. They disproportionately benefit those who lack these basic resources.

Under certain conditions, these policies have a rather lesser distinct role, however. In heterogenous economy with imperfect credit market, redistributive public policies may also have efficiency benefit as they could substitute for the missing credit market. They serve as means to relax the resource constraints that impede certain investment. The efficiency effect of complementary public investment, on the other hand, is compromised as it

Figure 8. Effects of a 5% increase in complementary public investment, when $\rho = -0.15$. 
aggravates existing inequality. It disproportionately benefits the individuals who have access to those private goods. Internet provision by the public sector may disproportionately benefit those individuals who own laptop, for instance.

This paper has developed a growth model that features this issue. The model was calibrated for real economies and employed to examine the effects of public investment policy on wealth mobility and growth. The source of endogenous growth is public capital that combines with private capital in a constant return to scale production function that makes the cumulative marginal product of total capital in the long run constant. Public investment could be complementary or substitutable to private investment. Endogenous inequality is rather generated due to missing credit and insurance markets. When individuals cannot perfectly ensure themselves from future income uncertainty and, the credit market is imperfect, wealth mobility and inequality emerge and persist. The dynamics of aggregate variables and wealth distribution are jointly determined. Imperfection in capital markets and existence of diminishing returns to private investment imply a suboptimal level of private investment in this inegalitarian society. The paper showed that public investment policy that forwards efficiency constitute of both complementary and substitutable public investments. However, those that forward both equity and efficiency constitute of substitutable public investment though their marginal growth effect is diminished.

A. Approximation

We need only to log-linearize the nonlinear term in (7b), 

\[ f (\phi_i^t) \equiv \ln (\alpha \exp (\rho \phi_i^t) + 1 - \alpha), \]

at a stationary point, (see also Campbell 1994, appendix A.1). A natural choice for the stationary point is the steady state of the economy \( E [\phi^i] = \phi \). Then, Taylor’s approximation of \( f (\phi_i^t) \) is

\[ f (\phi_i^t) \approx f (\phi) + f' (\phi) (\phi_i^t - \phi) \]

where

\[ f' (\phi) = \frac{\alpha \rho \exp (\rho \phi)}{\alpha \exp (\rho \phi) + (1 - \alpha)} \]  

(A.1.1)

For \( \rho \phi \) small,

\[ f' (\phi) \approx \frac{\alpha \rho (1 + \rho \phi)}{\alpha (1 + \rho \phi) + (1 - \alpha)} = \frac{\alpha \rho (1 + \rho \phi)}{\alpha \rho \phi + 1} \]

Thus,

\[ f (\phi_i^t) \approx \varphi + \rho (1 - \omega) \phi_i^t \]  

(A.1.2)

where \( \varphi \equiv \ln (\alpha \exp (\rho \phi) + (1 - \alpha)) - \rho (1 - \omega) \) and \( \omega \equiv \frac{1 - \alpha}{\alpha \rho \phi + 1} \). Substituting (A.1.2) back into (7b), we obtain its log-linear equivalent, (7b’):

\[ \ln h_{t+1}^i \approx \ln (Aa(1 - \tau)) BG_t + \ln \xi_{t+1}^i + \varphi + (1 - \omega) \phi_i^t \]  

(7b')
Through aggregating (7b'), we obtain (10b), i.e.,

\[
E[\ln h_{t+1}^i] = \ln H_{t+1} - \frac{1}{2} \sigma_{t+1}^2
\]
\[
= \ln (Aa(1 - \tau))BG_t + \varrho - \frac{\nu^2}{2} + (1 - \omega)E[\phi_t^i]
\]

since \(E[\phi_t^i] = \phi_t - \frac{1}{2}\sigma_t^2\) and \(\sigma_{t+1}^2 \equiv \text{var}[\ln h_{t+1}^i] = \nu^2 + (1 - \omega)^2 \sigma_t^2\). Thus,

\[
\ln H_{t+1} = \ln (Aa(1 - \tau))BG_t + \varrho + (1 - \omega)\phi_t - \frac{1}{2} \sigma_t^2 (1 - \omega)(\omega)
\]  \hspace{1cm} (10b)

With respect to the approximation and aggregation of (5), \(G_{t+1} = \tau E[y_t^i(h_t^i)]\), first note that \(y_t^i(h_t^i)\) is log-normally distributed since \(h_t^i\) is log-normal. Then,

\[
\ln E[y_t^i] = E[\ln y_t^i] + \frac{1}{2} \text{var}[\ln y_t^i]
\]  \hspace{1cm} (A.1.3)

From (4), (5), and (A.1.3), we obtain

\[
\ln G_{t+1} = \ln AG_t + \frac{1}{\rho} E[\ln (\alpha \exp(\rho \psi_t^i) + 1 - \alpha)]
\]
\[
+ \frac{1}{2\rho^2} \text{var}[\ln (\alpha \exp(\rho \psi_t^i) + 1 - \alpha)]
\]  \hspace{1cm} (A.1.4)

By substituting the approximation in (A.1.1) into (A.1.4), we obtain its log-linear equivalent (11b):

\[
\ln G_{t+1} \approx \ln AG_t + (1 - \omega)\phi_t + \varrho - \frac{1}{2} (1 - \omega)\omega \sigma_t^2
\]  \hspace{1cm} (11b)

B. Proofs

B.1. Stability of equilibria

The Jacobian matrix corresponds to the system of wealth mobility and capital ratio dynamics of the aggregate economy, (8a) and (13a), is given by,

\[
\varpi_{cd} = \begin{bmatrix}
(1 - (1 - \alpha) \eta)^2 & 0 \\
-\frac{1}{2} (1 - \alpha) ((\eta - \alpha) - (1 - \alpha) \eta^2) & (1 - \alpha)(1 - \eta)
\end{bmatrix}
\]  \hspace{1cm} (B.1.1)

whereas the Jacobian associated to the individuals' dynamics, (8a), (12a) and \(\xi_{t+1}^i = \xi_t^i\),
(see footnote 18), is

$$
\varpi_{cd}^i = \begin{pmatrix}
(1 - (1 - \alpha) \eta)^2 & 0 & 1 \\
\alpha (1 - \alpha) & (1 - \eta (1 - \alpha)) & 0 \\
0 & 0 & 1
\end{pmatrix}
$$

(B.1.2)

The det and tr associated to them are given by:

$$
\det (\varpi_{cd}) = (1 - (1 - \alpha) \eta)^2 (1 - \alpha) (1 - \eta) < 1 \quad \text{(B.1.3)}
$$

$$
\tr (\varpi_{cd}) = (1 - (1 - \alpha) \eta)^2 + (1 - \alpha) (1 - \eta) \quad \text{(B.1.4)}
$$

$$
\det (\varpi_{cd}) = (1 - (1 - \alpha) \eta)^2 (1 - \alpha) (1 - \eta) - \alpha (1 - \alpha) < 1 \quad \text{(B.1.5)}
$$

$$
\tr (\varpi_{cd}) = (1 - (1 - \alpha) \eta)^2 + (1 - \alpha) (1 - \eta) + 1 \quad \text{(B.1.6)}
$$

Then, considering (4'), it is easy to see that $|\det (\varpi_{cd})| < 1 \& |\det (\varpi_{cd}) + 1| < |\tr (\varpi_{cd})|$. Therefore, the steady state at aggregate is a sink (see De la Croix 2002, appendix A.3.4). It is straightforward to see that the system at individual is also sink path stable.

Similarly, with respect to the CES case, we obtain the Jacobian matrix corresponds to (8b), (12b), and $\xi_{t+1}^i = \xi_t^i$,

$$
\varpi_{ces}^i = \begin{pmatrix}
(1 - \omega)^2 & 0 & 1 \\
\omega (1 - \omega) & (1 - \omega) & 0 \\
0 & 0 & 1
\end{pmatrix}
$$

(B.1.7)

where the det and tr are given by,

$$
\det (\varpi_{ces}) = (1 - \omega)^2 (1 - \omega) - \omega (1 - \omega) < 1 \quad \text{(B.1.8)}
$$

$$
\tr (\varpi_{ces}) = (1 - \omega)^2 + (1 - \omega) + 1 \quad \text{(B.1.9)}
$$

Then, given (4'), $\omega \in (0, 1)$. And, $|\det (\varpi_{ces})| < 1 \& |\det (\varpi_{ces}) + 1| < |\tr (\varpi_{ces})|$. Therefore, the system of wealth mobility and capital dynamics at individual in the CES case is also sink path stable.

**B.2. Existence and uniqueness $\phi$**

We prove here existence and uniqueness of $\phi^i$, the individual capital ratio in (7b), considering (5). From (3), (4), (5) and (6), we obtain

$$
\varphi_{t+1}^i = \frac{B \varphi_t^i (a(1 - \tau))^{\eta} A_{\eta-1} \varphi_t^i \left( f (\varphi_t^i)^{(1)} \right)^{\eta}}{E[f (\varphi_t^i)]}
$$

(B.2.1)

where $\varphi^i \equiv \frac{h^i}{c_t}; f (\varphi_t^i) \equiv (\alpha \exp (\varphi_t^i)^{\alpha} + (1 - \alpha))^\frac{1}{\alpha} > 0; f (\varphi_t^i)^{(1)} \equiv (\alpha + (1 - \alpha) (\varphi_t^i)^{-\alpha})^\frac{1}{\alpha} > 0$.

In the steady state,
\[ \psi \equiv \frac{B \xi^i (a(1 - \tau))^n}{\tau} A^{n-1} f \left( \left( \varphi^i \right)^{-1} \right)^n - \mathbb{E} \left[ f \left( \varphi^i \right) \right] = 0 \] (B.2.2)

Note first that \( \frac{\partial}{\partial \varphi} \mathbb{E} \left[ f \left( \varphi^i \right) \right] = f'_\varphi \left( \varphi^i \right) \). Then, it is easy to see \( \psi'_\varphi < 0 \) because \( f'_\varphi \left( \varphi^i \right) > 0 \) and \( f'_\varphi \left( \left( \varphi^i \right)^{-1} \right) < 0 \). Moreover, \( \psi > 0 \) for sufficiently small \( \varphi^i \), whereas \( \psi < 0 \) for sufficiently large \( \varphi^i \). For instance, \( \lim_{\varphi^i \to 0} \mathbb{E} \left[ f \left( \varphi^i \right) \right] = (1 - \alpha)^\frac{1}{2} \) and \( \lim_{\varphi^i \to \infty} \mathbb{E} \left[ f \left( \varphi^i \right) \right] = \infty \).

**B.3. Proposition 3**

Based on (13b), and noting \( \varphi^i(1) \), we obtain,

\[ \sigma^2_\tau \left( \tau \right) = -\omega'_\varphi \frac{2v^2 \left( 1 - \omega \right)}{\omega \left( 2 - \omega \right)} = \rho \phi'_\tau \tau \] (B.3.1)

where \( \tau \equiv v^2 \frac{(1 - \omega)\omega}{2 - \omega} \frac{\alpha}{1 - \alpha} \); \( \omega'_\varphi = -\frac{\alpha}{1 - \alpha} \rho \omega^2 \phi'_\tau \left( \tau \right) \). From (12b'), \( \phi'_\tau < 0 \). Therefore, \( \sigma^2_\tau \left( \tau \right) \leq 0 \) when \( \rho \geq 0 \).

**B.4. Growth rate**

We now examine whether the second derivative of (14b) with respect to \( \tau \) is negative. We can rewrite (14b) using \( \omega \left( \tau \right) \equiv \frac{1 - \alpha}{\alpha \rho \phi(\tau) + 1} \),

\[ \gamma = \ln A \tau + \alpha \frac{\rho \phi(\tau) + 1}{\alpha \rho \phi(\tau) + 1} \phi(\tau) - \frac{v^2}{2} \alpha \frac{\rho \phi(\tau) + 1}{2 \alpha \rho \phi(\tau) + 1 + \alpha} \] (B.4.1)

where \( \phi(\tau) \) is given by (13b).

In case of a representative agent economy with complete market \( \sigma^2 = v^2 = 0 \), the growth rate becomes

\[ \gamma|_{v^2=0} = \ln A \tau + \alpha \frac{\rho \phi(\tau) + 1}{\alpha \rho \phi(\tau) + 1} \phi(\tau) \] (B.4.2)

The growth maximizing tax rate \( \tau^* \) with respect to (B.4.1) is given by the implicit equation,

\[ \gamma'_\tau = \frac{\epsilon (\tau^*, \rho, \alpha)}{0} = \frac{1}{\tau} + \phi'_\tau \left( 1 + \frac{1}{(\alpha \rho \phi + 1)^2} \right) - \frac{v^2}{2} \alpha \rho \phi'_\tau \frac{1 - \alpha}{(\alpha \rho \phi + 1 + \alpha)^2} \] (B.4.3)

The second derivative of (B.4.3) is given by,

\[ \gamma''_{\tau\tau} = -\frac{1}{\tau^2} + \phi''_{\tau\tau} \theta + 2 \alpha \rho (\phi'_\tau)^2 \left( 1 - \alpha \right) (\alpha \rho \phi + 1) - \frac{v^2}{2} \pi \] (B.4.4)
where \( \theta \equiv 1 + \frac{\alpha - 1}{(\alpha \rho \phi + 1)^2} \) and \( \pi \equiv \rho (\phi'_\tau)^2 \frac{\alpha (1 - \alpha)}{(\alpha \rho \phi + 1 + 1 + \alpha)} \left( -(2 \tau - 1) - \frac{2 \alpha \rho}{\alpha \rho \phi + 1 + \alpha} \right) \).

Let’s first examine whether \( \gamma'_{\tau \tau} |_{v^2=0} < 0 \), (i.e., when \( \sigma^2 = v^2 = 0 \)). Then, from (13b), we have

\[
\phi'_\tau = \frac{1}{\tau (\tau - 1)} < 0 \quad \text{and} \quad \phi''_{\tau \tau} = -(\phi'_\tau)^2 (2 \tau - 1) \tag{B.4.5}
\]

Substituting this into (B.4.4), we obtain,

\[
\gamma'_{\tau \tau} |_{v^2=0} = -\frac{1}{\tau^2} + \phi''_{\tau \tau} \theta + \alpha \rho (\phi'_\tau)^2 (2 (1 - \alpha) (\alpha \rho \phi + 1)) = (\phi'_\tau)^2 \rho \tag{B.4.6}
\]

where

\[
\zeta \equiv -(\tau - 1)^2 - (2 \tau - 1) \theta + \alpha \rho (2 (1 - \alpha) (\alpha \rho \phi + 1)) \tag{B.4.7}
\]

Apparently, \( \gamma'_{\tau \tau} < 0 |_{v^2=0} \) if \( \zeta < 0 \). Since \( \theta \in (0, 1) \), (considering \( 4'' \)), and \( -(\tau - 1)^2 - (2 \tau - 1) < 0 \), then \( -(\tau - 1)^2 - (2 \tau - 1) \theta < 0 \). If \( \rho < 0 \), the last term in (B.4.7) is also negative, therefore \( \gamma'_{\tau \tau} < 0 |_{v^2=0} \).

If \( \rho > 0 \), however, the necessary and sufficient condition \( \gamma'_{\tau \tau} < 0 |_{v^2=0} \) is that

\[
-\tau^2 - (2 \tau - 1) \frac{\alpha - 1}{(\alpha \rho \phi + 1)^2} < -\alpha \rho (2 (1 - \alpha) (\alpha \rho \phi + 1)) \tag{B.4.8}
\]

Thus, \( \rho (> 0) \) should be sufficiently small for (B.4.8) to be satisfied. This is intuitive. As the elasticity of substitution is getting larger and hence the degrees of substitutability getting higher, from growth maximizing perspective, it is rather efficient to substitute factors and avoid higher taxation that creates inefficiency, consequently, making more difficult finding a positive growth maximizing tax rate.

If the conditions for \( \gamma'_{\tau \tau} < 0 |_{v^2=0} \) are satisfied, the general case that \( v^2 \neq 0 \) is rather simple because \( \gamma'_{\tau \tau} < 0 |_{v^2=0} \) since \( v^2 \in (0, \infty) \).

**B.5. proposition 5**

We have just shown that \( \gamma |_{v^2 \neq 0} \) and \( \gamma |_{v^2=0} \) are both concave under certain conditions. Under these conditions, considering (B.4.3) and (B.4.5), the slope of \( \gamma |_{v^2 \neq 0} \leq \gamma |_{v^2=0} \) if \( \rho \leq 0 \), and hence \( \max \{ \gamma |_{v^2 \neq 0} \} \leq \max \{ \gamma |_{v^2=0} \} \). Therefore, \( \tau^* |_{v^2 \neq 0} \leq \tau^* |_{v^2=0} \).

References


